# A Subspace of Maximal Dimension of Schmidt Rank at least 3


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- In quantum information theory, quantum states of a particle called *qubit* are used to encode the data. A qubit is typically an element (of unit length) of the two dimensional complex Hilbert space $\mathbb{C}^2$ with the usual inner product.
- We use the notation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Then a qubit is not only either of $|0\rangle$ and $|1\rangle$, but it can be any (normalized) $\mathbb{C}$-linear combination of $|0\rangle$ and $|1\rangle$ which is called *superposition*.
- The Hilbert space $\mathbb{C}^2$ is known as $1$-*qubit* Hilbert space.
- In general, data is encoded as a state in the $n$-*qubit space* $(\mathbb{C}^2)^{\otimes^n}$ which is $n$-fold tensor product of the $1$-qubit space.

- A *state* (also known as the density matrices) is a positive matrix (with complex entries) whose trace is equal to 1.
- Consider the bipartite Hilbert space $\mathcal{H} := \mathbb{C}^m \otimes \mathbb{C}^n$. A unit vector $|v\rangle \in \mathcal{H}$ (or the positive matrix $|v\rangle \langle v|$ of trace $1$) is called a *pure state*.
- A pure state $|v\rangle \in \mathcal{H}$ is said to be a *product state* if it can be written as $|v\rangle = |v_1\rangle \otimes |v_2\rangle \, (= |v_1\rangle |v_2\rangle)$ for some $|v_1\rangle \in \mathbb{C}^m$ and $|v_2\rangle \in \mathbb{C}^n$; otherwise it is called *entangled*.
- For $x_i \in \{0, 1\}$, $i = 1, 2$, let $|x_1 x_2\rangle = |x_1\rangle |x_2\rangle = |x_1\rangle \otimes |x_2\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$.
- The (pure) state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

  is a famous 2-qubit entangled state, which is known as *Bell state* or *EPR pair*.

- States which are not pure are also referred to as *mixed states*.
- A mixed state $\rho$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ (that is, an element of $M_m \otimes M_n$) is called *separable* if it is a convex combination of pure product states; otherwise it is called *entangled*.
- Entanglement is the key property of quantum systems which is responsible for the higher efficiency of quantum computation and tasks like teleportation, super-dense coding, etc.

# Schmidt Rank and Schmidt Number

- In the Schmidt decomposition $|\psi\rangle = \sum_{j=1}^{k} \alpha_j |u_j\rangle \otimes |v_j\rangle$ of a pure state $|\psi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ where $\{|u_j\rangle : 1 \leqslant j \leqslant k\}$ and $\{|v_j\rangle : 1 \leqslant j \leqslant k\}$ are orthonormal sets in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, and $\alpha_j$'s are nonnegative real numbers satisfying $\sum_j \alpha_j^2 = 1$, the minimum number of terms required in the summation is known as the *Schmidt rank* of $|\psi\rangle$.
- The *Schmidt number* of a state $\rho$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ is defined to be the least natural number $k$ such that $\rho$ can be decomposed to the form $\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j|$ for some pure states $|\psi_j\rangle$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ and a probability distribution $\{p_j\}$, with Schmidt rank of $|\psi_j\rangle \leqslant k$ for all $j$.

- The Schmidt rank of vectors (pure states) and Schmidt number of states in a bipartite finite dimensional Hilbert space are measures of entanglement.
- For example, if Schmidt rank is $1$, then it is a product vector; otherwise it is entangled. Similarly, if Schmidt number is $1$, then it is a separable state; otherwise it is entangled.

**Theorem:**

Let $\mathcal{S}$ be a subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ which does not contain any vector of Schmidt rank lesser or equal to $k$. Then any state $\rho$ supported on $\mathcal{S}$ has Schmidt number at least $k+1$.

- It is proved by T. Cubitt, A. Montanaro, and A. Winter that the dimension of any subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$, with Schmidt rank (of all vectors in the subspace) greater than or equal to $k$ is bounded above by $(m-k+1)(n-k+1)$. Also, this bound is attained.
- The special case when $k=2$, the above result is first proved by K. R. Parthasarathy, N. R. Wallach separately.
- B. V. Rajarama Bhat proves this result for $k=2$ in a different approach.
- Motivated by the analysis done in Bhat's paper, we present an alternate approach to the above mentioned result by T. Cubitt, A. Montanaro, and A. Winter for the case when $k=3$.

- Fix an infinite dimensional Hilbert space generated by the orthonormal basis $\{|e_0\rangle, |e_1\rangle, \ldots\}$, and identify $\mathbb{C}^m$ and $\mathbb{C}^n$ with span$\{|e_0\rangle, |e_1\rangle, \ldots, |e_{m-1}\rangle\}$ and span$\{|e_0\rangle, |e_1\rangle, \ldots, |e_{n-1}\rangle\}$, respectively.
- Define $N = m + n - 2$, and for $2 \leqslant d \leqslant N - 2$ define

$$\mathcal{S}^{(d)} = \text{span}\{|e_{i-1}\rangle \otimes |e_{j+1}\rangle - 2|e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leqslant i \leqslant m-2, 1 \leqslant j \leqslant n-2, i+j = d\},$$

$$\mathcal{S}^{(0)} = \mathcal{S}^{(1)} = \mathcal{S}^{(N-1)} = \mathcal{S}^{(N)} = \{0\}, \text{ and } \mathcal{S} := \bigoplus_{d=0}^{N} \mathcal{S}^{(d)}.$$

**Theorem:**

Let $m$ and $n$ be natural numbers such that $3 \leqslant \min\{m, n\}$. The space $\mathcal{S}$ (subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$) defined above does not contain any vector of Schmidt rank $\leqslant 2$ and $\dim \mathcal{S} = (m-2)(n-2)$.

## Lemma

*The columns of the* $(t+2) \times t$ *matrix*

$$A_t = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ -2 & 1 & \dots & 0 & 0 \\ 1 & -2 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -2 & 1 \\ 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

*are linearly independent such that any linear combination of these columns has at least 3 nonzero entries.*

# Sketch of the Proof

- For any element $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$, we have $|v\rangle = \sum_{ij} c_{ij}|e_i\rangle \otimes |e_j\rangle$.
- We identify $\mathbb{C}^m \otimes \mathbb{C}^n$ with the space $M_{m\times n}$ of $m \times n$ matrices by the isometric isomorphism $\phi : \mathbb{C}^m \otimes \mathbb{C}^n \longrightarrow M_{m\times n}$ defined by $\phi(|v\rangle) = [c_{ij}]_{m\times n}$.
- An element of $\mathbb{C}^m \otimes \mathbb{C}^n$ has Schmidt rank at least $r$ if and only if the corresponding $m \times n$ matrix is of rank at least $r$.
- Also, it is known that a matrix has rank at least $r$ if and only if it has a nonzero minor of order $r$.
- So, it is enough to construct a set of $(m-2)(n-2)$ linearly independent matrices, which are image under $\phi$ of a basis of $\mathcal{S}$, such that any linear combination of these matrices has a nonzero minor of order 3.

- Label the anti-diagonals of any $m \times n$ matrix by non-negative integers $k$ such that the first anti-diagonal from upper left (of length one) is labelled $k = 0$ and value of $k$ increases from upper left to lower right.
- Recall that for $2 \leqslant k \leqslant N-2$, $\mathcal{S}^{(k)}$ is generated by the set

$$B_k = \{|e_{i-1}\rangle \otimes |e_{j+1}\rangle - 2\,|e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leqslant i \leqslant m-2, \\ 1 \leqslant j \leqslant n-2, i+j=k\}.$$

- For $2 \leqslant k \leqslant N-2$, let $\widetilde{B}_k$ denote the image of $B_k$ under the map $\phi$. Note that any element of $\widetilde{B}_k$ is the matrix obtained from the $m \times n$ zero matrix by replacing the $k^{th}$ anti-diagonal by $(0, \ldots, 0, 1, -2, 1, 0, \ldots, 0)$.

- Let the length of the $k^{th}$ anti-diagonal be denoted by $|k|$.
- For $2 \leqslant k \leqslant N-2$ and $t = |k| - 2$, from the lemma, $\widetilde{B}_k$ is a set of $t$ linearly independent matrices.
- Also, by the lemma it follows that if $M$ is a matrix obtained by taking any linear combination of matrices from $\widetilde{B}_k$, then $M$ has at least $3$ nonzero entries in the $k^{th}$ anti-diagonal and the entries of $M$, other than those on $k^{th}$ anti-diagonal, are zeros.
- Since the determinant of the $3 \times 3$ submatrix with these $3$ nonzero elements in its principal anti-diagonal is clearly nonzero, any linear combination of the matrices in $\widetilde{B}_k$ has at least one nonzero order-$3$ minor, thus has rank at least $3$.

- Let $\widetilde{B} = \bigcup_{k=2}^{N-2} \widetilde{B}_k$.
- Since elements from different $\widetilde{B}_k$'s have different nonzero anti-diagonal, $\widetilde{B}$ is linearly independent. Thus $B = \bigcup_{k=2}^{N-2} B_k$ is a basis for $\mathcal{S}$.
- Let $C$ be a matrix obtained by an arbitrary linear combination from the elements of $\widetilde{B}$.
- Let $\kappa$ be the largest $k$ for which the linear combination involves an element from $\widetilde{B}_k$. The $\kappa^{th}$ anti-diagonal of $C$ has at least $3$ nonzero elements.

- The $3 \times 3$ submatrix of $C$ with these 3 nonzero elements in the principal anti-diagonal has nonzero determinant.
- Thus the rank of $C$ is at least 3.
- We conclude that $\mathcal{S}$ does not contain any vector of Schmidt rank $\leqslant 2$.
- The dimension of $\mathcal{S}$ is equal to the cardinality of $B$, which is calculated to be $(m-2)(n-2)$.

From the above theorem, it follows that the basis

$$B = \bigcup_{k=2}^{N-2} \{ |e_{i-1}\rangle \otimes |e_{j+1}\rangle - 2\,|e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leqslant i \leqslant m-2, \\ 1 \leqslant j \leqslant n-2, i+j=k \},$$

of $\mathcal{S}$ is of Schmidt rank $3$ (i.e., all the elements have Schmidt rank $3$).

# References

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# Thank You