Asymptotic expansions of approximate eigenvalues of integral operator with algebraic multiplicity m .

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Sketch of the talk

1. Notations and preliminaries.
2. Eigenvalue problems.
3. **Operator Approximations:**
   Iterated Galerkin Approximations.
4. **Asymptotic expansions:**
   Eigenvalue with algebraic multiplicity 1.
5. **Asymptotic expansions:**
   Eigenvalue with algebraic multiplicity $m > 1$.
6. Numerical examples.

Notations

1. X : complex Banach space.
2. $BL(X)$: Bounded linear operators from X to X.
3. $T \in BL(X)$.
4. Resovent : $re(T) = \left\{z \in \mathbb{C} : (T - zI)^{-1} \in BL(X)\right\}$.
5. Spectrum : $sp(T) = \mathbb{C} \setminus re(T)$.
6. T : compact linear operator.

$$(zI - T_1)^{-1} - (zI - T_2)^{-1} = (zI - T_1)^{-1}(T_1 - T_2)(zI - T_2)^{-1}.$$

$T : X \to X$ Looking for 1 dimensional invariant subspace:

$$M = span\ \{x\}$$

gives

$$Tx = \lambda x.$$

In general looking for m dimensional invariant subspace

$$M = span\ \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$$

gives

$$T\varphi_j = \theta_{1,j}\varphi_1 + \theta_{2,j}\varphi_2 + \ldots + \theta_{m,j}\varphi_m$$

if

$$\underline{T}[\varphi_1, \varphi_2, \ldots, \varphi_m] := [T\varphi_1, T\varphi_2, \ldots, T\varphi_m]$$

Thus

$$\underline{T}\underline{\varphi} = \underline{\varphi}\Theta$$

Eigenvalue problem

$$T_n \varphi_n = \lambda_n \varphi_n.$$

Let

$$T_n : \text{finite rank and continuous.}$$

Then

- The eigenvalue problem $\rightarrow$ Matrix eigenvalue problem.

Description of the Iterated Galerkin method

For any $n \in \mathbb{N}$, let $h = \frac{1}{n}$ and

$$t_i = (i-1)h, \; i = 1, \ldots, (n+1),$$

Uniform Partition :

$$0 = t_1 < t_2 < \cdots < t_{n+1} = 1$$

X_n : piecewise polynomials degree $r-1$, $X = L^2[0,1]$,

$$\langle x, y \rangle = \int_0^1 x(t)\overline{y(t)}dt. \tag{0.1}$$

$\pi_n : L^2[0,1] \to X_n$ orthogonal projection.

$$T\pi_n\varphi_n = \lambda_n\varphi_n$$

or

$$\underline{T}\underline{\pi_n}\underline{\varphi_n} = \Theta_n\underline{\varphi_n}$$

Integral operator and Green's Function Type kernels

$$(Tx)(s) = \int_0^1 k(s,t)\, x(t) dt, \;\; s \in [0,1].$$

Definition

1. $k(\cdot,\cdot) \in C([0,1] \times [0,1])$.
2. $k(\cdot,\cdot) \in C^{2r+2}([0,1] \times [0,1] \setminus D)$ where
$$D = \{(s,t) \in (0,1) \times (0,1) : s = t\}.$$
3. $$k^{(i,j)}(s,t) = \left(\frac{\partial^{i+j}}{\partial s^i \partial t^j}\right) k(s,t)$$
$k^{(0,j)}(s,0+)$, $k^{(0,j)}(s,1-)$, $k^{(0,j)}(s,s+)$ and $k^{(0,j)}(s,s-)$, $s \in [0,1]$ exist for $j = 1,\ldots,2r+2$. A similar property holds for the first argument.

Note that T is compact.

$$(Tx)(s) = \int_0^1 k(s,t)\, x(t)dt,\ s \in [0,1].$$

1. Note that T^* is also compact with the same type of kernel.
2. The eigenfunctions of T and T^* have same smoothness properties.
3. Infact $\varphi, \varphi^* \in C^{2r+2}[0,1]$.

Let $\lambda \neq 0$ simple eigenvalue of T.
Γ be a positively oriented circle of radius ϵ.

Definition

$$P = -\frac{1}{2\pi i}\int_{\Gamma}(T - zI)^{-1}dz = -\frac{1}{2\pi i}\int_{\Gamma} R(z)dz.$$

$\Gamma \subset \rho(T\pi_n)$.

Definition

$$P_n = -\frac{1}{2\pi i}\int_\Gamma (T\pi_n - zI)^{-1}dz = -\frac{1}{2\pi i}\int_\Gamma R_n(z)dz$$

Theorem

Let λ be a nonzero eigenvalue of T with algebraic multiplicity m. Let T_n converge to T. Then there is a positive integer n_0 such that Γ is contained in $\rho(T_n)$ and there are m eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m}$ inside Γ. Also for each large n, $\underline{P_n\varphi}$ is an ordered basis of Range P_n if $\underline{\varphi}$ is an ordered basis of Range P, where P_n denotes the spectral projection associated with T_n.

Eigenvalue problem: Green's kernel

Lemma

Let λ be a simple nonzero eigenvalue of T with eigenvector φ and $T^*\varphi^* = \bar{\lambda}\varphi^*$. Let $\langle \varphi, \varphi^* \rangle = 1$. Then

$$(\lambda - \lambda_n)\langle P_n\varphi, \varphi^* \rangle = \langle T(I - \pi_n)P_n\varphi, \varphi^* \rangle. \tag{0.2}$$

Proof.

$$\begin{aligned}
(\lambda - \lambda_n)\langle P_n\varphi, \varphi^* \rangle &= \lambda\langle P_n\varphi, \varphi^* \rangle - \lambda_n\langle P_n\varphi, \varphi^* \rangle \\
&= \langle P_n\varphi, \bar{\lambda}\varphi^* \rangle - \langle \lambda_n P_n\varphi, \varphi^* \rangle \\
&= \langle P_n\varphi, T^*\varphi^* \rangle - \langle T\pi_n P_n\varphi, \varphi^* \rangle \\
&= \langle TP_n\varphi, \varphi^* \rangle - \langle T\pi_n P_n\varphi, \varphi^* \rangle \\
&= \langle T(I - \pi_n)P_n\varphi, \varphi^* \rangle.
\end{aligned}$$

□

Lemma

Let λ be a simple nonzero eigenvalue of T with eigenvector φ and $T^*\varphi^* = \bar{\lambda}\varphi^*$. Let $\langle \varphi, \varphi^* \rangle = 1$. Then

$$\begin{aligned} \langle T(I-\pi_n)P_n\varphi, \varphi^* \rangle &= \langle T(I-\pi_n)\varphi, \varphi^* \rangle \\ &+ \langle V_n(\lambda) T(I-\pi_n)\varphi, \bar{\lambda}(I-\pi_n)\varphi^* \rangle, \end{aligned}$$

where

$$V_n(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{R_n(z)}{\lambda - z}dz.$$

Lemma

Let λ be a simple nonzero eigenvalue of T with eigenvector φ, and $T^*\varphi^* = \bar{\lambda}\varphi^*$. Let $\langle \varphi, \varphi^* \rangle = 1$. Then

$$\langle P_n\varphi, \varphi^* \rangle = 1 + \frac{1}{2\pi i}\langle \int_\Gamma \frac{R(z)T(\pi_n - I)R_n(z)T(I - \pi_n)\varphi}{\lambda - z} dz, \varphi^* \rangle. \quad (0.3)$$

Theorem

Let $\alpha, \beta \in C^{2r+2}[0,1]$. Then

$$\int_0^1 \alpha(t)(I - \pi_n)\beta(t)dt = \psi_{2r}h^{2r} + O(h^{2r+2}), \quad (0.4)$$

Theorem

Let $\alpha, \beta \in C^{2r+2}[0,1]$. Then

$$\int_0^1 \alpha(t)(I - \pi_n)\beta(t)dt = \psi_{2r} h^{2r} + O(h^{2r+2}), \qquad (0.5)$$

$\langle T(I - \pi_n)x, y\rangle$, for $x, y \in C^{2r+2}[0,1]$.

Theorem

Let T be an integral operator with a kernel of class $\mathcal{C}(2r+2, 0)$. Let $x, y \in C^{2r+2}[0,1]$, then

$$\langle T(I - \pi_n)x, y\rangle = A_{2r} h^{2r} + O(h^{2r+2}), \qquad (0.6)$$

where A_{2r} is a scalar independent of h.

Note $\varphi, \varphi^* \in C^{2r+2}[0,1]$,

Proof.

$$
\begin{aligned}
\langle T(I-\pi_n)x, y\rangle &= \int_0^1 [T(I-\pi_n)x](s)\overline{y(s)}ds \\
&= \int_0^1 \left(\int_0^1 k(s,t)(I-\pi_n)x(t)\,dt\right)\overline{y(s)}ds \\
&= \int_0^1 \left(\int_0^s k_2(s,t)(I-\pi_n)x(t)\,dt\right)\overline{y(s)}ds \\
&+ \int_0^1 \left(\int_s^1 k_1(s,t)(I-\pi_n)x(t)\,dt\right)\overline{y(s)}ds.
\end{aligned}
$$

□

Proof.

Interchanging the order of integration, we get

$$\begin{aligned}
\langle T(I-\pi_n)x, y\rangle &= \int_0^1 \left(\int_t^1 k_2(s,t)\overline{y(s)}\, ds\right)(I-\pi_n)x(t)dt \\
&+ \int_0^1 \left(\int_0^t k_1(s,t)\overline{y(s)}\, ds\right)(I-\pi_n)x(t)dt \\
&= \int_0^1 \alpha_1(t)\,(I-\pi_n)x(t)dt \qquad (0.7)\\
&+ \int_0^1 \alpha_2(t)\,(I-\pi_n)x(t)dt, \qquad (0.8)
\end{aligned}$$

$\alpha_1, \alpha_2 \in C^{2r+2}[0,1]$. □

Theorem

Let T be an integral operator with a kernel of class $\mathcal{C}(2r+2,0)$. Let λ be a simple nonzero eigenvalue of T with eigenvector φ and $T^*\varphi^* = \bar{\lambda}\varphi^*$. Let $\langle \varphi, \varphi^* \rangle = 1$. Then

$$\langle T(I-\pi_n)P_n\varphi, \varphi^* \rangle = A_{2r}h^{2r} + O(h^{2r+2}), \tag{0.9}$$

where A_{2r} is a scalar independent of h.

Theorem

Let T be an integral operator with a kernel of class $\mathcal{C}(2r+2,0)$. Let λ be a simple nonzero eigenvalue of T with eigenvector φ, $T^*\varphi^* = \bar{\lambda}\varphi^*$. Let $\langle \varphi, \varphi^* \rangle = 1$.Then

$$\langle P_n\varphi, \varphi^* \rangle = 1 + O(h^{2r+2}). \tag{0.10}$$

$$(\lambda - \lambda_n) = \frac{\langle T(I-\pi_n)P_n\varphi, \varphi^* \rangle}{\langle P_n\varphi, \varphi^* \rangle}. \tag{0.11}$$

Lemma

Let λ be a nonzero eigenvalue of T with algebraic multiplicity m, and $\underline{T^*}\underline{\varphi^*} = \underline{\varphi^*}\Theta^*$. Let $\underline{\varphi}$ be a basis of Range P which is adjoint to $\underline{\varphi^*}$. Then

$$\lambda - \widehat{\lambda}_n = \frac{1}{m} trace(\underline{T}\ \underline{P_n}\underline{\varphi} - \underline{T\pi_n}P_n\underline{\varphi}, \underline{\varphi^*}) (\underline{P_n}\ \underline{\varphi}, \underline{\varphi^*})^{-1},$$

where P_n denotes the spectral projection associated with T_n and $\widehat{\lambda}_n$ denotes the arithemetic mean of the eigenvalues of T_n lying inside the curve Γ.

Proof.

Since $\underline{T_n P_n}\, \underline{\varphi} = \underline{P_n}\underline{\varphi}\Theta_n$ we have

$$
\begin{aligned}
\langle \underline{T}\, \underline{P_n}\, \underline{\varphi} - \underline{T_n P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle &= \langle \underline{T}\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle - \langle \underline{T_n}\, \underline{P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle \\
&= \langle \underline{P_n}\, \underline{\varphi}, \underline{T^*}\underline{\varphi^*} \rangle - \langle \underline{P_n}\, \underline{\varphi}\Theta_n, \underline{\varphi^*} \rangle \\
&= \langle \underline{P_n}\underline{\varphi}, \underline{\varphi^*}\Theta^* \rangle - \langle \underline{P_n}\underline{\varphi}\Theta_n, \underline{\varphi^*} \rangle \\
&= \Theta \langle \underline{P_n}\underline{\varphi}, \underline{\varphi^*} \rangle - \langle \underline{P_n}\underline{\varphi}, \underline{\varphi^*} \rangle \Theta_n \\
&= -\langle \underline{P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle (\Theta_n - \Theta) \\
&- \langle \underline{P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle \Theta + \Theta \langle \underline{P_n}\, \underline{\varphi}, \underline{\varphi^*} \rangle .
\end{aligned}
$$

Let

$$R(\Theta, z) := (\Theta - z\mathrm{I})^{-1}.$$

□

Proof.

Using the Resolvent identity,

$$
\begin{aligned}
(\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*}) &= (\underline{P}\, \underline{\varphi}, \underline{\varphi^*}) \\
&- (\frac{1}{2\pi} \int_\Gamma \underline{R_n(z)}(\underline{T} - \underline{T_n})\underline{\varphi} R(\Theta, z) dz, \underline{\varphi^*}) \\
&= I - (\frac{1}{2\pi} \int_\Gamma \underline{R_n(z)}(\underline{T} - \underline{T_n})\, \underline{\varphi} R(\Theta, z) dz, \underline{\varphi^*})
\end{aligned}
$$

$(\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})$ is invertible. Hence

$$
(\underline{T}\, \underline{P_n}\, \underline{\varphi} - \underline{T_n P_n}\, \underline{\varphi}, \underline{\varphi^*}) (\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})^{-1} =
$$

$$
(\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})(\Theta_n - \Theta)(\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})^{-1} - (\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})\Theta(\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})^{-1} + \Theta.
$$

$$
\text{trace}(\Theta - \Theta_n) = \text{trace}(\underline{T}\, \underline{P_n \varphi} - \underline{T_n} P_n\, \underline{\varphi}, \underline{\varphi^*}) (\underline{P_n}\, \underline{\varphi}, \underline{\varphi^*})^{-1}
$$

□

Lemma

Let λ be a nonzero eigenvalue of T with algebraic multiplicity m, and $\underline{T}^*\underline{\varphi}^* = \underline{\varphi}^*\Theta^*$. Let $\underline{\varphi}$ be a basis of (P) which is adjoint to $\underline{\varphi}^*$. Then

$$\langle \underline{T}(\underline{I} - \underline{\pi_n})\underline{P_n}\, \underline{\varphi}, \underline{\varphi}^* \rangle = \langle \underline{T}(\underline{I} - \underline{\pi_n})\underline{\varphi}, \underline{\varphi}^* \rangle -$$

$$\frac{1}{2\pi}\int_\Gamma \Theta \langle \underline{R_n(z)}\underline{T}(\underline{I} - \underline{\pi_n})\underline{\varphi}, (\underline{I} - \underline{\pi_n})\underline{\varphi}^* \rangle \langle \underline{R(z)}\underline{\varphi}, \underline{\varphi}^* \rangle dz.$$

Lemma

The norm of the integrand is

$$\|\Theta \langle \underline{R_n(z)}\underline{T}(\underline{I} - \underline{\pi_n})\underline{\varphi}, (\underline{I} - \underline{\pi_n})\underline{\varphi}^* \rangle \langle \underline{R(z)}\underline{\varphi}, \underline{\varphi}^* \rangle \|_F = O(h^{2r+2}).$$

Lemma

Let $\underline{T_n} = \underline{T}\underline{\pi_n}$, then

$$\langle \underline{P_n}\,\underline{\varphi}, \underline{\varphi^*} \rangle = I$$

$$+\frac{1}{2\pi}\int_{\Gamma} \langle \underline{R_n(z)}(\underline{T} - \underline{T_n})\underline{R(z)}(\underline{T} - \underline{T_n})\underline{\varphi}, \varphi^* \rangle \langle \underline{R(z)}\underline{\varphi}, \underline{\varphi^*} \rangle dz.$$

Corollary

Let T be an integral operator with a kernel of class $\mathbb{C}(2r+2, 0)$. Then

$$\langle (\underline{T} - \underline{T_n})\underline{\varphi}, \underline{\varphi^*} \rangle = Bh^{2r} + O(h^{2r+2}).$$

Theorem

Let λ be an eigenvalue of algebraic multiplicity m. Then

$$\lambda - \widehat{\lambda}_n = \alpha h^{2r} + O(h^{2r+2}),$$

$$k(s,t) = \begin{cases} s(1-t) & \text{if} \quad s \leq t, \\ t(1-s) & \text{if} \quad t < s. \end{cases}$$

We approximate the largest eigenvalue $\frac{1}{\pi^2}$ by λ_n

Iterated Galerkin method-piecewise constant polynomials

n	$\lvert\lambda - \lambda_n\rvert$	α_0	$\lvert\lambda - \lambda_n^{(1)}\rvert$	α_1
4	4.10×10^{-3}			
8	1.03×10^{-3}	2.00	6.34×10^{-7}	
16	2.56×10^{-4}	2.00	2.52×10^{-8}	4.66
32	6.40×10^{-5}	2.00	1.35×10^{-9}	4.22
64	1.60×10^{-5}	2.00	8.01×10^{-11}	4.06
128	4.00×10^{-6}	2.00	4.99×10^{-12}	4.02

$$\int_0^1 k(s,t)\,\varphi(t)dt = \lambda\,\varphi(s), \quad 0 \le s \le 1, \tag{0.12}$$

with

$$k(s,t) := \begin{cases} s/2 & \text{for } s \le t, \\ t - s/2 & \text{for } t < s, \end{cases}$$

$\dfrac{1}{(2j-1)^2\pi^2}$, $j = 1, 2, \ldots$ each of multiplicity 2.

Iterated Galerkin method-piecewise constant polynomials

n	$\lvert\lambda - \hat{\lambda}_n\rvert$	α_0	$\lvert\lambda - \hat{\lambda}_n^{(1)}\rvert$	α_1
32	8.04×10^{-3}			
64	2.03×10^{-3}	2.00	2.48×10^{-7}	
128	5.03×10^{-4}	2.00	1.53×10^{-8}	4.01
256	1.27×10^{-6}	2.00	9.58×10^{-10}	4.00
512	3.18×10^{-7}	2.00	5.99×10^{-11}	4.00
1024	7.90×10^{-8}	2.00	3.74×10^{-12}	4.00

(1) A. S. RANE, A note on asymptotic expansions for approximate eigenvalues of integral operators with Green's kernels in the case of iterated Galerkin method. Numerical Functional Analysis and Optimization, 36 (2015),1067-1086.

(2) A. S. RANE, Asymptotic Expansions for Approximate Eigenvalues of Integral Operators with Non-Smooth Kernels of Multiplicity $m > 1$. (Accepted for publication in Journal of Integral Equations and Applications.)

Thank you all