z-classes of reductive groups

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January 4, 2019.

Basic definitions

Let G be a group. A (non-empty) set X admitting an action of G is called a G-set.

Two G-sets X and Y are called G-isomorphic if there is a bijection $\phi : X \to Y$ such that

$$g\phi(x) = \phi(gx)$$

for all $g \in G$ and for all $x \in X$.

If $X = G \cdot x$ and $Y = G \cdot y$ are two transitive G-sets then they are G-isomorphic if and only if the stabilisers G_x and G_y are conjugate in G.

In particular, if $x, y \in G$ then the conjugacy classes of x and y in G are G-isomorphic if and only if the centralisers of x and y in G are conjugate.

z-equivalence

Let G be a group and let $x, y \in G$.

We say that x and y are z-equivalent if their centralisers in G, $Z(x)$ and $Z(y)$, are conjugate subgroups of G.

This defines an equivalence relation on G and the corresponding equivalence classes, the z-classes, are "isotypic components" of G under the conjugation action.

If G is abelian then it has a single z-class.

A dihedral group has three z-classes consisting respectively of central elements, non-central rotations and reflections, including the (countably) infinite dihedral group as well as the uncountable analogue, the group $O_2(\mathbb{R})$!

More examples?

The number of z-classes of $\mathfrak{S}_n$ (and $\mathfrak{A}_n$) are computed by Sushil Bhunia, Dilpreet Kaur and Anupam Kumar Singh.

These values for some small n are as follows:

1	2	3	4	5	6
1	1	3	5	6	10

The number for $\mathfrak{S}_n$ is

$$p(n) - p(n-2) + p(n-3) + p(n-4) - p(n-5)$$

where $p(m)$ denotes the number of partitions of $m \in \mathbb{N}$.

A finite group will have only finitely many z-classes but are there infinite groups with only finitely many z-classes, other than the infinite dihedral groups?

$GL_n(\mathbb{C})$ has only finitely many z-classes!

> **Jordan decomposition**: *If $g \in GL_n(\mathbb{C})$ then there are unique elements $g_s, g_u \in GL_n(\mathbb{C})$ such that g_s is semisimple, g_u is unipotent and*
>
> $$g_s g_u = g = g_u g_s.$$

Then, $Z(g) = Z(g_s) \cap Z(g_u)$.

The group $Z(g_s)$ is conjugate in $GL_n(\mathbb{C})$ to $\prod_{i=1}^{r} GL_{n_i}(\mathbb{C})$ with $n_1 + \cdots + n_r = n$ and $Z(g)$ is equal to the centraliser of g_u in the group $Z(g_s)$.

Finally, there are only finitely many conjugacy classes of unipotent elements in a general linear group. □

What about $GL_n(\mathbb{Q})$?

Let us first see the case $n = 2$.

If $K/\mathbb{Q}$ is a quadratic extention and $K = \mathbb{Q}(\alpha_K)$ then $K^\times$ admits an embedding in $GL_2(\mathbb{Q})$ as the centraliser of $\alpha_K \in GL_2(\mathbb{Q})$.

Since there are infinitely many non-isomorphic quadratic extensions of $\mathbb{Q}$ there are infinitely many non-conjugate element-centralisers in $GL_2(\mathbb{Q})$.

The case of a general n is no different. The punchline is that the arithmetic of the base field k determines the (in)finiteness of the number of z-classes of $GL_n(k)$.

We aim to prove this for all reductive groups.

Fields of type (F)

A field of type (F) is a perfect field k such that for any natural number n there are only finitely many field extensions of k of degree n (in a fixed algebraic closure of k).

Examples of such fields include

- a finite field,
- an algebraically closed field,
- $\mathbb{R}$,
- a p-adic field and
- $\mathbb{C}((t))$.

Non-examples of such fields include $\mathbb{Q}$, a number field and $\mathbb{F}_q((t))$.

Reductive group

Let k be a field and let G be a linear algebraic group defined over k.

We say that G is reductive if its unipotent radical is trivial.

The groups O_2 and GL_n seen above are examples of reductive groups.

Our main result is about reductive groups defined over fields of type (F).

Main result

Theorem (SMG - Anupam Kumar Singh)

Let G be a reductive group defined over a field k of type (F). Then the group $G(k)$ has only finitely many z-classes.

This result is complete in the sense that if k is a perfect field that is not a field of type (F) then there is some GL_n over k with infinitely many z-classes.

The proof of the above result uses Galois cohomology and the analogous result over an algebraic closure of k.

Thank you!