Approximate Solution of Urysohn Integral Equations with Smooth Kernels by Discrete Modified Projection Method

Gobinda Rakshit
(IIT Bombay)

Joint work with

Prof. Rekha P. Kulkarni

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Urysohn Integral equation :

$\mathscr{X} = L^\infty[a,b]$ and $\kappa(\cdot,\cdot,\cdot) \in C([a,b]\times[a,b]\times\mathbb{R})$.

$$\mathcal{K}(x)(s) = \int_a^b \kappa(s,t,x(t))\,dt, \qquad s \in [a,b].$$

Example : $\mathcal{K}(x)(s) = \int_0^1 \frac{dt}{s+t+x(t)}, \quad s \in [0,1].$

Main Equation : $x - \mathcal{K}(x) = f, \qquad f \in C[a,b],$

Assumptions :

- $x - \mathcal{K}(x) = f$ has a unique solution, say φ.
- 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$.

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Approximations

Interested in approximate solution of

$$x - \mathcal{K}(x) = f.$$

Exact solution : φ.
Approximate Equation :

$$x_n - \mathcal{K}_n(x_n) = f (or\ f_n),$$

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Quadrature Rule

$$\mathcal{K}(x)(s) = \int_a^b \kappa(s, t, x(t))dt.$$

In order to replace integrals by a numerical quadrature formula, define a

$$\textbf{Basic quadrature rule} : \int_0^1 f(t)\, dt \approx \sum_{i=1}^{\rho} \omega_i\, f(\mu_i),$$

$$\textbf{Fine partition :}\ a = s_0 < s_1 < \cdots < s_m = b,\ \tilde{h} = \frac{b-a}{m}.$$

For $r \geq 1$, exact for polynomials of degree $\leq 2r-1$.

$$\textbf{Composite quadrature rule:} \int_a^b f(t)dt \approx \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} \omega_i\, f(s_{j-1} + \mu_i \tilde{h}).$$

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Nyström Method.

$$\mathcal{K}(x)(s) = \int_a^b \kappa(s, t, x(t))dt.$$

$$\mathcal{K}_m(x)(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} \omega_i \ \kappa(s, s_{j-1} + \mu_i \tilde{h}, x(s_{j-1} + \mu_i \tilde{h})).$$

$$\mathcal{K}_m(x) \to \mathcal{K}(x).$$

$$\varphi - \mathcal{K}(\varphi) = f,$$

$$\varphi_m - \mathcal{K}_m(\varphi_m) = f.$$

If the integration error for the composite quadrature, is d ($\geq 2r$), then,

$$\|\varphi - \varphi_m\|_\infty = O(\tilde{h}^d), \quad \text{provided } f \in C^d[a, b].$$

Projection Methods

$\mathscr{X}_n$: space of piecewise polynomials w.r.t. a coarse partition of $[a, b]$.
$\mathscr{X}_n$: finite dimensional approximate space.

$$\text{Approximate equation : } x_n - \mathcal{K}_n(x_n) = f(or\ f_n).$$

Projection Methods :
π_n : orthogonal or an interpolatory projection onto $\mathscr{X}_n$.

$$\pi_n \to \mathcal{I} \ \text{ pointwise.}$$

$\mathcal{K}_n(x_n) = \pi_n \mathcal{K}(\pi_n x_n)$ or $\mathcal{K}(\pi_n x_n)$ or other variants.

Interpolatory Projection

Coarse partition : $a = t_0 < t_1 < \cdots < t_n = b,$

$$h = \frac{b-a}{n} = t_k - t_{k-1}.$$

$r \geq 1,\ \mathscr{X}_n$: Piecewise polynomials of degree $\leq r-1$ on $[t_{k-1}, t_k]$.
Choose r distinct points in each sub-interval and fit a polynomial of degree $\leq r-1$.

$$(\mathcal{Q}_n x)(\tau_i^k) = x(\tau_i^k), \quad i = 1, 2, \cdots, r \ \ and \ \ k = 1, 2, \cdots, n.$$

Interpolation points : $\tau_i^k, \quad i = 1, 2, \cdots, r \ \ and \ \ k = 1, 2, \cdots, n$

$$x \in C^r[a,b] \implies \|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq C_1 \left\| x^{(r)} \right\|_\infty h^r.$$

Proved Results for Linear Integral Equation

Collocation Method : [Atkinson, 1976]

Approximate Equation : $\varphi_n^C - \mathcal{Q}_n \mathcal{K} \varphi_n^C = \mathcal{Q}_n f$ with $\left\| \varphi_n^C - \varphi \right\|_\infty = O(h^r)$.

Iterated Collocation Method : [Sloan, 1976]

Approximate Equation : $\varphi_n^S - \mathcal{K} \mathcal{Q}_n \varphi_n^S = f$ with $\left\| \varphi_n^S - \varphi \right\|_\infty = O(h^{2r})$.

Modified Projection Method : [Kulkarni, 2003]

Approximate Equation : $\varphi_n^M - \mathcal{K}_n^M \varphi_n^M = f$ with $\left\| \varphi_n^M - \varphi \right\|_\infty = O(h^{3r})$

and $\tilde{\varphi}_n^M = \mathcal{K} \varphi_n^M + f \implies \left\| \tilde{\varphi}_n^M - \varphi \right\|_\infty = O(h^{4r})$,

where, $\mathcal{K}_n^M = \mathcal{K} \mathcal{Q}_n + \mathcal{Q}_n \mathcal{K} - \mathcal{Q}_n \mathcal{K} \mathcal{Q}_n$.

The Modified Projection Method and its Iterative Version

Collocation and iterated collocation method for Urysohn integral equation was done by Atkinson-Potra[1987].

$$\text{Exact equation} : \varphi - \mathcal{K}(\varphi) = f.$$

$$^{1}\text{Modified Projection} : \varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f,$$

$$\mathcal{K}_n^M(x) = \mathcal{K}(\mathcal{Q}_n x) + \mathcal{Q}_n \mathcal{K}(x) - \mathcal{Q}_n \mathcal{K}(\mathcal{Q}_n x).$$

$$\left\| \varphi - \varphi_n^M \right\|_\infty = O(h^{3r}).$$

$$\tilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f,$$

$$\left\| \varphi - \tilde{\varphi}_n^M \right\|_\infty = O(h^{4r}).$$

[1] For reference, see Grammont et al (A Galerkins perturbation type method of approximate a fixed point of a compact operator, 2011).

Why Discrete Method is chosen?

The Computation of φ_n^M and $\tilde{\varphi}_n^M$ needs evaluation of $\mathcal{K}(x)$.
In practice the integral needs to be replaced by a numerical quadrature formula, gives rise to discrete methods.

Aim : To propose a numerical quadrature formula which preserves the orders of convergence h^{2r}, h^{3r} and h^{4r} in the methods mentioned above.

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Discrete Modified Projection

Modified Projection : $\varphi_n^M - \mathcal{K}_n^M \varphi_n^M = f,$

$$\text{where} \quad \mathcal{K}_n^M = \mathcal{Q}_n \mathcal{K} + \mathcal{K} \mathcal{Q}_n - \mathcal{Q}_n \mathcal{K} \mathcal{Q}_n.$$

Discrete version : Replace $\mathcal{K}$ by the Nyström operator $\mathcal{K}_m$.

$$\tilde{\mathcal{K}}_n^M = \mathcal{Q}_n \mathcal{K}_m + \mathcal{K}_m \mathcal{Q}_n - \mathcal{Q}_n \mathcal{K}_m \mathcal{Q}_n,$$

$$z_n^M - \tilde{\mathcal{K}}_n^M z_n^M = f.$$

$$\begin{aligned} \mathcal{K}_m - \tilde{\mathcal{K}}_n^M &= \mathcal{K}_m - \mathcal{Q}_n \mathcal{K}_m - \mathcal{K}_m \mathcal{Q}_n + \mathcal{Q}_n \mathcal{K}_m \mathcal{Q}_n \\ &= (\mathcal{I} - \mathcal{Q}_n) \mathcal{K}_m (\mathcal{I} - \mathcal{Q}_n). \end{aligned}$$

Calculation of bounds.

$$(\mathcal{I}-\mathcal{K})\,\varphi = f, \quad \left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right) z_n^M = f,$$
$$\mathcal{K}_m - \tilde{\mathcal{K}}_n^M = (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,.$$

$$\left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right)(\varphi - z_n^M) = \varphi - \tilde{\mathcal{K}}_n^M \varphi - f$$
$$= \mathcal{K}\varphi - \tilde{\mathcal{K}}_n^M \varphi$$
$$= (\mathcal{K}\varphi - \mathcal{K}_m\varphi) + (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi.$$

$$\left\|\varphi - z_n^M\right\|_\infty \leq C_2\left[\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty + \|(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi\|_\infty\right].$$

$$\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty = O(\tilde{h}^d).$$

Linear Integral Equations

Calculation of bounds.

$$(\mathcal{I}-\mathcal{K})\,\varphi = f, \quad \left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right) z_n^M = f,$$
$$\mathcal{K}_m - \tilde{\mathcal{K}}_n^M = (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,.$$

$$\begin{aligned}\left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right)(\varphi - z_n^M) &= \varphi - \tilde{\mathcal{K}}_n^M\varphi - f \\ &= \mathcal{K}\varphi - \tilde{\mathcal{K}}_n^M\varphi \\ &= (\mathcal{K}\varphi - \mathcal{K}_m\varphi) + (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi.\end{aligned}$$

$$\left\|\varphi - z_n^M\right\|_\infty \leq C_2\left[\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty + \|(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi\|_\infty\right].$$

$$\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty = O(\tilde{h}^d).$$

Calculation of bounds.

$$(\mathcal{I}-\mathcal{K})\,\varphi = f, \quad \left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right) z_n^M = f,$$
$$\mathcal{K}_m - \tilde{\mathcal{K}}_n^M = (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,.$$

$$\begin{aligned}\left(\mathcal{I}-\tilde{\mathcal{K}}_n^M\right)(\varphi - z_n^M) &= \varphi - \tilde{\mathcal{K}}_n^M\varphi - f \\ &= \mathcal{K}\varphi - \tilde{\mathcal{K}}_n^M\varphi \\ &= (\mathcal{K}\varphi - \mathcal{K}_m\varphi) + (\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi.\end{aligned}$$

$$\left\|\varphi - z_n^M\right\|_\infty \le C_2\left[\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty + \|(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I}-\mathcal{Q}_n)\,\varphi\|_\infty\right].$$

$$\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty = O(\tilde{h}^d).$$

Iterated Discrete Modified Projection

$$\text{Approximate solution :} \quad \tilde{z}_n^M = \mathcal{K}_m z_n^M + f.$$

$$\left\|\varphi - \tilde{z}_n^M\right\|_\infty \leq \|\varphi - \varphi_m\|_\infty + \left\|\varphi_m - \tilde{z}_n^M\right\|_\infty.$$

$$\|\varphi - \varphi_m\|_\infty = O(\tilde{h}^d).$$

$$\begin{aligned}
\varphi_m - \tilde{z}_n^M &= \mathcal{K}_m(\mathcal{I} - \mathcal{K}_m)^{-1}(\mathcal{K}_m - \tilde{\mathcal{K}}_n^M)(\mathcal{I} - \tilde{\mathcal{K}}_n^M)^{-1} f \\
&= (\mathcal{I} - \mathcal{K}_m)^{-1}\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n) z_n^M \\
&= (\mathcal{I} - \mathcal{K}_m)^{-1}\Big\{\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)(z_n^M - \varphi) \\
&\qquad + \mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)\varphi\Big\}.
\end{aligned}$$

Role of Gauss Points in Discrete Methods.

To obtain : $\|\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)x\|_\infty = O(h^{2r}),$

$$\mathcal{K}_m(x - \mathcal{Q}_n x)(s) = \tilde{h} \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} \omega_i \ \kappa(s, \zeta_i^{(k-1)p+\nu}) \ (x - \mathcal{Q}_n x)(\zeta_i^{(k-1)p+\nu}).$$

$(x - \mathcal{Q}_n x)(t) = x[\tau_1^k, \tau_2^k, \cdots, \tau_r^k, t] \ \prod_{j=1}^{r}(t - \tau_j^k), \quad t \in [t_{k-1}, t_k],$

Use the discrete version of $\displaystyle\int_0^1 t^j \psi(t) \ dt \ = 0, \quad (0 \le j \le r-1)$

$$i.e. \quad \frac{1}{p} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} \omega_i \ \left(\frac{\nu - 1 + \mu_i}{p}\right)^j \psi\left(\frac{\nu - 1 + \mu_i}{p}\right) = 0,$$

where $\psi(t) = (t - q_1) \cdots (t - q_r)$.

Crucial Relations.

- $$\|(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq C_1 \left\|x^{(r)}\right\|_\infty h^r,$$
- $$\frac{1}{p}\sum_{\nu=1}^{p}\sum_{i=1}^{\rho}\omega_i \left(\frac{\nu-1+\mu_i}{p}\right)^j \psi\left(\frac{\nu-1+\mu_i}{p}\right)=0,$$
- $$\|\mathcal{K}_m(\mathcal{I}-\mathcal{Q}_n)(x)\|_\infty \leq C_2 \|\kappa\|_{r,\infty}\|x\|_{2r,\infty} h^{2r},$$
- $$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}_m(\mathcal{I}-\mathcal{Q}_n)(x)\|_\infty \leq C_3 \|\kappa\|_{2r,\infty}\|x\|_{2r,\infty} h^{3r},$$
- $$\|\mathcal{K}_m(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}_m(\mathcal{I}-\mathcal{Q}_n)(x)\|_\infty \leq C_4 \|\kappa\|_{r,\infty}\|\kappa\|_{3r,\infty}\|x\|_{2r,\infty} h^{4r}.$$

Orders of convergence.

Recall, Exact solution : φ,
Modified Solution : φ_n^M,

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O(h^{3r}).$$

$$\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty = O(\tilde{h}^d).$$

Discrete Modified Solution : z_n^M, and

$$\left\|\varphi - z_n^M\right\|_\infty \leq C_5 \left[\|\mathcal{K}\varphi - \mathcal{K}_m\varphi\|_\infty + \|(\mathcal{I} - \mathcal{Q}_n)\,\mathcal{K}_m\,(\mathcal{I} - \mathcal{Q}_n)\,\varphi\|_\infty\right].$$

$$\Rightarrow \left\|\varphi - \mathbf{z_n^M}\right\|_\infty = \mathbf{O}(\max\{\mathbf{\tilde{h}^d}, \mathbf{h^{3r}}\}).$$

Iterated Discrete Modified Projection.

Iterated Modified Projection Solution : $\tilde{\varphi}_n^M$,
Discrete iterated Modified Projection Solution : $\tilde{z}_n^M$,
Recall Nyström Solution : φ_m and $\|\varphi - \varphi_m\|_\infty = O(\tilde{h}^d)$.

$$\left\|\varphi - \tilde{\varphi}_n^M\right\|_\infty = O(h^{4r}).$$

$$\left\|\varphi - \tilde{z}_n^M\right\|_\infty \leq \|\varphi - \varphi_m\|_\infty + \left\|\varphi_m - \tilde{z}_n^M\right\|_\infty .$$

$$\|\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m(\mathcal{I} - \mathcal{Q}_n)(\varphi)\|_\infty \leq C_6 \|\kappa\|_{r,\infty} \|\kappa\|_{3r,\infty} \|\varphi\|_{2r,\infty} h^{4r}.$$

$$\therefore \quad \left\|\varphi - \tilde{\mathbf{z}}_{\mathbf{n}}^{\mathbf{M}}\right\|_\infty = \mathbf{O}(\max\{\tilde{\mathbf{h}}^{\mathbf{d}}, \mathbf{h}^{\mathbf{4r}}\}).$$

Discrete Modified Projection Method

Approximate Equation : $x - \tilde{\mathcal{K}}_n^M(x) = f,$

$$\tilde{\mathcal{K}}_n^M(x) = \mathcal{Q}_n \mathcal{K}_m(x) + \mathcal{K}_m(\mathcal{Q}_n x) - \mathcal{Q}_n \mathcal{K}_m(\mathcal{Q}_n x).$$

Assumption : 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$.

$\mathcal{K}_m'(\varphi) \to \mathcal{K}'(\varphi)$ in collectively compact fashion.

$\Rightarrow \left[\mathcal{I} - \mathcal{K}_m'(\varphi)\right]^{-1}$ exists and uniformly bounded.

$$\left\|\mathcal{K}_m'(\varphi) - \left(\tilde{\mathcal{K}}_n^M\right)'(\varphi)\right\| \to 0 \text{ as } n \to \infty.$$

$\Rightarrow \mathcal{I} - \left(\tilde{\mathcal{K}}_n^M\right)'(\varphi)$ is invertible and uniformly bounded.

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Order of Convergence

Define : $B_n(x) = x - \left[\mathcal{I} - \left(\tilde{\mathcal{K}}_n^M\right)'(\varphi)\right]^{-1}\left[x - \tilde{\mathcal{K}}_n^M(x) - f\right].$

$$\left\|z_n^M - \varphi\right\|_\infty \leq \frac{3}{2}\left\|\left[\mathcal{I} - \left(\tilde{\mathcal{K}}_n^M\right)'(\varphi)\right]^{-1}\left(\mathcal{K}(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi)\right)\right\|_\infty.$$

$$\mathcal{K}(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi) = \left[\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\right] + \left[\mathcal{K}_m(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi)\right].$$

$\|\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\|_\infty = O(\tilde{h}^d)$ and

$$\begin{aligned}\mathcal{K}_m(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi) &= (\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}_m(\varphi) - \mathcal{K}_m(\mathcal{Q}_n\varphi)) \\ &= -(\mathcal{I} - \mathcal{Q}_n)\left[\mathcal{K}_m(\mathcal{Q}_n\varphi) - \mathcal{K}_m(\varphi) - \mathcal{K}_m'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\right] \\ &\quad - (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m'(\varphi)(\mathcal{Q}_n\varphi - \varphi).\end{aligned}$$

Urysohn Integral equations

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_m'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\right\|_\infty = O(h^{3r}),$$

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\Big[\mathcal{K}_m(\mathcal{Q}_n\varphi) - \mathcal{K}_m(\varphi) - \mathcal{K}_m'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\Big]\right\|_\infty = O(h^{3r}),$$

$$\Rightarrow \left\|\mathcal{K}_m(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi)\right\|_\infty = O(h^{3r}).$$

$$\mathcal{K}(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi) = \Big[\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\Big] + \Big[\mathcal{K}_m(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi)\Big],$$

$$\|\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\|_\infty = O(\tilde{h}^d)$$

$$\left\|z_n^M - \varphi\right\|_\infty \leq \frac{3}{2}\left\|\left[\mathcal{I} - \left(\tilde{\mathcal{K}}_n^M\right)'(\varphi)\right]^{-1}\left(\mathcal{K}(\varphi) - \tilde{\mathcal{K}}_n^M(\varphi)\right)\right\|_\infty .$$

$$\Rightarrow \left\|\mathbf{z_n^M} - \varphi\right\|_\infty = \mathbf{O}\left(\max\left\{\mathbf{\tilde{h}^d}, \mathbf{h^{3r}}\right\}\right).$$

Iterated Discrete Modified Projection Method

$$\tilde{z}_n^M = \mathcal{K}_m(z_n^M) + f.$$

$$\begin{aligned} \tilde{z}_n^M - \varphi &= \tilde{z}_n^M - \varphi_m + \varphi_m - \varphi \\ &= \mathcal{K}_m(z_n^M) - \mathcal{K}_m(\varphi_m) + (\varphi_m - \varphi). \end{aligned}$$

$$\|\varphi_m - \varphi\|_\infty = O(\tilde{h}^d).$$

$$\tilde{B}_n(x) \\ = \varphi_m - \left\{\mathcal{I} - \mathcal{K}'_m(\varphi_m)\right\}^{-1}\left[\mathcal{K}_m(\varphi_m) - \mathcal{K}'_m(\varphi_m)\varphi_m - \tilde{\mathcal{K}}_n^M(x) + \mathcal{K}'_m(\varphi_m)x\right].$$

which gives, $\left\|\mathbf{\tilde{z}_n^M} - \varphi\right\|_\infty = \mathbf{O}\left(\max\left\{\mathbf{\tilde{h}^d}, \mathbf{h^{4r}}\right\}\right).$

Advantage

To get same order of convergence, we need to solve systems much higher dimension in Nyström method than the discrete iterated modified projection method.
We can optimize the composite integration rule so that the order of convergence preserve.

We have defined the composite quadrature in a fine partition with mesh size $\tilde{h}$, where $\tilde{h} \leq h$. Then, in the case of Green's function type of kernel we have the error in the Nyström approximation as

$$\|\mathcal{K}(x) - \mathcal{K}_m(x)\|_\infty = O(\tilde{h}^2).$$

So, in this case, we can also preserve the order of convergence for suitable choice of $\tilde{h}$.

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THANK YOU.