

Diagonal subalgebras of bigraded algebras

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Joint work with H. Ananthnarayan and Vivek Mukundan

© Diamond Jubilee Symposium of IIT Bombay,
January 4-6, 2019

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Some algebraic objects have naturally **bigraded structure**,

- ▶ Associate to it graded algebras (**diagonal subalgebras**) via a diagonal functor.
 - ▶ Study their (commutative) algebraic properties.

Part I - Diagonal subalgebras

Diagonal subalgebras

- i) c and e denote positive integers,
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- ▶ **Koszul property** of the (c, e) -**diagonal subalgebra** R_Δ .
- ▶ **Cohen-Macaulay property** of the (c, e) -**diagonal subalgebra** R_Δ .

Diagonal subalgebras of Rees algebra

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- ii) The *Rees algebra* of I is the subalgebra of $k[x_1, \dots, x_n, t]$ defined as

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Remark. $\mathcal{R}(I)_\Delta$ is the homogeneous co-ordinate ring of the twisted quartic curve in the projective space \mathbb{P}^3 .

An example of Diagonal subalgebra of Rees algebra

Proof of Remark.

$$i) \mathcal{R}(I) = k[x_1, x_2, x_1^3 t, x_2^3 t] \cong k[x_1, x_2, y_1, y_2]/(x_1^3 y_2 - x_2^3 y_1) = B/K.$$

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- ii) $\mathcal{R}(I)_\Delta = B_\Delta/K_\Delta$, where $B_\Delta = k[x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2]$ and K_Δ is $\left((x_1 y_1)^2 (x_1 y_2) - (x_2 y_1)^3, (x_2 y_1)(x_2 y_2)^2 - (x_1 y_2)^3, (x_1 y_1)(x_1 y_2)^2 - (x_2 y_1)^2 (x_2 y_2) \right)$.
- iii) Define $z_0 \mapsto x_1 y_1, z_1 \mapsto x_1 y_2, z_2 \mapsto x_2 y_1, z_3 \mapsto x_2 y_2$, then

$$\frac{k[z_0, \dots, z_3]}{(z_1 z_2 - z_0 z_3)} \cong k[x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2] = B_\Delta$$

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- v) The vanishing locus of a twisted quartic curve defined from $\mathbb{P} \rightarrow \mathbb{P}^3$ by the map $[x_0, x_1] \mapsto [x_0^4, x_0^3 x_1, x_0^2 x_1^2, x_0 x_1^3, x_1^4]$ are the polynomials $z_1 z_2 - z_0 z_3, z_1^3 - z_2 z_3^2, z_2^3 - z_0^2 z_1, z_0 z_1^2 - z_2^2 z_3$.

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$$\text{Sym}(I) = \frac{S[t_1, \dots, t_r]}{J_\phi}, \text{ where } J_\phi = ([t_1 \ t_2 \ \dots \ t_r] \cdot \phi)$$

and ϕ is the presentation matrix of I , i.e., $S^p \xrightarrow{\phi} S^r \rightarrow I \rightarrow 0$.

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- vi) $\text{Sym}(R^m)_\Delta$ and $\text{Sym}(I)_\Delta$ are diagonal subalgebras.

Part II - Cohen-Macaulay and Koszul properties

Linear Algebra and Commutative Algebra

Finitely generated (f.g.) module over a ring \longleftrightarrow f.g. vector space over a ring.

Usually f.g. modules have **generators and relations among them** instead of having basis as in Linear algebra.

Let R be any commutative ring with 1. Then R^n is a **vector space** of rank n over R with basis e_1, e_2, \dots, e_n . We say R^n is a **free R -module**. **Modules having basis are rare.**

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Example. Let $R = k[x, y]/(x^2, y^3)$. Then R is a module over $k[x, y]$ with **generators $\underline{x}, \underline{y}$ and relations $x^2 = 0, y^3 = 0$ and $fx^2 + gy^3 = 0$** for some f, g in $k[x, y]$. (For instance, take $f = y^3$ and $g = -x^2$).

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Free modules are most fundamental (in other words, most elementary) objects in the theory of modules in commutative algebra.

Free resolution

- ▶ Let $R = k[x_1, \dots, x_n]$ be a polynomial ring and M a f.g. R -module.

D. Hilbert (1890): Used **generators of module M and relations among them** to describe the properties of module M , by approximating it with free R -modules. This process of approximation by free module is known as **free resolution**.

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Free resolution of M over R :

$$\dots \rightarrow R^{\beta_2} \xrightarrow{\partial_2 = \begin{pmatrix} \text{relations on} \\ \text{the relations} \\ \text{in } \partial_1 \end{pmatrix}} R^{\beta_1} \xrightarrow{\partial_1 = \begin{pmatrix} \text{relations on} \\ \text{the generators} \\ \text{of } M \end{pmatrix}} R^{\#M} \xrightarrow{\begin{pmatrix} \text{generators} \\ \text{of } M \end{pmatrix}} M \rightarrow 0,$$

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- ▶ **Poincaré-Betti series** of M is $P_M^R(t) = \sum_{i \geq 0} \beta_i \cdot t^i$
(Assuming that free resolution is minimal).
- ▶ **Hilbert series** of M is $H_M^R(t) = \sum_{i \geq 0} \dim_R M_i \cdot t^i$.

Koszul algebras

- ▶ $R = \bigoplus_{i \geq 0} R_i$ is a graded ring with $R_0 = k$ (e.g. $R = S/I$).
- ▶ Let $\mathfrak{m}_R = \bigoplus_{i \geq 1} R_i$ be the unique homogeneous maximal ideal of R .
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Let \mathbf{F} be the minimal graded free resolution of $k = R/\mathfrak{m}_R$ over R :

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Using Mumford (1966) definition of regularity, R is **Koszul** if

$$\text{reg}_R(k) := \sup_{i \in \mathbb{N}} \{j - i : \dim_k \text{Tor}_i^R(k, k)_j \neq 0\} = 0.$$

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Löfwall (1983): R is **Koszul** if and only if $H_R(t) \cdot P_k^R(-t) = 1$.

Few definitions and facts

- ▶ $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k .
- ▶ $I = (f_1, f_2, \dots, f_r) \subset S$, where f_i 's are homogeneous polynomials.
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- ▶ $R = S/I$ is defined by **Gröbner basis of quadrics**, if w.r.t some coordinate system of S_1 and some term order τ on S , I can be **deformed to an ideal generated by monomials of degree 2** without changing the **Hilbert series** of R .

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- ▶ $I = (f_1, f_2, \dots, f_r) \subset S$ is generated by **regular sequence** if for each i , the image of f_{i+1} is a nonzero divisor in $S/(f_1, \dots, f_i)$.
- ▶ $R = S/I$ is **complete intersection** if I is generated by regular sequence.

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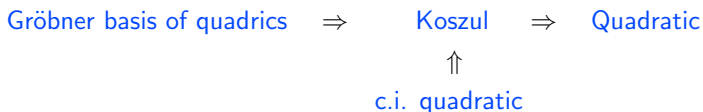
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- ▶ $I = (f_1, f_2, \dots, f_r) \subset S$ is generated by **regular sequence** if for each i , the image of f_{i+1} is a nonzero divisor in $S/(f_1, \dots, f_i)$.
- ▶ $R = S/I$ is **complete intersection** if I is generated by regular sequence.
- ▶ We say R is **Cohen-Macaulay** if the length of maximal regular sequence in R is same as the dimension of R , i.e., $\dim R = \text{depth } R$.

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- i) Mumford (1969) - Any projective variety $X \subset \mathbb{P}^n$, its Veronese embedding $v_{d_0}(X) \subset \mathbb{P}^{N_0}$ is cut out by quadrics, for $d_0 \gg 0$.
- ii) Backlin (1986) - If R is any standard graded k -algebra, then Veronese ring $R^{(d_1)}$ is even Koszul, for $d_1 \geq d_0 \gg 0$.
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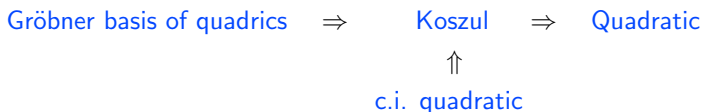
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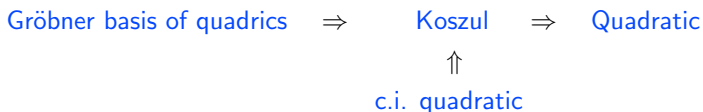
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- (c) Anick (1980), Roos and Sturmfels (1998) - Poincaré series can be irrational.

Results

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Then $J \subset R$ is *geometric m -residual intersection* of \mathfrak{m}_S , if $\text{ht}(J) \geq m$, and $\text{ht}(I_n(\phi)) \geq m - n + 1$.

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Thank you for your attention!