On Continuity and Well-Posedness of the Restricted Center Multifunction

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Outline

Abstract

Given a finite dimensional subspace V and a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of $\mathcal{C}_0(T)$, where T is a locally compact Hausdorff space, we survey here some interesting results on continuity and well-posedness of the *restricted center multifunction* $C_V : \mathcal{F} \rightrightarrows V$. In particular, we are concerned with a Haar-like *intrinsic characterization* of finite dimensional subspaces V of $\mathcal{C}_0(T)$ that yields lower semicontinuity of the multifunction C_V as well as Hausdorff strong uniqueness of restricted centers in V. The equivalence of Hausdorff continuity and Hausdorff Lipschitz continuity of the multifunction C_V is also examined here.

Introduction

Given a nonempty subset V of a metric space X and a function $I : X \to (-\infty, \infty]$ which is a proper extended real-valued function, let us consider well-posedness of the following abstract minimization problem:

$$\min I(v), v \in V,$$

which we denote by (V, I). Let

$$v_V(I) := \inf\{I(v) : v \in V\}$$

denote the *optimal value function*.

We assume I to be lower bounded on V, i.e., $v_V(I) > -\infty$, and let $\arg\min_V(I)$ denote the (possibly void) set

$$\{v \in V : I(v) = v_V(I)\}$$

of optimal solutions of problem (V, I).

For $\epsilon \geq 0$, let us also denote by $\epsilon\text{-}\arg\min_V(I)$ the nonempty set

$$\{v \in V : I(v) \leq v_V(I) + \epsilon\}$$

of ϵ-approximate minimizers of I.

Introduction continued

Recall that problem (V, I) is said to be

- **Tikhonov well-posed** if I has a unique global minimizer on V towards which every **minimizing sequence** (i.e., a sequence $\{v_n\} \subset V$, such that $I(v_n) \to v_V(I)$) converges.

Put differently,

- there exists a point $v_0 \in V$ such that $\arg\min_V(I) = \{v_0\}$, and whenever a sequence $\{v_n\} \subset V$ is such that $Iv_n \to Iv_0$,
  one has $v_n \to v_0$.

Tikhonov well-posedness

The concept of Tikhonov well-posedness has been extended to minimization problems admitting non-unique optimal solutions. For our purpose here, the most appropriate well-posedness notion for such problems is the one introduced in Bednarczuk and Penot(1992) :

- Problem (V, I) is called **metrically well-set** (or **M-well set**) if $\arg\min_V(I) \neq \emptyset$ and for every minimizing sequence $\{v_n\}$, one has

$$\mathrm{dist}(v_n, \arg\min_V(I)) \to 0 \text{ as } n \to \infty.$$

(Here $\mathrm{dist}(v, S)$ denotes the distance of v from the set S.)
Equivalently, it is easily seen that

- Problem (V, I) is M-well set if and only if $\arg\min_V(I) \neq \emptyset$ and the multifunction

$$\epsilon \rightrightarrows \epsilon - \arg\min_V(I)$$

is upper Hausdorff semicontinuous (uHsc) at $\epsilon = 0$.

Tikhonov well-posedness and M-well setness

Tikhonov well-posedness as well as M-well setness of problem (V, I) are conveniently characterized in terms of the notion of a firm function (or a forcing function). A function $c : T \to [0, \infty)$ is called

- a **firm function** or a **forcing function** if

$$0 \in T \subset [0, \infty), c(0) = 0 \text{ and } t_n \in T, c(t_n) \to 0 \Rightarrow t_n \to 0.$$

Characterization

It is well known that

- *problem* (V, I) *is Tikhonov well-posed if and only if there exists a firm function c and a point* $v_0 \in V$ *such that*

$$I(v) \geq I(v_0) + c[d(v, v_0)], \quad \text{for all } v \in V.$$

Likewise, it is well known that *if I is a proper lower semicontinuous function then*

- *problem* (V, I) *is M-well set if and only if* $\arg\min_V(I) \neq \emptyset$ *and I is firmly conditioned, i.e., there exists a firm function c on* $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ *such that*

$$I(v) \geq v_V(I) + c\,\mathrm{dist}(v, \arg\min_V(I)), \ \textit{for all } v \in V.$$

Strong uniqueness of best approximation SUBA

In the classical Chebyshev theory as well as in the more recent theory of best approximants in normed linear spaces, there has been a lot of interest in studying "strong unicity" of best approximants :

- An element $v_0 \in V$, a finite dimensional linear subspace of a normed linear space X, is called a **strongly unique best approximant** (SUBA) to x in V if there exists a constant $\lambda = \lambda(x), 0 < \lambda \leq 1$, such that

$$\|x - v\| \geq \|x - v_0\| + \lambda\|v - v_0\|, \text{ for all } v \in V.$$

Put differently,

- the strong uniqueness of a best approximant $v_0 \in V$ to x is precisely the Tikhonov well-posedness of problem (V, I_x) where $I_x(v) := \|x - v\|, v \in V$, with the associated firm function being linear: $c(t) = \lambda t,\ t \in T$.
- The problem (V, I_x) is also said to be **linearly conditioned** in this case.

Metric projection multifunction

Given a finite dimensional subspace V of a normed linear space X and $x \in X$, let us denote by $P_V(x)$ the (nonempty) set

$$\{v_0 \in V : \|x - v_0\| = \mathrm{dist}(x, V)\}$$

of **best approximants** to x in V.
In this case

- the multifunction
$$X : x \rightrightarrows P_V(x)$$
of X into V is called the **metric projection multifunction.**

Let us recall that V is said to be

- **Chebyshev** if $P_V(x)$ is a sigleton for each $x \in X$.

Hausdorff strong uniqueness of metric projection

In case V is non-Chebyshev, Li (1989) introduced the following definition:

- The metric projection multifunction $P_V : X \rightrightarrows V$ is said to be **Hausdorff strongly uniquely** at $x \in X$ if $P_V(x) \neq \emptyset$, and there exists a constant $\lambda_V(x) > 0$, such that

$$\|x - v\| \geq \operatorname{dist}(x, V) + \lambda_V(x) \operatorname{dist}(v, P_V(x)),$$

  for all $v \in V$.

Note that

- Hausdorff strong uniqueness of the multifunction P_V at x is precisely M-well setness of the problem (V, l_x) with the associated firm function c_x being linear: $c_x(t) = \lambda_V(x)t$.

In this case

- problem (V, l_x) is also said to be **linearly conditioned**.

Best simultaneous approximation

Consider the problem of approximating simultaneously a *data set* in a given space by a single element of an approximating family. One way to treat this problem is to cover the given data set (assumed to be bounded) by a ball of minimal radius among those centered at the points of the approximating family.

- The problem of *best simultaneous approximation* in this sense coincides with problem (V, I_F), where V, a finite dimensional subspace of a normed linear space X, is the approximating family, and F, a nonempty bounded subset of X, is the data set.

The objective function in this problem is $I_F : V \to \mathbb{R}$, which measures 'worstness' of an element $v \in V$ as a representer of F, defined by

$$I_F(v) = r(F; v), \text{ where } r(F; v) := \sup\{\|f - v\| : f \in F\}.$$

Best simultaneous approximant or restricted center

The optimal value function $v_V(I_F)$ in this case is denoted by $r_V(F)$. Thus the 'intrinsic error' in the problem of approximating simultaneously all the elements $f \in F$ by the elements of V is the number

$$r_V(F) := \inf\{r(F; v) : v \in V\},$$

called the **Chebyshev radius** of F in V. It is the minimal radius of a ball (if one such exists) centered at a point in V and covering F. The centers of all such balls are precisely the elements of the set $\arg\min_V(I_F)$ which in this case will be denoted by $C_V(F)$.

- A typical element of the set

  $$C_V(F) := \{v_0 \in V : r(F; v_0) = r_V(F)\}$$

  is called a **best simultaneous approximant** or a **restricted center** of F in V.

Restricted center multifunction

- When the bounded sets F are allowed to range over a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of X, the multifunctions $C_V : \mathcal{F} \rightrightarrows V$, with values $C_V(F)$, $F \in \mathcal{F}$, is called the **restricted center multifunction.**
- Note that in case F is a singleton $\{x\}$, $x \in X$, $r_V(F)$ is the distance of x from V, denoted by $\mathrm{dist}(x, V)$, and $C_V(F)$ is precisely the set $P_V(x)$ of all best approximants to x in V.

Strongly unique best simultaneous approximation SUBSA

Let $F \in \mathcal{F}$. Analogously, as in the case of a SUBA,

- an element $v_0 \in V$ is called a **strongly unique best simultaneous approximant** (**SUBSA**) to F in V if there exists a constant $\lambda = \lambda_V(F) > 0$ such that

$$r(F; v) \geq r(F; v_0) + \lambda \|v - v_0\|, \text{ for all } v \in V.$$

Likewise, in case $C_V(F)$ is not a singleton,

- the set F is said to admit **Hausdorff strongly unique best simultaneous approximant** (**H-SUBSA**) in V if $C_V(F) \neq \emptyset$, and there exists a constant $\lambda = \lambda_V(F) > 0$ such that for all $v \in V$,

$$r(F; v) \geq r_V(F) + \lambda \operatorname{dist}(v, C_V(F)).$$

Property SUBSA and property H-SUBSA

Clearly, F admits a SUBSA (resp. a H-SUBSA) in V If and only if problem (V, I_F) is Tikhonov well-posed (resp. M-well set) and linearly conditioned.

- The triplet $(X, V, \mathcal{F})$ is said to satisfy **property SUBSA** (resp. **property H-SUBSA**) if F admits SUBSA (resp. H-SUBSA) in V for every $F \in \mathcal{F}$.

Although uniqueness of best simultaneous approximants was studied previously in many articles, surprisingly, strong uniqueness was not treated in these articles. Apparently, in a general framework, strong uniqueness of best simultaneous approximation was studied for the first time in Laurent and Pai (1998). More recently in Indira and Pai (2005), Hausdorff strong uniqueness of best simultaneous approximation was explored.

Preliminaries

Throughout the following, X will be a real normed linear space which for the most part will be the Banach space $\mathcal{C}_0(T, U)$, where T is a locally compact Hausdorff space and U is a strictly convex (real) Banach space, and V will be a finite dimensional subspace of X.
Let us recall that

- $\mathcal{C}_0(T, U)$ consists of all continuous functions $f : T \to U$ vanishing at infinity, i.e, a continuous function f is in $\mathcal{C}_0(T, U)$ if and only if, for every $\epsilon > 0$, the set $\{t \in T : \|f(t)\| \geq \epsilon\}$ is compact.
- The space $\mathcal{C}_0(T, U)$ is endowed with the norm:

$$\|f\| := \max\{\|f(t)\| : t \in T\}, \quad f \in \mathcal{C}_0(T, U).$$

In case $U = \mathbb{R}$ or $\mathbb{C}$, we denote $\mathcal{C}_0(T, U)$ by $\mathcal{C}_0(T)$.

Preliminaries continued

With X as a normed linear space,

- we denote by $CL(X)$ (resp. $CLB(X)$, resp. $K(X)$) the class of all nonempty closed (resp. nonempty closed and bounded, resp. nonempty compact) subsets of X.

Let us also recall that

- a set $F \in CLB(X)$ is said to be *sup-compact* w.r.t. V if for each $v_0 \in V$, every maximizing sequence $\{f_n\}$, i.e., a sequence $\{f_n\} \subseteq F$ such that $\lim_n \|f_n - v_0\| = r(F; v_0)$, has a convergent subsequence converging in F.

Preliminaries continued

Clearly, if F is sup-compact w.r.t. V, then the set

$$\mathcal{Q}_{F,v_0} := \{f_0 \in F : \|f_0 - v_0\| = r(F; v_0)\}$$

of all *remotal points* of v_0 in F is non-void for each $v_0 \in V$. Let

$$s\text{-}K_V(X) := \{F \in CLB(X) : F \text{ is sup-compact w.r.t } V \text{ and } r_V(F) > r_X(F)\}.$$

In the sequel, for some of the results to follow, we will take $\mathcal{F} = s\text{-}K_V(X)$ which contains the family $K_V(X)$ of all nonempty compact subsets F of X satisfying the same restriction $r_V(F) > r_X(F)$.

Semi-continuities of multifunctions

We recall that

- the **lower** (resp. **upper**) **Vietoris topology** τ_V^- (resp. τ_V^+) on $CL(X)$ is the one generated by all sets of the form

$$V^- := \{A \in CL(X) : A \cap V \neq \emptyset\}$$

(resp.
$$V^+ := \{A \in CL(X) : A \subset V\})$$

as V runs over all open subsets of X.

If T is a topological space, by a

- **multifunction** $\Gamma : T \rightrightarrows X$, we mean a set-valued function from T to $CL(X)$.
- A multifunction $\Gamma : T \rightrightarrows X$ is said to be **lower semicontinuous** (resp. **upper semicontinuous**) abbreviated lsc (resp. usc) if is continuous as a function from T to $CL(X)$ equipped with τ_V^- (resp. τ_V^+).

Derived submultifunction

It is convenient also to recall here

- the notion of the *derived submultifunction* $T^* : \mathcal{F} \rightrightarrows V$ of T defined by

$$T^*(F) := \{v \in T(F) : \lim_n d(v, T(F_n)) = 0,$$
$$\text{for every sequence } F_n \text{ in } \mathcal{F} \text{ convergent to } F\}.$$

It follows immediately from the definitions that T *is lsc if and only if* $T = T^*$.

Hausdorff semicontinuities of multifunctions

Recall also that

- **lower** (resp. **upper**) **Hausdorff topology** τ_H^- (resp. τ_H^+) on $CL(X)$ is the one for which a neighbourhood base at $A_0 \in CL(X)$ consists of classes of the type $\{A \in CL(X) : A_0 \subset B_\epsilon(A)\}$ (resp. $\{A \in CL(X) : A \subset B_\epsilon(A_0)\}$).

Here $B_\epsilon(A_0)$ denotes the set $\{x \in X : \operatorname{dist}(x, A_0) < \epsilon\}$. A multifunction $\Gamma : T \rightrightarrows X$ is said to be

- **upper Hausdorff semicontinuous** (resp. **lower Hausdorff semicontinuous**), abbreviated uHsc (resp. lHsc), if it is continuous as a function from T to $CL(X)$ equipped with τ_H^+ (resp. τ_H^-).

Hausdorff continuity

It is said to be

- **Hausdorff continuous** if it is both uHsc as well as lHsc.

For the most part, we are concerned here with $\mathrm{CLB}(X)$ which is equipped with

- the Hausdorff metric H defined by

$$H(A,B) := \max\{e(A,B), e(B,A)\}, \quad A,B \in CLB(X).$$

Here $e(A,B) := \sup\{\,\mathrm{dist}(a,B) : a \in A\}$ denotes the *excess* of A over B.

Whenever $\mathcal{F} \subset CLB(X)$, we shall regard $\mathcal{F}$ as a metric space endowed with the induced Hausdorff metric topology.

A Sufficient Condition for Lower Semicontinuity of the Multifunction C_V

Throughout this section X will be a (real) normed linear space whose normed dual will be denoted by X^*, and V will be a finite dimensional subspace of X. The weak* or $\sigma(X^*, X)$-topology of X^* will be denoted by w^*. Let $\mathrm{Ext}\,(B(X^*))$ denote the set of all extreme points of the closed unit ball $B(X^*)$ of X^*. For the sake of brevity, let us denote

$$\mathcal{E}_{X^*} := \overline{\mathrm{Ext}}^{w^*}(B(X^*)),$$

the closure being taken in the w^*-topology. Also, for $x \in X$, let

$$\mathcal{E}_x := \{x^* \in \mathcal{E}_{X^*} : |x^*(x)| = \|x\|\},$$

denote the set of all *critical functionals*. Clearly, $\mathcal{E}_x$ is nonempty and w^*-compact subset of X^* for each $x \in X$.

Useful notations

For $A \subseteq X$,

- we denote by $A^\perp$ the *annihilator* of A :

$$A^\perp := \{x^* \in X^* : x^*(A) = \{0\}\}.$$

- For $f \in X$, let

$$\begin{aligned} \mathcal{E}_{f-A} &= \cap_{\alpha \in A} \mathcal{E}_{f-\alpha} \\ &= \{x^* \in \mathcal{E}_{X^*} : |x^*(f-\alpha)| = \|f-\alpha\|, \forall \alpha \in A\}. \end{aligned}$$

Notations continued

For F in $\mathrm{CLB}(X)$,

- let us denote by $\mathcal{G}_F$, the subspace

$$\mathcal{G}_F := \operatorname{span}\{v_1 - v_2 : v_1, v_2 \in C_V(F)\}.$$

- Note that

$$\mathcal{G}_F^{\perp} = \bigcap_{v \in C_V(F)} \{v - v_0\}^{\perp},$$

  for any fixed $v_0 \in C_V(F)$. Hence, for $F \in \mathrm{CLB}(X)$ such that $0 \in C_V(F)$, we have $\mathcal{G}_F^{\perp} = C_V(F)^{\perp}$.

We will denote the relative interior of $C_V(F)$ by $\operatorname{relint} C_V(F)$.

Lemma

Lemma

Let V be a finite dimensional subspace of a normed space X, and let $F \in \mathrm{CLB}(X)$ be sup-compact w.r.t. V. If $v_0 \in \mathrm{relint}\, C_V(F)$, then

$$\mathcal{E}_{f_0 - v_0} = \mathcal{E}_{f_0 - C_V(F)} \subseteq \mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}, \tag{1}$$

for every $f_0 \in \mathcal{Q}_{\mathcal{F},\sqsubseteq,}$. Also

$$\mathcal{Q}_{F,v_0} = \bigcap_{v \in C_V(F)} \mathcal{Q}_{F,v}. \tag{2}$$

Theorem

Theorem

Let X, V and F be as in Lemma 3.1. If

$$\mathcal{E}_{f_0 - v_0} \subseteq \mathrm{int}\,(\mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}), \tag{3}$$

the interior being taken in the induced w^*-topology of $\mathcal{E}_{X^*}$, for every $f_0 \in \mathcal{Q}_{F,v_0}$, whenever $v_0 \in \mathrm{relint}\, C_V(F)$, then the multifunction $C_V : s\text{ - }K_V(X) \rightrightarrows V$ is lsc at F.

Remark

Since C_V is additive modulo V:$C_V(F - v_0) = C_V(F) - v_0$ for $v_0 \in V$, we can say that if

$$\mathcal{E}_{f_0} \subseteq \mathrm{int}\,(C_V^{\perp}(F) \cap \mathcal{E}_{X^*}) \tag{4}$$

for every $f_0 \in \mathcal{Q}_{F,0}$ whenever $0 \in \mathrm{relint}\, C_V(F)$, then the multifunction $C_V : s\text{ - }K_V(X) \rightrightarrows V$ is lsc at F.

Lower Semicontinuity of C_V in the Space $C_0(T, U)$

Let $X = \mathcal{C}_0(T, U)$, where T is a locally compact Haussdorff space, and U is a strictly convex (real) Banach space. Throughout the remainder, V will be a finite dimensional subspace of X. For $X = \mathcal{C}_0(T, U)$, $f \in X$ and $\mathcal{A} \subseteq X$, let

$$Z(\mathcal{A}) := \{t \in T : \alpha(t) = 0 \text{ for all } \alpha \in \mathcal{A}\}.$$

For $\alpha \in \mathcal{A}$, let

$$E(f - \alpha) := \{t \in T : \|f(t) - \alpha(t)\| = \|f - \alpha\|\},$$

denote the set of all *critical points* of the function $f - \alpha$. Also let

$$\begin{aligned} E(f - \mathcal{A}) &:= \cap\{E(f - \alpha) : \alpha \in \mathcal{A}\} \\ &= \{t \in T : \|f(t) - \alpha(t)\| = \|f - \alpha\| \text{ for all } \alpha \in \mathcal{A}\}. \end{aligned}$$

useful lemma

The next lemma is an analogue of Lemma 3.1 for the present case. Let us recall that we are denoting by $\mathcal{G}_F$ the subspace $\operatorname{span}\{v_2 - v_1 : v_1, v_2 \in C_V(F)\}$ of V, and by $\operatorname{relint} C_V(F)$, the relative interior of $C_V(F)$.

Lemma

Let X, V and F be as in the last theorem. If $v_0 \in \operatorname{relint} C_V(F)$, then

$$E(f - v_0) = E(f - C_V(F)) \subseteq Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$. Also $\mathcal{Q}_{F,v_0} = \bigcap_{v \in C_V(F)} \mathcal{Q}_{F,v}$.

Remark

Note that $Z(\mathcal{G}_F) = \cap\{Z(v - v_0) : v \in C_V(F)\}$ for any fixed $v_0 \in C_V(F)$. Hence, the conclusion of the lemma can be restated as follows:
If $0 \in \operatorname{relint} C_V(F)$, then

$$E(f_0) = E(f_0 - C_V(F)) \subseteq Z(C_V(F))$$

for every $f_0 \in \mathcal{Q}_{F,0}$.

An Intrinsic Characterization of Lower Semicontinuity of the Multifunction C_V

As before, let $X = \mathcal{C}_0(T, U)$ and V be a finite dimensional subspace of X. The next lemma involves perturbation of sets. For F, G in $K(X)$, and $S \subseteq T$,

- we write $F|_S = G|_S$ if for every $f \in F$, there is a $g \in G$ such that $f|_S = g|_S$, and conversely.

Lemma

Let $F \in K(X)$ be such that $0 \in \operatorname{relint} C_V(F)$ and $r_V(F) = 1$. Let O be any open neighbourhood of $Z(C_V(F))$. If $G \in K(X)$ is such that $G|_O = F|_O$ and $\sup_{g \in G} \|g\| = 1$, then $0 \in C_V(G)$ and $C_V(G) \subseteq \operatorname{span} C_V(F)$.

B-M-W theorem for metric projection

Let us now recall the following well known result for lower semicontinuity of metric projection due to Blatter, Morris and Wulbert(1968).

Theorem

Let $X = \mathcal{C}(T)$ and V be a finite dimensional subspace of X. the metric projection multifunction $P_V : X \rightrightarrows V$ is lsc if and only if $Z(P_V(f))$ is open for every f in $\mathcal{C}(T)$ for which $0 \in P_V(f)$.

We are now ready to state the first main characterization theorem for lower semicontinuity of C_V in Pai(2009). This extends Theorem 2 of Indira and Pai(2005) and Theorems 6 and 9 of Brosowski and Wegmann(1973).

Lower semicontinuity of C_V

Theorem

Let V be a finite dimensional subspace of $\mathcal{C}_0(T, U)$.

(i) *If the multifunction $C_V := K(X) \rightrightarrows V$ is lsc for all $F \in K(X)$ with $0 \in \operatorname{relint} C_V(F)$, then*

$$E(g - v_0) \subseteq \ \mathit{int}\, Z(\mathcal{G}_G)$$

for every $G \in K(X)$, $v_0 \in \operatorname{relint} C_V(G)$ and $g \in \mathcal{Q}_{G,v_0}$.

(ii) *The multifunction $C_V := K(X) \rightrightarrows V$ is lsc at $F \in K(X)$ if*

$$E(f - v_0) \subseteq \ \mathit{int}\, Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$, whenever $v_0 \in \operatorname{relint} C_V(F)$.

Global necessary and sufficient conditions

We can now state a global necessary and sufficient condition for lower semicontinuity of C_V as follows.

Theorem

Let V be a finite dimensional subspace of $X = \mathcal{C}_0(T, U)$. Then the multifunction $C_V : K(X) \rightrightarrows V$ is lsc if and only if for each $F \in K(X)$, we have

$$E(f - v_0) \subseteq \; int\, Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$, whenever $v_0 \in \operatorname{relint} C_V(F)$.

The next theorem partially extends Theorem 4.5 of Blatter, Morris, and Wulbert(1968).

Theorem

Let X, V as in the last theorem. If for every $F \in K(X)$ with $0 \in \operatorname{relint} C_V(F)$, the set $Z_F := Z(C_V(F))$ is open, then the multifunction $C_V : K(X) \rightrightarrows V$ is lsc.

Characterization of Lower Semicontinuity of C_V using Haar-like Conditions

Our main goal here is to give an intrinsic characterization of finite dimensional subspaces V of $\mathcal{C}_0(T,U)$ for which the restricted center multifunction $C_V : K(X) \rightrightarrows V$ is lsc.
Let us recall that a finite dimensional subspace V of $\mathcal{C}_0(T)$ is called

- a *Haar subspace*
(or that it is said to satisfy *Haar condition*) if for each $v \in V \setminus \{0\}, card\ Z(v) \leq \dim V - 1$.

It is easily seen that

- V is a Haar subspace of dimension n if and only if $\dim V|_S = n$ for every subset S of T such that $card\ (S) = n$.

Here $card(S)$ denotes the cardinality of S and $V|_S := \{v|_S : v \in V\}$.

Let us also recall the generalized Haar condition introduced by Zukhovitskii and Stechkin (1960) for a finite dimensional subspace V of $\mathcal{C}_0(T, U)$.
Consider the following properties of V.
(T_m) For each $v \in V \setminus \{0\}$, there are at most m zeros in T.
(P_m) For each set of m distinct points $t_i \in T$ and m elements $u_i \in U$, there exists at
least one $v \in V$, such that

$$v(t_i) = u_i, \quad i = 1, \ldots, m.$$

An n-dimensional subspace V of $\mathcal{C}_0(T, U)$ is said to satisfy

- the *generalized Haar condition* if either $\dim U = k \leq n$ and V satisfies conditions (T_m) and (P_m) where $m \in$ is the unique integer satisfying

$$mk < n \leq (m+1)k,$$

or $\dim U > n$, and V satisfies condition (T_0).

Properties (Li) and (Li′)

For finite dimensional subspaces V of $\mathcal{C}_0(T)$, the following extension of *Haar condition* is due to W. Li (1989).

Definition

V is said to satisfy property (Li) if for every $v \in V \setminus \{0\}$,

$$card \, \mathrm{bd}\, Z(v) \leq \ dim\{p \in V : p|_{\mathrm{int}\ Z(v)} = 0\} - 1.$$

For finite dimensional subspaces V of $\mathcal{C}_0(T, U)$, the following variant of the *generalized Haar condition* is also due to W. Li(1989 b).

Definition

A finite dimensional subspace V of $\mathcal{C}_0(T, U)$ is said to satisfy property (Li′) if for every $v \in V \setminus \{0\}$,

$$card \, \mathrm{bd}\, Z(v) \leq (\dim U)^{-1} \cdot \dim\{p|_{\mathrm{bd}\, Z(v)} : p \in V \text{ and } p|_{\mathrm{int}\ Z(v)} = 0\}.$$

Note that if T is connected, then property (Li) coincides with the Haar condition. Moreover, in case $\dim U = k \leq n = \dim V$, the property (Li′) is implied by the generalized Haar condition.

A technical lemma

We require the following lemma due to W. Li(1989 b) to prove the next theorem.

Lemma

For a finite dimensional subspace V of $\mathcal{C}_0(T, U)$, the following statements are equivalent.

(i) *The metric projection multifunction $P_V : X \rightrightarrows V$ is lsc.*

(ii) *V satisfies property (Li′).*

(iii) *For any set $\{t_i : 1 \leq i \leq m\} \subseteq T$, if there exist $u_i^* \in U^* \setminus \{0\}, 1 \leq i \leq m$ such that*

$$\sum_{i=1}^{m} u_i^*(v(t_i)) = 0, \quad v \in V,$$

then for any $v \in V$ with $\{t_i : 1 \leq i \leq m\} \subseteq Z(v)$, we have

$$\{t_i : 1 \leq i \leq m\} \subseteq \operatorname{int} Z(v).$$

Intrinsic characterization of finite dimensional subspaces

The next theorem in Pai(2009) gives an intrinsic characterization of finite dimensional subspaces V of $\mathcal{C}_0(T, U)$ for which the restricted center multifunction C_V is lsc. It extends Theorem 3 of Indira and Pai(2005).

Theorem

For a finite dimensional subspace V of $\mathcal{C}_0(T, U)$, the following statements are equivalent.

(i) *The multifunction $C_V : K_V(X) \rightrightarrows V$ is lsc.*

(ii) *V satisfies property (Li').*

Restricted Center Multifunction in $C_0(T)$

Throught this section, we take $X = \mathcal{C}_0(T)$ and V a finite dimensional subspace of X.Recall that

- V is called a **Haar subspace** if for each $v \in V \setminus \{0\}$, $card\ Z(v) \leq \dim V - 1$.

Here we use the notation *card* (A) to denote the cardinality of A and $Z(v)$ to denote the set of all zeros of v. Let

$$\Omega_V(X) := \{F \in CLB(X) : r_X(F) < r_V(F)\}.$$

It is convenient to restate here the following theorem from Pai(2000) which summarizes the main characteristizations of Haar subspaces of $\mathcal{C}_0(T)$ in terms of best simultaneous approximants of sets.

Characterization of Haar subspaces

Theorem

For a finite dimensional subspace V of $\mathcal{C}_0(T)$, the following statements are equivalent.

(i) *V is Haar.*

(ii) *The triplet $(X, V, \Omega_V(X))$ satisfies property SUBSA.*

(iii) *For each $F \in \Omega_V(X)$, $C_V(F)$ is a singleton and the multifunction $C_V : \Omega_V(X) \rightrightarrows V$ is point-wise Hausdorff Lipschitz continuous, i.e., for each $F \in \Omega_V(X)$, there exists $\beta = \beta(F) \geq 2$ such that*

$$\|C_V(F) - C_V(G)\| \leq \beta H(F, G)$$

for every $G \in \Omega_V(X)$. Here if $C_V(F)$ is a singleton and $C_V(F) = \{v_0\}$, we simply write $C_V(F)$ for the element v_0.

Furthermore, if T is a connected metric space, then all the above statements are equivalent to:

(iv) *$U_V = SU_V$, where $U_V := \{F \in \Omega_V(X) : F$ has a unique best simultaneous approximant in $V\}$ and*

$$SU_V := \{F \in \Omega_V(X) : F \text{ has a SUBSA in } V\}.$$

Lipschitz uHsc

Next, we need to recall here the following result (cf. Pai and Indira(2004), Theorem 4.4.8).

Theorem

Let X be a normed linear space, V be a finite dimensional subspace of X and $\mathcal{F} \subset CLB(X)$. If the triplet $(X, V, \mathcal{F})$ satisfies property H-SUBSA, then the multifunction $C_V : \mathcal{F} \rightrightarrows V$ is pointwise Lipschitz uHsc. More precisely at each $F_0 \in \mathcal{F}$, we have

$$e(C_V(F), C_V(F_0)) \leq 2(\lambda_V(F_0))^{-1} H(F, F_0), \text{ for all } F \in \mathcal{F}.$$

Let us now recall once again property (Li) of the subspace V from Definition 4.1. Li(1989) has shown that this property of V is equivalent to Hausdorff Lipschitz continuity of the metric projection multifunction $P_V : X \rightrightarrows V$. This result was extended in Indira and Pai(2005) to the restricted center multifunction as follows.

Theorem

For a finite dimensional subspace V of $\mathcal{C}_0(T)$ the following statements are equivalent.

(i) *The multifunction $C_V : K_V(X) \rightrightarrows V$ is lsc.*

(ii) *V satisfies property (Li).*

We also need to recall here the following theorem which was established in Indira and Pai(2005).

Theorem

Let V be a finite dimensional subspace of $\mathcal{C}_0(T)$. If V satisfies property (Li) then the triplet $(\mathcal{C}_0(T), V, K_V(X))$ satisfies property H-SUBSA.

Pointwise Hausdorff Lipschitz Continuity of C_V.

We have seen in Theorem 5.2 that

- if V is a finite dimensional subspace of a normed linear space X and $\mathcal{F} \subset CLB(X)$, then H-SUBSA property of the triplet $(X, V, \mathcal{F})$ entails pointwise Lipschitz upper Hausdorff semicontinuity of the restricted center multifunction C_V.

It is shown in Pai and Indira(2007) that

- for a finite dimensional subspace V of $\mathcal{C}_0(T)$, property (Li), in fact, ensures Lipschitz continuity of the multifunction C_V.

Since by Theorem 5.4, property (Li) of V yields property H-SUBSA of the triplet $(\mathcal{C}_0(T), V, K_V(X))$, which in turn, gives pointwise Lipschitz upper Hausdorff semicontinuity of C_V, it is only necessary to establish pointwise Lipschitz lower Hausdorff semicontinuity of the multifunction C_V in this case. We reproduce in the next slide the results of Pai and Indira(2007) in this direction.

Equivalence of Hausdorff continuity and pointwise Haussdorff Lipschitz continuity of C_V

Lemma

The restricted center multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Lipschitz continuous at $F \in K_V(X)$ if and only if there exist constants $\lambda > 0$ and $\epsilon > 0$ such that $H(C_V(F), C_V(G)) \leq \lambda H(F, G)$ for all $G \in K_V(X)$ with $H(F, G) \leq \epsilon$.

Theorem

Let V be a finite dimensional subspace of $\mathcal{C}_0(T)$. If V satisfies property (Li) then the restricted center multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Hausdorff Lipschitz continuous on $K_V(X)$.

Equivalent statements for the multifunction C_V

We note that since for a finite dimensional subspace V of $\mathcal{C}_0(T)$, the multifunction C_V is compact-valued, lower semicontinuity is equivalent to lower Hausdorff semicontinuity for C_V. Hence we can summarize Theorems 5.2, 5.3, 5.4 and 5.5 into the next theorem.

Theorem

Let V be an n-dimensional subspace of $\mathcal{C}_0(T)$. Then the following statements are equivalent.

- **(i)** *V satisfies property (Li).*
- **(ii)** *The multifunction $C_V : K_V(X) \rightrightarrows V$ is Hausdorff continuous.*
- **(iii)** *The triplet $(X, V, K_V(X))$ satisfies property H-SUBSA.*
- **(iv)** *The multifunction $C_V : K_V(X) \rightrightarrows V$ is pointwise Hausdorff Lipchitz continuous.*

Comments

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- Bednarczuk, E., and J.P. Penot, Metrically well-set minimization problems, *Appl. Math. Optim.* **26** (1992), 273-285.
- Blatter, J., Morris P.D. and Wulbert,D.E., Continuity of the set-valued metric projection, *Math. Ann* **178** , 12–24(1968).
- Brosowski, B., and Wegmann,R., On the lower semicontinuity of the set-valued metric projection, *J. Approx. Theory* **8**, 84–100(1973).
- Dontchev, A.L., and T. Zolezzi, *Well-posed Optimization Problems*, Lecture Notes in Mathematics, No.1543, Springer-Verlag, Berlin, 1993.
- Indira K., and Pai,D., Hausdorff strong uniqueness in simultaneous approximation. Part I, in *Approximation Theory XI : Gatlinburg 2004,* Chui, C.K, Neamtu, M. and Schumaker, L.L. (eds.), 101–118, Nashboro Press, Brentwood, TN, USA, 2005.
- Laurent, P., and D. Pai, On simultaneous approximation, *Numer. Funct. Anal. and Optimiz.* **19** (1998), 1045-1064.
- Li, W., Strong uniqueness and Lipschitz continuity of metric projections: A generalization of the Classical Haar theory, *J. Approx. Theory* **56** , 164–184(1989).

- Li, W., An intrinsic characterization of lower semicontinuity of the metric projection in $C_0(T, X)$, *J. Approx. Theory* **57**, 136–149(1989).
- Mhaskar, H.N., and Pai,D.V., *Fundamentals of Approximation Theory*, CRC Press, Boca Raton, Florida, 2000.
- Pai, D.V., Strong uniqueness of best simultaneous approximation, *J. Indian Math. Soc. (N.S.)* **67**, 201–215(2000).
- Pai, D.V., and Indira, K., On well-posedness of some problems in approximation theory, in *Advances in Constructive Approximation*, Neamtu, M. and Saff, E.B. (eds.), 371–392, Nashboro Press, Brentwood, TN, 2004.
- Pai, D., and Indira,K., Hausdorff strong uniqueness in simultaneous approximation. Part II, in *Frontiers in Interpolation and Approximation*, Govil,N.K. Mhaskar,H.N., Mohapatra,R.N., Nashed, Z.and Szabados, J. (eds.), 365–380, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, USA, 2007.
- Pai, D.V., On lower semicontinuity of the restricted center multifunction, *ANTA, Analyse Numérique et Théorie de l'Approximation,* **38**(1),83–99, Academy of Sciences, Roumania, 2009.

Zuhovickii, S.I., and Steckin,S.B., On the approximation of abstract functions, *Amer. Math. Soc. Transl.* **16**, 401–406(1960).