Diagonal subalgebras of bigraded algebras

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Some algebraic objects have natually bigraded structure,

- Associate to it graded algebras (diagonal subalgebras) via a diagonal functor.
  - Study their (commutative) algebraic properties.

Part I - Diagonal subalgebras

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- Koszul property of the (c, e)-**diagonal subalgebra** R_Δ.
- Cohen-Macaulay property of the (c, e)-**diagonal subalgebra** R_Δ.

Diagonal subalgebras of Rees algebra

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Remark. $\mathcal{R}(I)_\Delta$ is the homogeneous co-ordinate ring of the twisted quartic curve in the projective space $\mathbb{P}^3$.

An example of Diagonal subalgebra of Rees algebra

Proof of Remark.

i) $\mathcal{R}(I) = \mathsf{k}[x_1, x_2, x_1^3 t, x_2^3 t] \;\cong\; \mathsf{k}[x_1, x_2, y_1, y_2]/(x_1^3 y_2 - x_2^3 y_1) = B/K.$

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$$\left((x_1y_1)^2(x_1y_2) - (x_2y_1)^3, (x_2y_1)(x_2y_2)^2 - (x_1y_2)^3, (x_1y_1)(x_1y_2)^2 - (x_2y_1)^2(x_2y_2)\right).$$

iii) Define $z_0 \mapsto x_1y_1,\ z_1 \mapsto x_1y_2,\ z_2 \mapsto x_2y_1,\ z_3 \mapsto x_2y_2$, then

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$$\mathcal{R}(I)_\Delta = \frac{\mathrm{k}[z_0, \dots, z_3]}{(z_1z_2 - z_0z_3, z_1^3 - z_2z_3^2,\ z_2^3 - z_0^2z_1,\ z_0z_1^2 - z_2^2z_3)}.$$

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v) The vanishing locus of a twisted quartic curve defined from $\mathbb{P} \longrightarrow \mathbb{P}^3$ by the map $[x_0, x_1] \mapsto [x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4]$ are the polynomials $z_1z_2 - z_0z_3, z_1^3 - z_2z_3^2, z_2^3 - z_0^2z_1, z_0z_1^2 - z_2^2z_3$.

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vi) $\mathrm{Sym}(R^m)_\Delta$ and $\mathrm{Sym}(I)_\Delta$ are diagonal subalgebras.

Part II - Cohen-Macaulay and Koszul properties

Linear Algebra and Commutative Algebra

Finitely generated (f.g.) module over a ring $\longleftrightarrow$ f.g. vector space over a ring.

Usually f.g. modules have generators and relations among them instead of having basis as in Linear algebra.

Let R be any commutative ring with 1. Then R^n is a vector space of rank n over R with basis $e_1, e_2, \ldots, e_n$. We say R^n is a free R-module. Modules having basis are rare.

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Example. Let $R = \mathrm{k}[x, y]/(x^2, y^3)$. Then R is a module over $\mathrm{k}[x, y]$ with generators $\underline{x}, \underline{y}$ and relations $x^2 = 0, y^3 = 0$ and $fx^2 + gy^3 = 0$ for some f, g in $\mathrm{k}[x, y]$. (For instance, take $f = y^3$ and $g = -x^2$).

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Free modules are most fundamental (in other words, most elementary) objects in the theory of modules in commutative algebra.

Free resolution

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D. Hilbert (1890): Used generators of module M and relations among them to describe the properties of module M, by approximating it with free R-modules. This process of approximation by free module is known as free resolution.

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Free resolution of M over R:

$$\cdots \to R^{\beta_2} \xrightarrow{\partial_2 = \left(\begin{smallmatrix}\text{relations on}\\ \text{the relations}\\ \text{in } \partial_1\end{smallmatrix}\right)} R^{\beta_1} \xrightarrow{\partial_1 = \left(\begin{smallmatrix}\text{relations on}\\ \text{the generators}\\ \text{of } M\end{smallmatrix}\right)} R^{\#M} \xrightarrow{\left(\begin{smallmatrix}\text{generators}\\ \text{of } M\end{smallmatrix}\right)} M \to 0,$$

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- Poincaré-Betti series of M is $\mathrm{P}_{\mathrm{M}}^{\mathrm{R}}(\mathrm{t}) = \sum_{\mathrm{i} \geq 0} \beta_{\mathrm{i}} \cdot \mathrm{t}^{\mathrm{i}}$
  (Assuming that free resolution is minimal).
- Hilbert series of M is $\mathrm{H}_{\mathrm{M}}^{\mathrm{R}}(\mathrm{t}) = \sum_{\mathrm{i} \geq 0} \dim_{\mathrm{R}} \mathrm{M}_{\mathrm{i}} \cdot \mathrm{t}^{\mathrm{i}}$.

Koszul algebras

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- Let $\mathfrak{m}_R = \oplus_{i \geq 1} R_i$ be the unique homogeneous maximal ideal of R.
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Let **F** be the minimal graded free resolution of $\mathrm{k} = R/\mathfrak{m}_R$ over R:

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Using Mumford (1966) definition of regularity, *R is Koszul* if

$$\mathrm{reg}_R(\mathsf{k}) := \sup_{i \in \mathbb{N}} \{ j - i \ : \ \dim_{\mathsf{k}} \mathrm{Tor}_i^R(\mathsf{k}, \mathsf{k})_j \neq 0 \} = 0.$$

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Löfwall (1983): R is Koszul if and only if $\mathrm{H_R(t)} \cdot \mathrm{P^R_k(-t)} = 1$.

Few definitions and facts

- $S = \mathrm{k}[x_1, \ldots, x_n]$ be a polynomial ring over a field k.
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- $R = S/I$ is complete interection if I is generated by regular sequence.
- We say R *is Cohen-Macaulay* if the length of maximal regular sequence in R is same as the dimension of R, i.e., $\dim R = \operatorname{depth} R$.

Motivation for studying Koszul algebras

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Then $J \subset R$ is *geometric m-residual intersection* of $\mathfrak{m}_S$, if $\operatorname{ht}(J) \geq m$, and $\operatorname{ht}(I_n(\phi)) \geq m - n + 1$.

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Thank you for your attention!