Some consequences of completeness in Analysis

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Introduction

Following are important concepts in Analysis.

- Compactness
- Connectedness
- Completeness

Compactness

"A great many propositions in Analysis are trivial for finite sets, true and reasonably simple for infinite compact sets; and either false or extremely difficult to prove for noncompact sets."

E. Hewitt, The rôle of compactness in analysis, Amer. Math. Monthly **67** (1960), 499–516. MR0120617

Completeness

Completeness can be used to draw global conclusions from local hypotheses.
Many such propositions are obvious and easy to prove for finite dimensional normed linear spaces, true and possible to prove(proofs can be difficult) for complete normed linear (Banach) spaces and either false or extremely difficult to prove for incomplete normed linear spaces.

Sets of first and second category

Recall that a subset E of a metric space X is said to be *nowhere dense* if the interior of its closure is empty, that is $(\overline{E})^0 = \emptyset$.

A subset A of X is said to be of *first category* if A can be expressed as a countable union of nowhere dense sets. Otherwise it is said to be of *second category*.

Baire Category Theorem

Every complete metric space is of second category.

Equivalent formulation:
In a complete metric space, intersection of any countable family of dense open sets is dense.

Some well known consequences

1. A Banach space can not have a denumerable(countably infinite) Hamel basis.
2. Every absolutely convergent series in a Banach space is convergent.
3. Uniform Boundenness Principle
4. Closed Graph Theorem, Open Mapping Theorem, Bounded Inverse Theorem

Seminorms

A real valued function p on a vector space X is called a *seminorm* on X if

1. $p(x+y) \leq p(x) + p(y)$ for every $x, y \in X$.
2. $p(\alpha x) = |\alpha| p(x)$ for every $x \in X$ and $\alpha \in \mathbb{C}$..

It follows from above that $p(0) = 0$ and $p(x) \geq 0$ for every $x \in X$.

Countably subadditive seminorm

A seminorm p on a normed linear space $(X, \|.\|)$ is said to be *countably subadditive* if

$$x = \sum_n x_n$$

in X, then

$$p(x) \leq \sum_n p(x_n).$$

It can be proved that if p is continuous , then p is countably subadditive.

Zabreiko's Theorem

Every countably subadditive seminorm on a Banach space is continuous.

P. P. Zabreĭko, A theorem for semiadditive functionals, Funkcional. Anal. i Priložen. **3** (1969), no. 1, 86–88. MR0241947

Polynomials

Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying the following: For every complex number z, there exists a natural number n_z such that

$$f^{(n_z)}(z) = 0$$

Does this imply that f must be a polynomial?

Proof

For each natural number n, define

$$E_n := \{z \in \mathbb{C} : f^{(n)}(z) = 0\}$$

Then each E_n is a closed set and $\mathbb{C} = \cup_n E_n$. Hence for some n, the interior of E_n is nonempty. Since $f^{(n)}$ is also an entire function, this implies that $f^{(n)}(z) = 0$ for all z.

Entire functions

A function $f : A \to B$ is said to be *finite to one* if the inverse image $f^{-1}(b)$ is a finite set for every $b \in B$. Thus a one to one function is a special case of a finite to one function. We consider the following question:

"What are finite to one entire functions? "

The answer to this question is very simple: Polynomials (and only polynomials). This is a consequence of a very famous theorem in Complex Analysis, namely the Big Picard Theorem.

A question

In his Presidential Address(Technical) to the Fifty Eighth Annual Conference of the Indian Mathematical Society in 1993, Prof. V. Kannan of University of Hyderabad asked whether it is possible to give an elementary proof of this characterization of polynomials among entire functions:

If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w)$ is a finite set for every $w \in \mathbb{C}$, then f must be a polynomial.

V. Kannan, *Formulas for functions*, The Mathematics Student **62**(1993) 241 - 270.

Continued

It is easy to prove this(actually a stronger version of this) by making use of the Big (Great) Picard Theorem (See Corollary 12.4.4, p. 303 of Conway). This Big Picard Theorem is usually not taught in the first introductory courses in complex analysis. The problem was to find a proof that does not use the Big Picard Theorem.

J. B. Conway, *Functions of One Complex Variable*, Narosa Publishing House, New Delhi, 1973.

Elementary proof

We give such a " relatively elementary" proof. In fact, we prove a somewhat stronger version. We use the following tools in this proof.

1. The open mapping theorem for analytic functions (Theorem 4.7.5 in Conway): The image $f(U)$ of an open set U under a nonconstant analytic function f is also open.
2. The Casorati-Weierstrass Theorem (Theorem 5.1.21 in Conway): Suppose Ω is an open set in $\mathbb{C}$, $a \in \Omega$ and f is an analytic function in $\Omega \setminus \{a\}$ with an essential singularity at a and $r > 0$ is such that $D := \{z \in \mathbb{C} : 0 < |z - a| < r\} \subseteq \Omega$. Then $f(D)$ is dense in $\mathbb{C}$.
3. The Baire category theorem : If X is a complete metric space, then the intersection of a countable family of dense open subsets of X is dense in X.

Continued

All these theorems are also nontrivial, but all these are usually taught in the first introductory courses. The first two theorems are taught in complex analysis and the third is taught in real analysis. On the other hand, the Big Picard Theorem is usually taught in an advanced course in complex analysis. Thus this " relatively elementary" proof can be understood by anyone who has taken one introductory course in Real Analysis and one introductory course in Complex Analysis.

S. H. Kulkarni, A relatively elementary proof of a characterization of polynomials, Indian J. Pure Appl. Math. **38** (2007), no. 4, 229–230. MR2349286

Some properties of entire functions

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and suppose there are constants $M > 0, \quad R > 0$ and a natural number n such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Then f is a polynomial of degree $\leq n$.
2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let g be defined by $g(z) = f(\frac{1}{z})$ for $z \in \mathbb{C} \setminus \{0\}$. Then
   2.1 g has a removable singularity at $z = 0$ (this is sometimes expressed by saying that f has a removable singularity at infinity) if and only if f is a constant.
   2.2 g has a pole of order m at $z = 0$ (this is sometimes expressed by saying that f has a pole of order m at infinity) if and only if f is a polynomial of degree m.

Theorem

We use the following notation: For $a \in \mathbb{C}$ and $r > 0$, the punctured disc with center at a and radius r is denoted by $D'(a, r)$. Thus,

$$D'(a, r) := \{z \in \mathbb{C} : 0 < |z - a| < r\}$$

Theorem

Suppose Ω is an open set in $\mathbb{C}$, $a \in \Omega$ and f is an analytic function in $\Omega \setminus \{a\}$ with an essential singularity at a. Then there exists a dense subset D of $\mathbb{C}$, such that for every $w \in D$ and for every neighbourhood V of a, $f^{-1}(w) \cap V$ is an infinite set.

Proof

Let V be a neighbourhood of a. There exists $n_0 \in \mathbb{N}$ such that $D'(a, \frac{1}{n}) \subseteq V$ for $n \geq n_0$. For such n, we define $D_n := f((D'(a, \frac{1}{n}))$. By the Casorati-Weierstrass Theorem, each D_n is dense in $\mathbb{C}$ and by the open mapping theorem each D_n is open. Let $D := \cap D_n$. Then D is dense in $\mathbb{C}$ by the Baire category theorem. Next let $w \in D$. Then $w \in D_n$ for all $n \geq n_0$, that is, there exists $z_n \in D'(a, \frac{1}{n})$ such that $f(z_n) = w$. It is easy to see that the set $\{z_n : n \geq n_0\}$ is infinite, because given any such z_n, one can find $m > n$ such that $\frac{1}{m} < |z_n - a|$. But then $|z_m - a| < \frac{1}{m} < |z_n - a|$.

Remark

The Big Picard Theorem says that the set D in fact contains all complex numbers with one possible exception. In that sense, the aboveTheorem may be called "Not So Big" or "Moderately Big" Picard Theorem.

Corollary

If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w)$ is a finite set for every w in some nonempty open subset U of $\mathbb{C}$, then f must be a polynomial.

Proof.

Consider g defined by $g(z) = f(\frac{1}{z})$ for $z \in \mathbb{C} \setminus \{0\}$. If f is not a polynomial, then g has an essential singularity at 0. Let D be the dense subset of $\mathbb{C}$ given by the above Theorem . Then $D \cap U$ is nonempty. Let $w \in D \cap U$. Then $g^{-1}(w)$ is an infinite set. But then, $f^{-1}(w) = \{\frac{1}{z} : z \in g^{-1}(w)\}$ is an infinite set. This is a contradiction. □

Remark

It may be noted that by making use of the Big Picard Theorem, it is possible to prove a much stronger result:
If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w_1)$ and $f^{-1}(w_2)$ are finite sets for two distinct complex numbers w_1 and w_2, then f must be a polynomial.
This is in fact Corollary 12.4.4 of Conway. It is not known whether this stronger result can be proved by elementary methods.

Nilpotent operator

Recall that a linear operator T on a vector space V is said to be *Nilpotent* if $T^n = 0$ for some natural number n.

We may consider a local version. For $x \in V$, we shall call T to be *Nilpotent at x* if there exists a natural number n_x such that $T^{n_x}(x) = 0$.

A Question

Suppose a linear operator T on a vector space V is nilpotent at every $x \in V$. Does this imply that T is nilpotent?

It is easy to see that the answer is affirmative if V is finite dimensional.

An Example

But in general, this need not happen.
Consider the space $V = c_{00}$ of all sequences $x = \{x_n\}$ such that $x_n = 0$ for all except finitely many n. Let L be the left shift operator on V defined by

$$L(\{x_1, x_2, \ldots\}) = \{x_2, x_3, \ldots\}.$$

Then L is nilpotent at every $x \in V$, but L is not a nilpotent operator.

Role of Completeness

Next let X be a Banach space and T be a bounded(continuous) linear operator on X such that T is nilpotent at every $x \in X$. Then T is nilpotent.

Proof

For each natural number n, let E_n denote the null space of T^n. Thus

$$E_n := \{x \in X : T^n(x) = 0\}$$

Then each E_n is a closed subspace of X and $X = \cup_n E_n$. Hence for some n, the interior of E_n is nonempty. This implies that $X = E_n$ and hence $T^n = 0$.

Completeness and Invertibility

Answers to many important questions in Functional Analysis depend upon knowing the following:

- whether a certain normed linear space is complete or/and
- whether a certain bounded linear map has a bounded linear inverse.

Completeness and Invertibility

Usually students do not think that these two important ideas in Functional Analysis, namely completeness and invertibility, have anything to do with each other. In this talk, we try to draw the attention to connections between these ideas.

S. H. Kulkarni, Completeness and invertibility, Math. Student **84** (2015), no. 3-4, 141–145. MR3467554

A well known theorem

The following well known theorem is given in many textbooks of Functional Analysis. (See for example,
B. Bollobás, *Linear analysis*, Cambridge Univ. Press, Cambridge, 1990. MR1087297 (92a:46001)

THEOREM 1: Let T be a bounded(continuous) linear map from a Banach space X to a normed linear space Y. Then the following are equivalent:

1. T has a bounded inverse.
2. T is bounded below and the range of T is dense in Y.

A natural question

It is natural to ask what happens if the hypothesis of completeness of X is dropped. Somehow, this question is not discussed in the textbooks. It is obvious that (1) would still imply (2) even without completeness. But the converse is false and it is easy to construct a counterexample. We give such an example.

An interesting observation

Further, it is interesting to note that (2) is equivalent to the following even without the completeness of X.

3. The transpose T' of T has a bounded inverse.

Even more interesting is the fact that the completeness of X is equivalent to the invertibility of every bounded linear map satisfying (2).

Definitions and notations

We recall a few standard notations, definitions and results that are used in the next section. Let X, Y be normed linear spaces. We denote by
$BL(X, Y)$ the set of all bounded linear operators from X to Y.
For an operator $T \in BL(X, Y)$, $N(T)$ denotes the null space of T and $R(T)$ denotes the range of T. Thus
$N(T) = \{x \in X : T(x) = 0\}$ and
$R(T) = \{T(x) : x \in X\}$.

Invertible operators

An operator T is said to be *bounded below* if there exists $\alpha > 0$ such that $\|T(x)\| \geq \alpha\|x\|$ for all $x \in X$.
Also T is said to be *invertible* if there exists $S \in BL(Y, X)$ such that $ST = I_X$, the identity map on X, and $TS = I_Y$.

Dual spaces

The *dual space* X' of X, is the set of all bounded linear functionals on X, that is, $X' = BL(X, \mathbb{K})$, where $\mathbb{K}$ is the underlying field of real or complex numbers.
For a subset $A \subseteq X$, the *annihilator* A^0 is the set of all continuous linear functionals that vanish on A, that is,
$A^0 := \{\phi \in X', \phi(a) = 0 \text{ for all } a \in A\}$.
If A is a subspace of X, then it follows by the Hahn-Banach Theorem, that A is dense in X, if and only if $A^0 = \{0\}$.

Transpose

The *transpose* T' of $T \in BL(X, Y)$ is the operator in $BL(Y', X')$ defined by
$(T'\psi)(x) := \psi(T(x))$ for all $x \in X$ and $\psi \in Y'$.
All the other notations (including the notations for sequence spaces c_{00}, ℓ^1 etc.) are as in the well known books of Bollobás and Limaye.
We make use of a well known result, namely,

$$(R(T))^0 = N(T')$$

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References

This can be found in any book on Functional Analysis, for example

1. B. Bollobás, *Linear analysis*, Cambridge Univ. Press, Cambridge, 1990. MR1087297 (92a:46001)
2. B. V. Limaye, *Functional analysis*, Second edition, New Age, New Delhi, 1996. MR1427262 (97k:46001)

An example

Let $X := (c_{00}, \|.\|_1)$, $Y := \ell^1$ and $T : X \to Y$ be given by $T(x) = x$ for $x \in X$. Clearly, T is bounded below, range of T is dense in Y, but T is not onto and hence not invertible.
More generally, we can consider the inclusion map from a proper dense subspace of a normed linear space.

Remark

Note that in the above example, though T is not invertible, its transpose T' is invertible. In fact, both the dual spaces X' of X and Y' of Y can be identified with ℓ^∞ in the usual way (See the book of Limaye for details.) and with respect to this identification T' becomes the identity operator on ℓ^∞.

Theorem

THEOREM 2: Let T be a bounded(continuous) linear map form a normed linear space X to a normed linear space Y. Consider the following statements:

1. T has a bounded inverse.
2. T is bounded below and the range of T is dense in Y.
3. T' is invertible.

Then (2) and (3) are equivalent and each is implied by (1).
If, in addition, X is a Banach space, then all the three statements are equivalent.

Proof

(1) implies (2): Obvious.
(2) implies (3): Since $R(T)$ is dense in Y, we have $\{0\} = (R(T))^0 = N(T')$. Thus T' is injective.
Further, since T is bounded below, there exists $\alpha > 0$ such that $\|T(x)\| \geq \alpha\|x\|$ for all $x \in X$.

Proof continued

Let $\phi \in X'$ $y \in R(T)$. There exists unique $x \in X$ such that $y = T(x)$.
Define ψ by $\psi(y) := \psi(T(x)) = \phi(x)$.
This defines ψ as a linear functional on $R(T)$. Further, $|\psi(y)| = |\psi(T(x))| = |\phi(x)| \leq \|\phi\| \|x\| \leq \|\phi\| \frac{1}{\alpha} \|T(x)\| = \|\phi\| \frac{1}{\alpha} \|y\|$.

Proof continued

This shows that ψ is bounded on $R(T)$ (which is dense in Y) and hence has a unique bounded extension to Y. We denote this extension also by the same symbol ψ. Thus $\psi \in Y'$ and $\phi = T'(\psi)$. This shows that T' is onto. Thus $T' : Y' \to X'$ is a bijection and is hence invertible by the Closed Graph Theorem as X', Y' are Banach spaces.

Proof continued

(3) implies (2): Let $x \in X$. By the Hahn-Banach Theorem, there exists $\phi \in X'$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$.
Further, since T' is onto, there exists $\psi \in Y'$ such that $\phi = T'(\psi)$.
Now
$\|x\| = \phi(x) = T'(\psi)(x) = \psi(T(x)) \leq \|\psi\|\|T(x)\| \leq \|(T')^{-1}\|\|\phi\|\|T(x)\|$.
This shows that T is bounded below.

Proof continued

Also since T' is injective, we have

$$\{0\} = N(T') = (R(T))^0.$$

This shows that $R(T)$ is dense in Y.
Finally, if X is a Banach space, then (2) implies (1) by Theorem 1 and hence all the three statements are equivalent.

A characterization of completeness

We may further note that completeness of X is , in fact, equivalent to the invertiblity of every bounded linear map satisfying (2). In other words, a normed linear space X is a Banach space if and only if every bounded linear map T from X to any normed linear space Y such that T is bounded below and the range of T is dense in Y, is invertible.
The only if part is already proved above.

A characterization of completeness

To prove the if part, consider $Y = X_c$, the completion of X. Then there is a linear isometry T of X onto a dense subspace Y_0 of Y. (See the book of Limaye for details.)
Obviously, this T is bounded below and $R(T) = Y_0$ is dense in Y. Hence by the hypothesis, T is invertible and, in particular, onto. Thus X is linearly isometric to Y and hence complete.

Spectrum

It is known that the invertibility of an operator is closely related to its spectrum.
Let X be a complex normed linear space and $T \in BL(X, X)$.
Recall that the *spectrum* $\sigma(T)$ of T is the set of all complex numbers λ such that $\lambda I - T$ is not invertible.
Applying the above Proposition to $\lambda I - T$, we obtain the known result that $\sigma(T') \subseteq \sigma(T)$ and the equality holds if X is a Banach space. (See Limaye's book) A natural question is whether the converse holds.

Spectral characterization of completeness?

In other words, can the completeness be also characterized in terms of spectra as follows:
A complex normed linear space X is complete if and only if $\sigma(T') = \sigma(T)$ for all $T \in BL(X, X)$?

Another formulation

Another formulation of the same question is as follows:
Given an incomplete normed linear space X, does there exist $T \in BL(X, X)$ such that T is bounded below, its range is dense in X and T is not invertible ?
Note that the above examples and remarks do not answer this question as the spaces X and Y considered there are different.

Thanks

THANK YOU.