Ill-Posed Operaror Equations and an Inverse Problem in PDE

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Ill-Posed Operator Equations

Let $T : X \to Y$ be a linear operator between normed linear spaces.

The problem of solving the operator equation

$$Tx = y \qquad (*)$$

for $y \in Y$ is said to be **ill-posed** if it does not have a unique solution which depends continuosly on the data y.

Theorem

If T is not bounded below, then $(*)$ is ill-posed.

In fact, if $y \in R(T)$ and $Tx = y$, then for every $\varepsilon > 0$, there exists $y_\varepsilon \in Y$ and $x_\varepsilon \in X$ such that $Tx_\varepsilon = y_\varepsilon$ and

$$\|y - y_\varepsilon\| \leq \varepsilon$$

but

$$\|x - x_\varepsilon\| \geq \frac{1}{\varepsilon}.$$

- Given any sequence (α_n) of poisitve real numbers such that $\alpha_n \to 0$ and $n \to \infty$, there there exists $y_n \in Y$ and $x_n \in X$ such that $Tx_n = y_n$ and

$$\|y - y_n\| \leq \alpha_n \quad \text{but} \quad \|x - x_\varepsilon\| \geq \frac{1}{\alpha_n}$$

for all $n \in \mathbb{N}$.

Proof of the theorem.

Since T is not bounded below, for every $\varepsilon > 0$, there exists $u_\varepsilon \in X$ with $\|u_\varepsilon\| = 1$ such that

$$\|Tu_\varepsilon\| < \varepsilon^2 \|u_\varepsilon\| = \varepsilon.$$

Let

$$x_\varepsilon = x + \frac{1}{\varepsilon} u_\varepsilon, \quad y_\varepsilon = Tx_\varepsilon.$$

Then $Tx_\varepsilon = Tx + \frac{1}{\varepsilon} Tu_\varepsilon$ so that

$$\|Tx_\varepsilon - Tx\| = \frac{1}{\varepsilon}\|Tu_\varepsilon\| < \varepsilon.$$

Note that

$$\|x_\varepsilon - x\| = \frac{1}{\varepsilon}\|u_\varepsilon\| = \frac{1}{\varepsilon}.$$

□

Corollary

If X is infinite dimensional and T is a compact operator, then $(*)$ is ill-posed.

Proof.

A compact operator on an infinite dimensiional normed linear space is not bounded below. □

- Recall that if $k(\cdot,\cdot) \in L^2(\Omega \times \Omega)$, where Ω is a bounded open set in $\mathbb{R}^k$, the operator T defined by

$$(Tx)(s) := \int_\Omega k(s,\zeta)x(\zeta)d\zeta, \quad x \in L^2(\Omega),$$

is a compact operator from the infinite dimensional space $L^2(\Omega)$ into itself.

An illustration

Let X and Y be Hilbert spaces and $T : X \to Y$ be a compact operator of infinite rank. Let

$$Tx := \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle v_n, \quad x \in X,$$

be a singular value decompsotion[1] of T.

Let $y \in Y$.

- For y to be in $R(T)$, it is necessary that

$$\sum_{n=1}^{\infty} \frac{|\langle y, v_n \rangle|^2}{\sigma_n^2} < \infty$$

and in that case, $Tx = y$, where

$$x = \sum_{n=1}^{\infty} \frac{\langle y, v_n \rangle}{\sigma_n} u_n \in N(T)^{\perp}.$$

[1] See M.T. Nair, *Functional Analysis: A First Course*, PHI-Learning.

For $k \in \mathbb{N}$, let $x_k = x + \frac{1}{\sqrt{\sigma}_k} u_k$. Then

$$\|Tx - Tx_k\| = \frac{1}{\sqrt{\sigma}_k}\|Tu_k\| = \sqrt{\sigma}_k,$$

but

$$\|x - x_k\| = \frac{1}{\sqrt{\sigma}_k}.$$

- As $k \to \infty$,

$$\|Tx - Tx_k\| \to 0 \quad \text{but} \quad \|x - x_k\| \to \infty.$$

Backward heat conduction problem

- Find $u(\cdot, t) \in L^2[0, \ell]$ such that

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \quad 0 < t < \tau, \tag{1}$$

$$u(0, t) = 0 = u(\ell, t), \tag{2}$$

from the knowledge of the final value

$$u(\cdot, \tau) := g.$$

Recall:

For $f \in L^2[0, \ell]$,

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f, \varphi_n \rangle \varphi_n(x), \qquad (*)$$

with

$$\lambda_n := \frac{n\pi c}{\ell}, \qquad \varphi_n(x) := \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi}{\ell} x\right)$$

satisfies the equations (1) and (2) and the initial condition

$$u(\cdot, 0) = f.$$

- Knowing $f := u(\cdot, 0)$, we otbain $u(\cdot, t)$ in a stable manner.

- From the knowledge of $g := u(\cdot, \tau)$, find $f := u(\cdot, t)$ for $0 \le t < \tau$.

From $(*)$,

$$u(x,\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 \tau} \langle f, \varphi_n \rangle \varphi_n(x). \qquad (**)$$

and

$$\langle f, \varphi_n \rangle = e^{\lambda_n^2 t} \langle u(\cdot, t), \varphi_n \rangle.$$

Hence, from $(**)$,

$$u(x,\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (\tau - t)} \langle u(\cdot, t), \varphi_n \rangle \varphi_n(x).$$

This is same as solving the operator equation:

$$Af = g,$$

where

$$A\varphi := \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t)} \langle \varphi, \varphi_n \rangle \varphi_n.$$

Note that A is a compact, positive, self adjoint operator on $L^2[0, \ell]$ with eigenvalues

$$\mu_n := e^{-\lambda_n^2(\tau - t)}.$$

- $\mu_n \to 0$ exponentially!
- The inverse problem is severly ill-posed!

General case

Ω: bounded domain in $\mathbb{R}^k$;

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \quad (x,t) \in \Omega \times [0, \tau],$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t \in [0, \tau).$$

In this case, $u(\cdot, 0) = f$ implies

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f, \varphi_n \rangle \varphi_n(x), \qquad (*)$$

with

$$-\Delta \varphi_n = \lambda_n \varphi_n, \quad n \in \mathbb{N};$$

and $\{\varphi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$.

- **BHCP:** From the knowledge of $g := u(\cdot, \tau)$ determine $f := u(\cdot, t)$.

Since a compact operator equation is ill-posed,

> *Small error in the data can cause large deviation in the solution.*

- One has to use some regularization method for obtaining stable approximation methods.

We shall consider one such method, the so called, *spectral cut-off method*.

Spectral cut-off method:

Recall: If $T : X \to Y$ is a compact operator of infinite rank, and if

$$T = \sum_{n=1}^{\infty} \sigma_n \langle \cdot, u_n \rangle v_n$$

is an SVD of V, and if $y \in R(T)$, then

$$x := \sum_{n=1}^{\infty} \frac{\langle y, v_n \rangle}{\sigma_n} u_n$$

belongs to $N(T)^{\perp}$ and $Tx = y$.
A natural way of approximating x would be to take a k-th cut-off of x:

$$x_k := \sum_{n=1}^{k} \frac{\langle y, v_n \rangle}{\sigma_n} u_n.$$

Clearly,

$$\|x - x_k\|^2 = \sum_{n=k+1}^{\infty} \frac{|\langle y, v_n\rangle|^2}{\sigma_n^2} \to 0$$

as $k \to \infty$.

Suppose the data y in noisy, i.e., we have $\tilde{y}$ in place of y. Then one may take the approximation as

$$\tilde{x}_k := \sum_{n=1}^{k} \frac{\langle \tilde{y}, v_n\rangle}{\sigma_n} u_n.$$

Then

$$x_k - \tilde{x}_k = \sum_{n=1}^{k} \frac{\langle y - \tilde{y}, v_n\rangle}{\sigma_n} u_n.$$

$$
\begin{aligned}
\|x-\tilde{x}_k\|^2 &= \|x-x_k\|^2 + \|x_k-\tilde{x}_k\|^2 \\
&= \|x-x_k\|^2 + \sum_{n=1}^{k} \frac{|\langle y-\tilde{y}, v_n\rangle|^2}{\sigma_n^2} \\
&\geq \|x-x_k\|^2 + \frac{|\langle y-\tilde{y}, v_k\rangle|^2}{\sigma_k^2}.
\end{aligned}
$$

Thus,

$$\|x-\tilde{x}_k\|^2 \geq \|x-x_k\|^2 + \frac{|\langle y-\tilde{y}, v_k\rangle|^2}{\sigma_k^2}$$

and

$$\|x-\tilde{x}_k\|^2 \leq \|x-x_k\|^2 + \frac{\delta^2}{\sigma_k^2}.$$

In particular, if $\tilde{y} := y + \delta v_k$, then $\|y - \tilde{y}\| = \delta$ and

$$\|x - \tilde{x}_k\|^2 = \|x - x_k\|^2 + \frac{\delta^2}{\sigma_k^2}. \qquad (*)$$

- For fixed δ, $\|x - \tilde{x}_k\|$ can be large (for large k).
- Rate of convergence cannot be assertained without having *source conditions* on x:

Estimate under source condition(s)

Recall

$$\|x - x_k\|^2 = \sum_{n=k+1}^{\infty} |\langle x, u_n\rangle|^2.$$

Suppose:

$$x \in R(T^*) \quad \text{so that} \quad x = T^* u. \qquad (**)$$

Then,

$$\begin{aligned} \|x - x_k\|^2 &= \sum_{n=k+1}^{\infty} |\langle T^* u, u_n\rangle|^2 = \sum_{n=k+1}^{\infty} |\langle u, T u_n\rangle|^2 \\ &= \sum_{n=k+1}^{\infty} \sigma_n^2 |\langle u, v_n\rangle|^2 \le \|u\|^2 \sigma_{k+1}^2. \end{aligned}$$

$(*)$ and $(**)$ imply:

$$\|x - \tilde{x}_k\|^2 \le \|u\|^2 \sigma_{k+1}^2 + \frac{\delta^2}{\sigma_{k+1}^2}$$

- Note that

$$x \in R(T^*) \iff \sum_{n=1}^{\infty} \frac{|\langle x, u_n \rangle|^2}{\sigma_n^2} < \infty.$$

Theorem

Suppose $x \in R(T^*)$ with $x = T^* u$ and k is such that $\delta \le \sigma_{k+1}^2$, then

$$\|x - \tilde{x}_k\|^2 \le (1 + \|u\|^2)\, \sigma_{k+1}^2.$$

More generally:

Theorem

If $x \in R((T^*T)^\nu)$ and k is such that $\delta \le \sigma_{k+1}^{2\nu+1}$, then

$$\|x - \tilde{x}_k\|^2 = O(\sigma_{k+1}^{2\nu+1}).$$

IVP for a Parabolic Problem

- H: a Hilbert space;
- $A : D(A) \subseteq H$: a densely defined positive self adjoint unbounded operator.

Given $\varphi_0 \in H$ and $f \in L^1([0,\infty), H)$, consider the initial value problem (IVP):

$$\frac{d}{dt}u(t) + Au(t) = f(t), \quad u(0) = \varphi_0. \tag{1}$$

Known[2]:

$$u(t) = e^{-tA}\varphi_0 + \int_0^t e^{-(t-s)A} f(s)\,ds.$$

[2]See: A. Pazzy, *Semigroups of Linear Operators and Applications to PDE*, Springer-Verlag, 1983

Here, the operator e^{-tA} is defined by

$$e^{-tA}\varphi := \int_0^\infty e^{-t\lambda} dE_\lambda \varphi.$$

where $\{E_\lambda : \lambda \geq 0\}$ is the resolution of identity of A.

- $\|e^{-tA}\varphi\|^2 := \int_0^\infty e^{-2t\lambda} d\|E_\lambda \varphi\|^2 \leq \int_0^\infty d\|E_\lambda \varphi\|^2 = \|\varphi\|^2.$

Recall spectral theorem:

There exists a resolution of identity $\{E_\lambda : \lambda \geq 0\}$ such that

$$A\varphi = \int_0^\infty \lambda dE_\lambda \varphi, \quad \varphi \in D(A),$$

and in that case $D(A) := \{\varphi \in H : \int_0^\infty \lambda^2 d\|E_\lambda \varphi\|^2\}$.

- For any continuous function $g : [0, \infty) \to \mathbb{R}$,

$$g(A)\varphi := \int_0^\infty g(\lambda) dE_\lambda \varphi, \quad \varphi \in D(g(A))$$

is a self adjoint operator with

$$D(g(A)) := \{\varphi \in H : \int_0^\infty |g(\lambda)|^2 d\|E_\lambda \varphi\|^2\}.$$

- For $\varphi \in D(g(A)$,

$$\|g(A)\varphi\|^2 := \int_0^\infty |g(\lambda)|^2 d\|E_\lambda \varphi\|^2.$$

In particular, we have the following:

- For $t \geq 0$

$$e^{tA}\varphi := \int_0^\infty e^{t\lambda} dE_\lambda \varphi, \quad \varphi \in D(e^{tA}),$$

  where

$$D(e^{tA}) := \{\varphi \in H : \int_0^\infty e^{2t\lambda} d\|E_\lambda \varphi\|^2\}$$

  and

$$\|e^{tA}\varphi\|^2 = \int_0^\infty e^{2t\lambda} d\|E_\lambda \varphi\|^2\} \geq \|\varphi\|^2.$$

- e^{tA} is one-one, onto (since self adjoint), and has bounded inverse.

- $R(e^{-tA}) = D(e^{tA}) \quad \forall\, t \geq 0.$

Let

$$S(t) := e^{-tA}, \quad t \geq 0.$$

Then

- $\{S(t) : t \geq 0\}$ is a strongly continuous semigroup on H with

$$\left\| \frac{S(t)\varphi - \varphi}{t} + A\varphi \right\| \to 0 \quad \text{as} \quad t \to 0.$$

- $-A$ is the infinitesimal generator of $\mathcal{S}$, i.e.,

$$-A\varphi := \lim_{t \to 0} \frac{S(t)\varphi - \varphi}{t}, \quad \varphi \in D(-A)$$

where

$$D(-A) := \{\varphi \in H : \lim_{t \to 0} \frac{S(t)\varphi - \varphi}{t} \text{ exists}\}.$$

Final value problem (FVP)

Let $\tau > 0$, $\varphi_\tau \in H$ and $f \in L^1([0,\tau], H)$.

Consider the final value problem (FVP):

$$\frac{d}{dt}u(t) + Au(t) = f(t), \quad u(\tau) = \varphi_\tau. \tag{2}$$

Suppose u is a solution of (2). Then

$$u(t) = e^{-tA}\varphi_0 + \int_0^t e^{-(t-s)A} f(s)ds, \quad u(\tau) = \varphi_\tau,$$

where $\varphi_0 := u(0)$. In particular,

$$\varphi_\tau = u(\tau) = e^{-\tau A}\varphi_0 + \int_0^\tau e^{-(\tau-s)A} f(s)ds.$$

Theorem

Let $\varphi_\tau \in H$ and $f \in L^1([0,\tau], H)$. If the FVP (2) has a solution $u(\cdot)$ with $u(\tau) = \varphi_\tau$, then

$$\psi(t) := \varphi_\tau - \int_t^\tau e^{-(\tau-s)A} f(s)ds$$

belongs to $D(e^{(\tau-t)A})$ and

$$u(t) = e^{(\tau-t)A}\psi(t).$$

In the above, if $f = 0$, then

$$u(t) = e^{(\tau-t)A}\varphi_\tau.$$

- Since A is an unbounded operator,

  Small error in φ_τ can lead to larege error in the solution.

Proof of theorem.

Let $\varphi_0 := u(0)$. Then

$$u(t) = e^{-tA}\varphi_0 + \int_0^t e^{-(t-s)A} f(s)ds. \tag{3}$$

In particular,

$$\varphi_\tau = e^{-\tau A}\varphi_0 + \int_0^\tau e^{-(\tau-s)A} f(s)ds.$$

Since $e^{-\tau A}\varphi_0 \in D(e^{\tau A})$,

$$\varphi_\tau - \int_0^\tau e^{-(\tau-s)A} f(s)ds = e^{-\tau A}\varphi_0 \in D(e^{\tau A}).$$

Hence,

$$\varphi_0 = e^{\tau A}\Big(\varphi_\tau - \int_0^\tau e^{-(\tau-s)A} f(s)ds\Big).$$

□

Continues.

Therefore, (3) implies

$$
\begin{aligned}
u(t) &= e^{-tA}e^{\tau A}\Big(\varphi_\tau - \int_0^\tau e^{-(\tau-s)A}f(s)ds\Big) + \int_0^t e^{-(t-s)A}f(s)ds \\
&= e^{(\tau-t)A}\Big(\varphi_\tau - \int_0^\tau e^{-(\tau-s)A}f(s)ds\Big) + \int_0^t e^{-(t-s)A}f(s)ds.
\end{aligned}
$$

That is,

$$
\begin{aligned}
e^{-(\tau-t)A}u(t) &= \Big(\varphi_\tau - \int_0^\tau e^{-(\tau-s)A}f(s)ds\Big) \\
&\quad + e^{-(\tau-t)A}\int_0^t e^{-(t-s)A}f(s)ds \\
&= \varphi_\tau - e^{-(\tau-t)A}\int_0^\tau e^{-(t-s)A}f(s)ds \\
&\quad + e^{-(\tau-t)A}\int_0^t e^{-(t-s)A}f(s)ds.
\end{aligned}
$$

Continues.

Hence,

$$e^{-(\tau-t)A}u(t) = \varphi_\tau - e^{-(\tau-t)A}\int_t^\tau e^{-(t-s)A}f(s)ds.$$

Thus,

$$\varphi_\tau - \int_t^\tau e^{-(\tau-s)A}f(s)ds \in D(e^{(\tau-t)A})$$

and

$$u(t) = e^{(\tau-t)A}\Big(\varphi_\tau - \int_t^\tau e^{-(\tau-s)A}f(s)ds\Big).$$

□

In view of the above theorem, we introdice the following definition.

Definition

If $\varphi_\tau \in H$ and $f \in L^1([0,\tau], H)$ are such that

$$\psi(t) := \varphi_\tau - \int_t^\tau e^{-(\tau-s)A} f(s)ds \quad \forall\, t \in [0,\tau)$$

belongs to $D(e^{(\tau-t)A})$, then $u(\cdot)$ defined by

$$u(t) = e^{(\tau-t)A}\psi(t)$$

is called the **mild solution** of the FVP (2).

FVP as operator equation

The problem of finding a mild solution $u(\cdot)$ of the FVP with $u(\tau) = \varphi_\tau$ can be posed as a problem of solving the operator equation

$$\mathcal{A}_t u(t) = \psi(t), \tag{4}$$

where

$$\mathcal{A}_t\varphi := e^{-(\tau-t)A}\varphi, \quad \varphi \in H,$$

$$\psi(t) := \varphi_\tau - \int_t^\tau e^{-(\tau-s)A} f(s)ds.$$

Note that:

- $\mathcal{A}_t$ is an injective bounded self adjoint operator.
- $R(\mathcal{A}_t) = D(e^{(\tau-t)A})$ is dense in H.
- $\mathcal{A}_t^{-1} = e^{(\tau-t)A} : R(\mathcal{A}_t) \to H$ is not continuous.

Hence, (4) is ill-posed.

- A mild solution is not necessary to be a solution of the FVP:

Theorem

Let $\varphi_\tau \in D(e^{\tau A})$ and let $u : [0, \tau] \to H$ be defined by $u(t) = e^{(\tau - t)A}\varphi_\tau$, $t \geq 0$. Then u is a solution of the FVP

$$u_t + Au(t) = 0, \quad u(\tau) = \varphi_\tau$$

if and only if $\varphi_\tau \in D(Ae^{\tau A})$.

Proof.

For $t \geq 0$ and $h > 0$,

$$\frac{u(t+h) - u(t)}{h} = \frac{e^{(\tau - t - h)A}\varphi_\tau - e^{(\tau - t)A}\varphi_\tau}{h} = \frac{e^{-hA}u(t) - u(t)}{h}.$$

Since $-A$ is the infinitesimal generator of the semigroup $\{e^{-hA} : h \geq 0\}$,

$$\lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \text{ exists} \iff u(t) \in D(-A)$$
$$\iff \varphi_\tau \in D(Ae^{\tau A}).$$

Thus, $u'(t)$ exists for every $t \geq 0$ iff $e^{\tau A}\varphi_\tau \in D(-A)$ iff $\varphi_\tau \in D(Ae^{\tau A})$, and in that case $u'(t) = -Au(t), \; u(\tau) = \varphi_\tau$. □

Truncated spectral regularization (TRS)

Let φ_τ and $f \in L^1([0,\tau], H)$.

Recall that, the mild solution of the FVP is

$$u(t) = e^{(\tau-t)A}\psi(t) = \int_0^\infty e^{(\tau-t)\lambda} dE_\lambda(\psi(t)) \tag{5}$$

whenever $\psi(t) := \varphi_\tau - \int_t^\tau e^{-(\tau-s)A} f(s)ds$ belongs to $D(e^{\tau A})$.

> *Since small error in the data (φ_τ, f) can lead to large error in the solution $u(\cdot)$, we have to look for a regularized solution which depends continuously on the data (φ_τ, f).*

Looking at the expression in (5) for the mild solution, we define such a regularized solution as

$$u_\beta(t) = \int_0^\beta e^{(\tau - t)\lambda} dE_\lambda(\psi(t)) \tag{6}$$

for each $\beta > 0$.

The following theorem shows that $u_\beta(\cdot)$ is an approximation of $u(\cdot)$ for large β.

Theorem

Under the assumption $\psi(t) \in D(e^{\tau A})$,

$$\|u(t) - u_\beta(t)\| \to 0 \quad as \quad \beta \to \infty.$$

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$$\|u(t) - u_\beta(t)\| \to 0 \quad as \quad \beta \to \infty.$$

Proof.

Since

$$\|u(t)\|^2 = \int_0^\infty e^{2(\tau - t)\lambda} d\|E_\lambda(\psi(t))\|^2 < \infty,$$

we obtain

$$\|u(t) - u_\beta(t)\|^2 = \int_\beta^\infty e^{2(\tau - t)\lambda} d\|E_\lambda(\psi(t))\|^2 \to 0 \quad \text{as} \quad \beta \to \infty.$$

□

Now, we show that

- $u_\beta(\cdot)$ is stable under perturbations in the data (φ_τ, f).

Suppose $\tilde{\varphi}_\tau \in H$ and $\tilde{f} \in L^1([0,\tau], H)$ are the noisy data, in place of the actual data φ_τ and f, respectively.

Let

$$\tilde{u}_\beta(t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda(\tilde{\psi}(t)),$$

where

$$\tilde{\psi}(t) := \tilde{\varphi}_\tau - \int_t^\tau e^{-(\tau-s)A}\tilde{f}(s)ds.$$

Theorem

Let $\varphi_\tau, \tilde{\varphi}_\tau \in H$ and $f, \tilde{f} \in L^1([0,\tau], H)$. The for each $t \in [0,\tau]$ and $\beta > 0$,

$$\begin{aligned} \|u_\beta(t) - \tilde{u}_\beta(t)\| &\leq e^{(\tau - t)\beta} \|\psi(t) - \tilde{\psi}(t)\| \\ &\leq e^{(\tau - t)\beta} (\|\varphi_\tau - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1). \end{aligned}$$

Suppose $\|\varphi_\tau - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1 \leq \delta$ for some $\delta > 0$. Then we obtain

$$\|u_\beta(t) - \tilde{u}_\beta(t)\| \leq e^{(\tau - t)\beta}\delta.$$

Observation:

- For a fixed $\beta > 0$,

$$\|u_\beta(t) - \tilde{u}_\beta(t)\| \to 0 \quad \text{as} \quad \delta \to 0.$$

- $u_\beta(t)$ is stable under perturbations in the data (φ_τ, f).

From the above theorem we have

Theorem

Let $\varphi_\tau, \tilde{\varphi}_\tau \in H$ and $f, \tilde{f} \in L^1([0,\tau], H)$ such that

$$\|\varphi_\tau - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1 \le \delta$$

for some $\delta > 0$. The for each $t \in [0,\tau]$ and $\beta > 0$,

$$\|u(t) - \tilde{u}_\beta(t)\| \le \|u(t) - u_\beta(t)\| + e^{(\tau - t)\beta}\delta.$$

$$\beta \approx \frac{1}{\tau - t} \log\left(\frac{1}{\delta^p}\right), \quad 0 < p < 1,$$

$\Rightarrow$

$$\|u(t) - \tilde{u}_\beta(t)\| = o(1) \quad \text{as} \quad \delta \to 0.$$

Proof of Theorem.

We observe that

$$\psi(t) - \tilde{\psi}(t) = \varphi_t - \tilde{\varphi}_\tau) - \int_t^\tau e^{-(\tau-s)A}\tilde{(}f(s) - \tilde{f}(s))ds$$

and

$$u_\beta(t) - \tilde{u}_\beta(t) = \int_0^\beta e^{(\tau-t)\lambda} dE_\lambda(\psi(t) - \tilde{\psi}(t)).$$

Note that

$$\begin{aligned}
\|\psi(t) - \tilde{\psi}(t)\| &\leq \|\varphi_t - \tilde{\varphi}_\tau\| + \int_t^\tau \|e^{-(\tau-s)A}\| \, \|f(s) - \tilde{f}(s)\| ds \\
&\leq \|\varphi_t - \tilde{\varphi}_\tau\| + \int_t^\tau \|f(s) - \tilde{f}(s)\| ds \\
&\leq \|\varphi_t - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1.
\end{aligned}$$

□

Continues.

Hence,

$$\begin{aligned}\|u_\beta(t) - \tilde{u}_\beta(t)\|^2 &= \int_0^\beta e^{2(\tau - t)\lambda} dE_\lambda \|\psi(t) - \tilde{\psi}(t)\|^2 \\ &\leq e^{2(\tau - t)\beta} \|\psi(t) - \tilde{\psi}(t)\|^2\end{aligned}$$

Thus,

$$\begin{aligned}\|u_\beta(t) - \tilde{u}_\beta(t)\| &\leq e^{2(\tau - t)\beta} \|\psi(t) - \tilde{\psi}(t)\| \\ &\leq e^{2(\tau - t)\beta} (\|\varphi_t - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1).\end{aligned}$$

□

Continues.

Hence,

$$\begin{aligned}\|u_\beta(t) - \tilde{u}_\beta(t)\|^2 &= \int_0^\beta e^{2(\tau - t)\lambda} dE_\lambda \|\psi(t) - \tilde{\psi}(t)\|^2 \\ &\leq e^{2(\tau - t)\beta} \|\psi(t) - \tilde{\psi}(t)\|^2\end{aligned}$$

Thus,

$$\begin{aligned}\|u_\beta(t) - \tilde{u}_\beta(t)\| &\leq e^{2(\tau - t)\beta} \|\psi(t) - \tilde{\psi}(t)\| \\ &\leq e^{2(\tau - t)\beta} (\|\varphi_t - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1).\end{aligned}$$

□

Next we obtain an estimate for the error under an additional smoothness assumption on $u(\cdot)$.

Theorem

Suppose $\varphi_\tau \in H$ *and* $f \in L^1([0,\tau], H)$ *are such that for each* $t \in [0,\tau)$, $\psi(t) \in D(e^{\tau A})$ *and there exists a monotonically increasing continuous function* $h_t(\dot{}) : [0,\tau] \to [0,\infty)$ *such that*

(i) $h_t(\lambda) \to \infty$ *as* $\lambda \to \infty$,

(i) $u(t) \in D(h_t(A))$,

(ii) $\|h_t(A)u(t)\| \le \rho_t$ *for some* $\rho_t > 0$.

Then

$$\|u(t) - u_\beta(t)\| \le \frac{\rho_t}{h_t(\beta)}.$$

Proof.

Recall that

$$u(t) = e^{(\tau - t)A}\psi(t).$$

Hence,

$$\begin{aligned}
\|u(t) - u_\beta(t)\|^2 &= \int_\beta^\infty e^{2(\tau-t)\lambda} d\|E_\lambda(\psi(t))\|^2 \\
&= \int_\beta^\infty \frac{1}{h_t(\lambda)^2} h_t(\lambda)^2 e^{2(\tau-t)\lambda} d\|E_\lambda(\psi(t))\|^2 \\
&\leq \frac{1}{h_t(\beta)^2} \int_\beta^\infty h_t(\lambda)^2 e^{2(\tau-t)\lambda} d\|E_\lambda(\psi(t))\|^2
\end{aligned}$$

□

Continues.

By the assumption,

$$\begin{aligned}\int_0^\infty h_t(\lambda) e^{(\tau - t)\lambda} d\|E_\lambda(\psi(t))\|^2 &= \|h_t(A) e^{(\tau-t)A}\psi(t)\|^2 \\ &= \|h_t(A)u(t)\|^2 \leq \rho_t^2.\end{aligned}$$

Hence, we have

$$\|u(t) - u_\beta(t)\| \leq \rho_t / h_t(\beta).$$

□

Combining the last two theorems, we obtain the following.

Combining the last two theorems, we obtain the following.

Theorem

Suppose $\tilde{\varphi}_\tau$ *and* $\tilde{f}$ *are noisy data such that*

$$\|\varphi_\tau - \tilde{\varphi}_\tau\| + \|f - \tilde{f}\|_1 \le \delta$$

for some noise level $\delta > 0$. *Then*

$$\|u_\beta(t) - \tilde{u}_\beta(t)\| \le e^{(\tau - t)\beta}\delta.$$

If $\rho_t > 0$ *and* $h_t(\cdot)$ *are as in last theorem, then we have*

$$\|u(t) - \tilde{u}_\beta(t)\| \le \frac{\rho_t}{h_t(\beta)} + e^{(\tau - t)\beta}\delta.$$

Parameter choice strategy

Theorem

Let

$$\xi_t(\lambda) := h_t(\lambda)e^{(\tau - t)\lambda}, \quad \lambda > 0$$

and

$$\beta = \beta_t := \xi_t^{-1}(\rho/\delta).$$

Then

$$\|u(t) - \tilde{u}_\beta(t)\| \le \frac{2\rho}{h(\xi_t^{-1}(\rho/\delta))}.$$

In particular,

$$\|u(t) - \tilde{u}_\beta(t)\| \to 0 \quad \textit{as} \quad \delta \to 0.$$

Proof.

Note that

$$\frac{\rho_t}{h_t(\beta)} = e^{(\tau-t)\beta}\delta \iff \xi_t(\beta) := h_t(\beta)e^{(\tau-t)\beta} = \frac{\rho_t}{\delta}$$
$$\iff \beta = \xi_t^{-1}(\rho_t/\delta).$$

Thus, for the choice of $\beta = \xi_t^{-1}(\rho_t/\delta)$,

$$\begin{aligned} \|u(t) - \tilde{u}_\beta(t)\| &\leq \frac{\rho_t}{h_t(\beta)} + e^{(\tau-t)\beta}\delta \\ &\leq \frac{2\rho}{h(\xi_t^{-1}(\rho/\delta))} \end{aligned}$$

Since $h(\xi_t^{-1}(\rho_t/)) \to \infty$ as $\to 0$.

$$\|u(t) - \tilde{u}_\beta(t)\| \to 0 \quad \text{as} \quad \to 0.$$

□

Remarks on optimality

Recall that the operator $\mathcal{A}_t : H \to H$ defined by

$$\mathcal{A}_t \varphi := e^{-(\tau - t)A}\varphi, \quad \varphi \in H$$

is injective, continuous, self adjoint, with $R(\mathcal{A}_t)$ dense in H.

Therefore,

- $u(\cdot)$ is a generalized solution

$$\mathcal{A}_t u(t) = \varphi_\tau - \int_t^\tau e^{-(\tau - s)A} f(s)ds \qquad (7)$$

  if and only if it is a solution.

Let $u(t)$ be the solution of (7) and let $u_\alpha^L(\cdot)$ be the Lavrentive regularized solution, i.e.,

$$(\mathcal{A}_t + \alpha I)u_\alpha^L(t) = \psi(t) := \varphi_\tau - \int_t^\tau e^{-(\tau - s)A} f(s)ds.$$

Then, from the standard theory, we know that

$$\|u(t) - u_\alpha^L(t)\| \to 0 \quad \text{as} \quad \alpha \to 0$$

and

$$\|u_\alpha^L(t) - \tilde{u}_\alpha^L(t)\| \le \frac{\delta}{\alpha}. \tag{8}$$

Note that the estimate

$$\|u_\beta(t) - \tilde{u}_\beta(t)\| \le \delta e^{(\tau - t)\beta}$$

obtained eariler is same as (8) if we take β such that

$$e^{(\tau - t)\beta} = \frac{1}{\alpha}.$$

That is,

$$\beta = \frac{1}{\tau - t} \ln\left(\frac{1}{\alpha}\right).$$

Next, suppose

$$u(t) = \mathcal{A}_t v(t) \quad \text{with} \quad \|v(t)\| \leq \rho_t, \tag{9}$$

equivalently,

$$u(t) \in D(e^{(\tau - t)A}) \quad \text{with} \quad \|e^{(\tau - t)A} u(t)\| \leq \rho_t. \tag{10}$$

Then we have the estimate

$$\|u(t) - u_\alpha^L(t)\| \leq \rho_t \alpha.$$

Under the choice $\beta := \frac{1}{\tau - t} \ln\left(\frac{1}{\alpha}\right)$, the above estimate takes the form

$$\|u(t) - u_\alpha^L(t)\| \leq \rho_t e^{-(\tau - t)\beta}. \tag{11}$$

This is same as the estimate obtained earlier for $\|u(t) - u_\beta(t)\|$ under the (10) .

Thus, we can conclude:

If $h_t(\lambda) := e^{(\tau - t)\lambda}$, then the the estimate obtained under TSR is same as the order optimal rate possible for the Lavrentive regularization for the source condition (9), if

$$\beta = \frac{1}{\tau - t} \ln\left(\frac{1}{\alpha}\right) \quad \text{and} \quad \alpha = \sqrt{\delta/\rho},$$

that is, if

$$\beta := \frac{1}{2(\tau - t)} \ln\left(\frac{\rho}{\delta}\right).$$

Similar conclusion can be made for a general $h_t(\cdot)$ as well.

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Thank you for your attention