Edge Ideals: Their algebraic and combinatorial invariants

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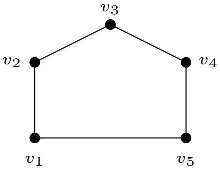

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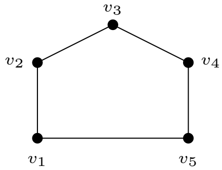

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I_G is the Stanley-Reisner ideal of the corresponding indepenence complex of G.

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A graph G is chordal if it has no induced cycle of length 4 or more and G is said to be co-chordal if its complement is chordal.

Interplay between the Algebraic and Combinatorial Properties

A resolution of an ideal I is an exact sequence of the form

$$0 \to R^{\beta_n} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to I \to 0.$$

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Then $\operatorname{reg}(M) := \max\{b_j - j \mid j = 0, \ldots, n\}$.

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$\nu(G)$ - number of edges at a distance 3 or more, called induced matching number.

$co - chord(G)$ - minimum number of co-chordal subgraphs of G required to cover all vertices and edges of G.

Regularity of different classes

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For several classes of graphs, the regularity is computed:

1. $\mathrm{reg}(I(C_n)) = \begin{cases} \nu(C_n) + 1 & \text{if } n \equiv 2 (mod\, 3); \\ \nu(C_n) + 2 & \text{if } n \equiv 0, 1 (mod\, 3). \end{cases}$
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3. G is weakly chordal $\Rightarrow$ $\nu(G) = co - chord(G)$ (Busch-Dragan-Sritharan (2010)) $\Rightarrow$ $\mathrm{reg}(I(G)) = \nu(G) + 1$.

Asymptotic linearity of regularity

If $I \subset K[x_1, \ldots, x_n]$ is an ideal generated by homogeneous polynomials, then there exists $P, Q \in \mathbb{Z}$ such that for $s \gg 0,\ \operatorname{reg}(I^s) = Ps + Q$. (Kodiyalam; Cutkosky-Herzog-Trung 1999).

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There are no general answers known for these questions. In fact, finding an explicit description of Q is a highly challenging task in general.

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Placing these questions in the context of Edge Ideals, one may ask:

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Fröberg (1990) + Herzog-Hibi-Zheng (2004): If $co - chord(G) = 1$, then $I(G)^s$ has a linear resolution for all $s \geq 1$ and hence $\operatorname{reg}(I(G)^s) = 2s$ for all.

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2. $\operatorname{reg}(I(G)^s) \leq 2s + \frac{1}{2}(\nu(G) + \min\{|X|, |Y|\}) - 1$, where $V(G) = X \sqcup Y$ is a bipartition of G.

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(J-Narayanan-Selvaraja 2016): Let G be a bipartite graph. If G is unmixed, whiskered, weakly chordal or P_6-free, then $\mathrm{reg}(I(G)^s) = 2s + \nu(G) - 1$ for all $s \geq 1$.

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We also have obtained some classes of graphs for which $\operatorname{reg}(I(G)^s) = 2s + co - chord(G) - 1$ for all $s \geq 1$.

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(J-Selvaraja, 2018) proved that if G is a very well-covered graph, then for all $q \geq 1$, $\mathrm{reg}(I(G)^q) = 2q + \nu(G) - 1$.

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Question

Does there exists a combinatorial invariant $\rho(G)$ such that for any graph G, $\operatorname{reg}(I(G)^q) \leq 2q + \rho(G)$ for all $q \gg 0$.

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(J-Selvaraja, 2018) $\operatorname{reg}(I(G)^q) \leq 2q + \zeta(G) - 1$ for all $q \geq 1$, where $\zeta(G)$ is called star-packing number introduced by Hà and Woodroofe.

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(J-Selvaraja, Banerjee-Bayerslan-Hà, 2018) If G is vertex decomposable, then $\operatorname{reg}(I(G)^q) \leq 2q + \operatorname{reg}(I(G)) - 2$ for all $q \geq 1$.

Thank you!