Spectral Bounds for Vanishing of Cohomology of a Simplicial Complex

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Definition (Graph)

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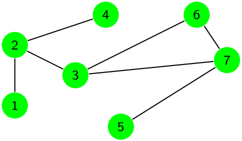
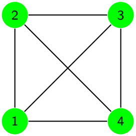
Example:



G



K_4

The degree of a vertex $v \in V(G)$ is $deg(v) := |\{w|(v,w) \in E(G)\}|$.

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Definition (Laplacian)

The (unnormalized) Laplacian of a graph G is the $V(G) \times V(G)$ matrix $L(G)$ given by

$$L(G)(x,y) := \begin{cases} deg(x) & x = y, \\ -1 & (x,y) \in E(G), \\ 0 & otherwise. \end{cases}$$

Definition (Clique complex)

Let G be a graph. The clique complex $X(G)$ of G is the simplicial complex, whose simplices are those subsets of $V(G)$ which spans a complete subgraph.

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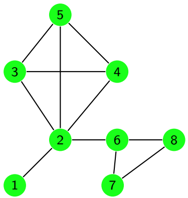

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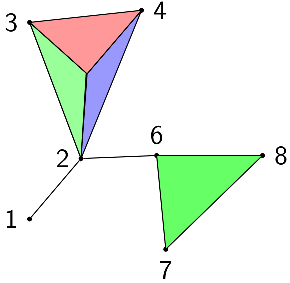

$X(G)$

Theorem (Aharoni, Berger & Meshulam, 2005)

Let G be a graph on n vertices. Let $\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$ denote the eigenvalues of $L(G)$. If $\lambda_2(G) > \frac{kn}{k+1}$, then $\widetilde{H}^k(X(G); \mathbb{R}) = 0$.

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- for a simplicial complex X, whose k-skeleton is a clique complex.

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- for a simplicial complex X, whose k-skeleton is a clique complex.
- for a simplicial complex X, which is a subcomplex of a clique complex Y having the same 1-skeleton as of X.

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$$D_k(X) := \max_{\sigma \in X(k)} |\{w \mid \sigma \cup \{w\} \notin X(k+1) \text{ and any } (k+1)\text{-subset of } \sigma \cup \{w\} \text{ is a } k\text{-simplex}\}|.$$

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Theorem (Yogeshwaran & S.)

Let G be the 1-skeleton of X and let k-skeleton of X is the clique complex of G. If $\lambda_2(G) > \frac{k|V(G)|}{k+1} + (k + \frac{1}{k+1})D_k(X)$, then $\widetilde{H}^k(X;\mathbb{R}) = 0$.

Let X' be a subcomplex of X. For $k \geq 1$, let

$$S_k(X, X') := \max_{\sigma \in X'(k)} |\{\tau \in X(k+1) \setminus X'(k+1) \mid \sigma \subset \tau\}|.$$

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Theorem (Yogeshwaran & S.)

Let X be the clique complex of G and the 1-skeleton of X' is G. If $\lambda_2(G) > \frac{k|V(G)|}{k+1} + \frac{k+2}{k+1} S_k(X, X')$, then $\widetilde{H}^k(X'; \mathbb{R}) = 0$.

Outline of Proof...

For $k \geq 0$, let

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$$\Delta_k(X) := \delta_{k-1}(X)\delta_{k-1}^*(X) + \delta_k^*(X)\delta_k(X) : C^k(X;\mathbb{R}) \to C^k(X;\mathbb{R}).$$

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Theorem (Simplicial Hodge Theorem)

For $k \geq 0$, $Ker\ \Delta_k(X) \cong \widetilde{H}^k(X;\mathbb{R})$.

Outline of Proof...

Let $\mu_k(X)$ be the minimum eigenvalue of $\Delta_k(X)$.

Theorem (Yogeshwaran & S.)

Let X be a simplicial complex on n vertices. For $k \geq 1$,

$$k\mu_k(X) \geq (k+1)\mu_{k-1}(X) - n - (k(k+1)+1)\sum_{j=2}^{k+1} D_k(X,j),$$

where

$$D_k(X,j) := \max_{\sigma \in X(k)} |\{u | \sigma \cup \{u\} \notin X(k+1) \textit{ and } \exists \textit{ exactly } j \textit{ vertices } v_1, \dots, v_j \in \sigma \textit{ such that } (\sigma \setminus \{v_i\}) \cup \{u\} \in X(k)\ \forall\ 1 \leq i \leq j\}|.$$

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If k-skeleton of X is clique complex, then $D_k(X,j) = 0\ \forall\ j \leq k$ and $D_k(X,k+1) = D_k(X)$.

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If $\mathbb{I}$ denotes the $|V(G)| \times |V(G)|$ matrix with all entries 1, then $\mathbb{I} + L(G)$ represents $\Delta_0(X)$ with respect to the standard basis. In particular the minimal eigenvalue of $\Delta_0(X)$ (*i.e.*, $\mu_0(X)$) is $\lambda_2(G)$.

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Theorem (Yogeshwaran & S.)

Let X be a simplicial complex and X' be a subcomplex of X'. For $k \geq 1$,

$$\mu_k(X') \geq \mu_k(X) - (k+2)S_k(X, X').$$

Application...

The neighborhood of a vertex v, $N(v) := \{w \mid (v, w) \in E(G)\}$.

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*The **neighborhood complex** $\mathcal{N}(G)$ of a graph G is the simplicial complex:*

- *simplices are all those subsets of $V(G)$ which have a common neighbor.*

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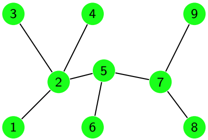

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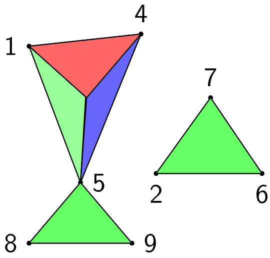

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The Erdös-Rényi random graph $G(n, p)$ with edge-probability p is the graph:

- $V(G(n, p)) = \{1, \ldots, n\}$,
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Let $k \geq 1$. If $p = \left(\frac{(k+2)\log n + c_n}{n}\right)^{\frac{1}{k+2}}$ with $c_n \to \infty$, then with high probability $\widetilde{H}^i(\mathcal{N}(G(n,p);\mathbb{R})) = 0$ for $i \leq k$.

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References

[1] R. Aharoni, E. Berger and R. Meshulam, *Eigenvalues and homology of flag complexes and vector representations of graphs*. Geom. Funct. Anal. 15, no. 3, 555–566, 2005.

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[3] S. Shukla and D. Yogeshwaran, *Spectral bounds for vanishing of cohomology and the neighborhood complex of a random graph*, (submitted).

Thank you