Smooth Structures on Quaternionic Projective Spaces

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(Joint work with Samik Basu)

Jan 4, 2019

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- The set of oriented diffeomorphism classes of homotopy m-spheres is denoted by Θ_m.

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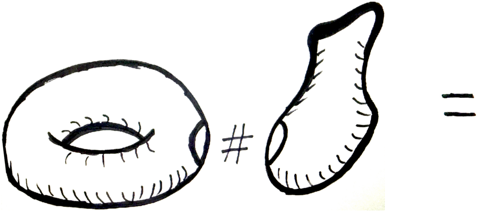

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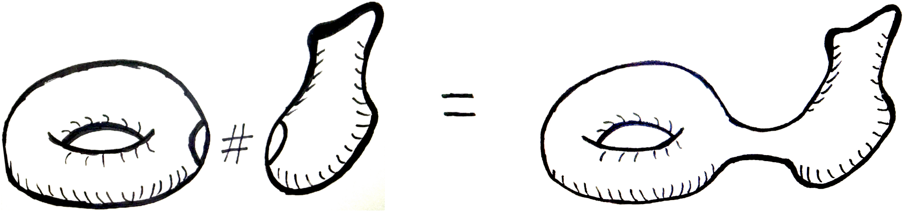

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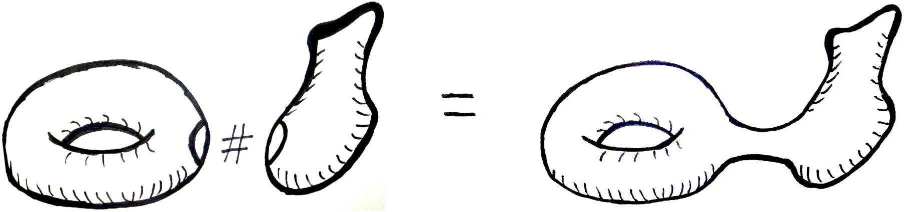

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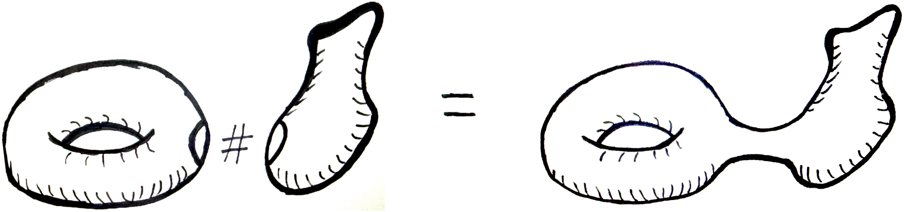

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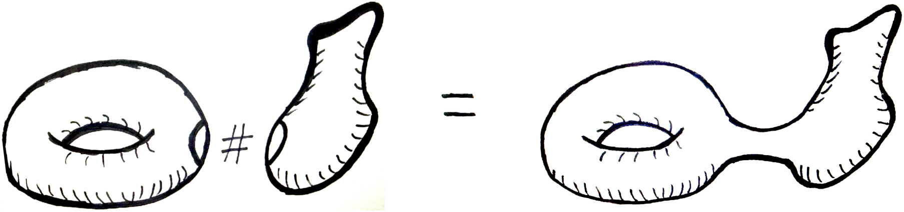

This operation is well defined up to orientation preserving diffeomorphism. Thus we obtain a commutative, associative semigroup $\mathcal{M}_m$ of oriented diffeomorphism classes; with the class of $\mathbb{S}^m$ as an identity element, $M^m \# \mathbb{S}^m \cong M^m$.

Invertibility: $M^n \# N^n \cong \mathbb{S}^n$ for some N^n?

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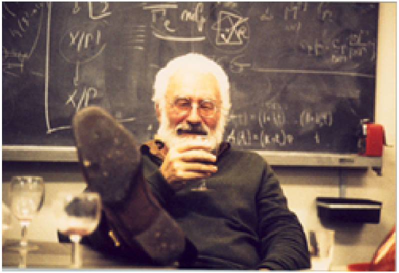
Theorem (Kervaire and Milnor + Perelman)

Each Θ_m is a finite abelian groups for $m \neq 4$.



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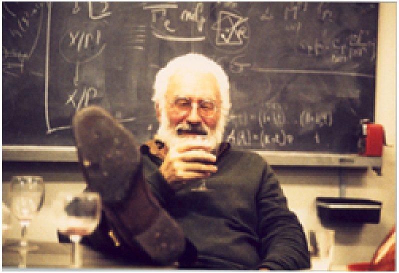
Group of homotopy spheres, 1963



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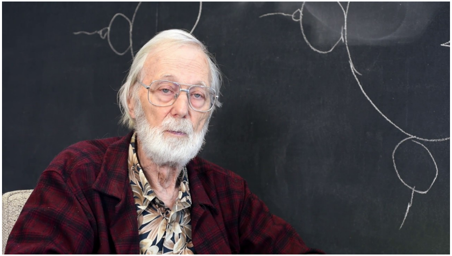

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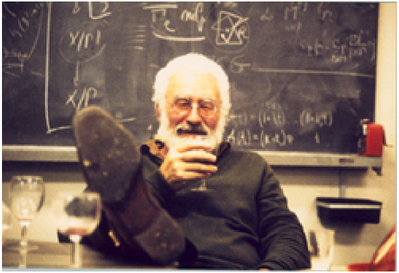

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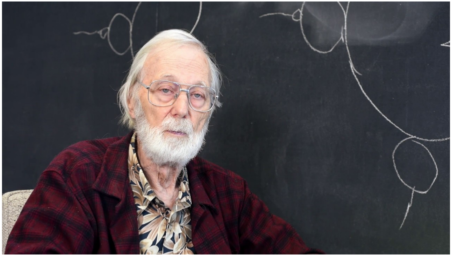

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m	1	2	3	4	5	6
Θ_m	0	0	0	?	0	0

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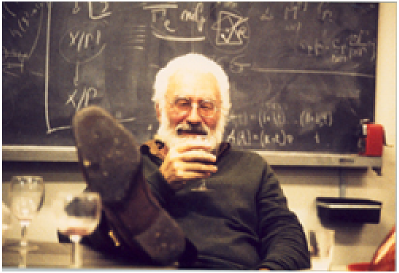

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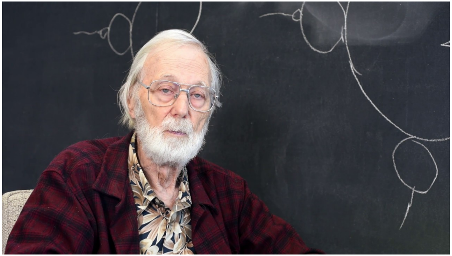

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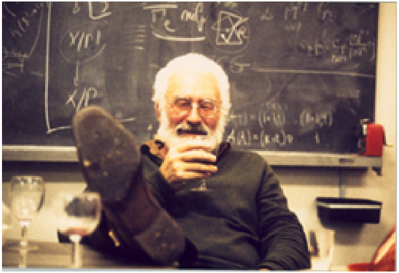

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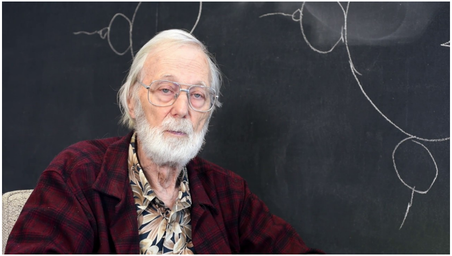

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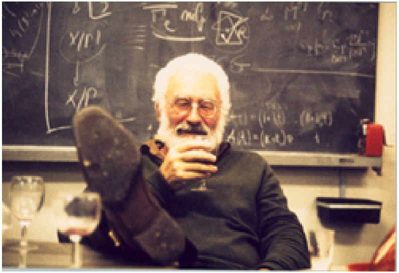

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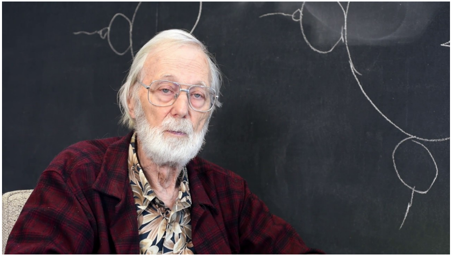

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- For ($m \leq 3$), the cardinality $|\mathcal{S}(M^m)| = 1$ (Rado(1925) and Moise (1952)).
- For closed 4-manifolds, Cheeger (1970) showed that there are at most countably many distinct smooth structures. There are many simply connected closed m-manifolds M^m such that $|\mathcal{S}(M^m)| = \infty$ (for instance, R. Friedman-J. W. Morgan (1988) for $M = \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$).

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- $\mathcal{C}(\mathbb{S}^m) = \Theta_m$ $(m \geq 5)$ (Kirby-Siebenmann, 1977).
- $$\mathcal{C}(\mathbb{T}^m) \cong \oplus_i H^i(\mathbb{T}^m; \pi_i(TOP/O))$$

  and
  $$\mathcal{S}(\mathbb{T}^m) = \oplus_i H^i(\mathbb{T}^m; \pi_i(TOP/O))/GL_m(\mathbb{Z}).$$

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- TOP/O is the fiber of the natural fibration $BO \mapsto BTOP$. Moreover, it is an infinite loop space (Boardman and Vogt, 1973); the Atiyah-Hirzebruch spectral sequence applies to compute $[M, TOP/O]$, which is H^0 of a generalized cohomology theory.

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- $\Theta_m \curvearrowright \mathcal{C}(M)$ by $([\Sigma], [N, f]) \longmapsto [N\#\Sigma, f]$.

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- For example, If M is a closed smooth manifold homotopy equivalent to $\mathbb{RP}^7$, then Θ_7 acts freely on

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- For infinitely many values of $n \geq 9$, I have obtained (jointly with Samik Basu, 2017) that the map $f_{\mathbb{CP}^{4n+1}}^*: \Theta_{8n+2} \to \mathcal{C}(\mathbb{CP}^{4n+1})$ is not injective.

Smooth Structures on $\mathbb{HP}^n$

Theorem (Samik Basu + R., 2018)

(i) $\mathcal{C}(\mathbb{HP}^3) \cong \mathbb{Z}_2$.

(ii) $\mathcal{C}(\mathbb{HP}^4) = \{[\mathbb{HP}^4 \# \Sigma] \mid \Sigma \in \Theta_{16}\}$.

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We have thus observed that for $n \leq 5$ and Σ^{4n} is an exotic sphere, then $\mathbb{HP}^n \# \Sigma^{4n} \ncong \mathbb{HP}^n$.

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We have thus observed that for $n \leq 5$ and Σ^{4n} is an exotic sphere, then $\mathbb{HP}^n \# \Sigma^{4n} \not\cong \mathbb{HP}^n$.

Theorem (Samik Basu + R., 2018)

For infinitely many n, there exists an exotic sphere Σ^{4n} *such that* $\mathbb{HP}^n \# \Sigma^{4n}$ *is diffeomorphic to* $\mathbb{HP}^n$.

Interesting problems

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Does an exotic torus $\mathbb{T}^{2n}\#\Sigma^{2n}$ $(n > 2)$ *carry any complex structures ?*

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Does an exotic torus $\mathbb{T}^{2n}\#\Sigma^{2n}$ $(n > 2)$ *carry symplectic structures?*

This is a generalization of a similar problem, posed by Benson and Gordon (1988) for kähler manifolds and B. Hajduk and A. Tralle (2008) conjectured the following:

Conjecture

There are no symplectic structures on exotic tori.

Interesting problems

- Instead of comparing the diffeomorphism type of M when taking the connected sum $M\#\Sigma$ with an exotic sphere Σ, one might try to compare the group of diffeomorphisms of M and $M\#\Sigma$.
- Using recent progress in manifold theory by Galatius and Randal-Williams, together with computations in stable homotopy theory, I am very much interested to know the behaviour of the cohomology $H^*(BDiff^+(M))$ and homotopy groups $\pi_k(BDiff^+(M))$ of $B^+Diff(M)$ when replacing M with $M\#\Sigma$.
- Recently, Manuel Krannich (2018) have discussed about the cohomology ring $H^*(BDiff^+((\mathbb{S}^n \times \mathbb{S}^n)^{\#g}); \mathbb{Z})$ and showed that $H^1(BDiff^+((\mathbb{S}^n \times \mathbb{S}^n)^{\#g}); \mathbb{Z})$ and $H^1(BDiff^+((\mathbb{S}^n \times \mathbb{S}^n)^{\#g}\#\Sigma); \mathbb{Z})$ cannot be isomorphic for certain exotic spheres Σ.

References

- With Samik Basu : *Inertia Groups and Smooth Structures on Quaternionic Projective Space* (Submitted).
- With Samik Basu : *Inertia Groups of Higher-Dimensional Complex Projective Spaces*, Algebraic & Geometric Topology 18 (2018) 387-408.
- Ramesh Kasilingam, *Topological Rigidity Problems*, Journal of Advanced Studies in Topology, Vol. **7**, Issue 4 (2016), 161-204.
- Ramesh Kasilingam, *A Survey of Smooth and PL-Rigidity Problems on Locally Symmetric Spaces*, Journal of Advanced Studies in Topology , Vol. **7**, Issue 4 (2016), 205-250.

Thank you for your attention.