Taut Foliations of 3-manifolds

Tejas Kalelkar
Indian Institute of Science Education and Research, Pune

4th January, 2019

Definition

A *codimension one foliation* $\mathcal{F}$ of a 3-manifold M is a union of disjoint connected surfaces L_i, called the *leaves of* $\mathcal{F}$, in M such that:

1. $\cup_i L_i = M$, and
2. there exists an atlas $\mathcal{A}$ on M with respect to which $\mathcal{F}$ satisfies the following local product structure:
   - for every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in $\mathcal{A}$ about p such that $U \approx \mathbb{R}^3$ and the restriction of $\mathcal{F}$ to U is the union of planes given by $z =$ constant.

Definition

A *codimension one foliation* $\mathcal{F}$ of a 3-manifold M is a union of disjoint connected surfaces L_i, called the *leaves of* $\mathcal{F}$, in M such that:

1. $\cup_i L_i = M$, and
2. there exists an atlas $\mathcal{A}$ on M with respect to which $\mathcal{F}$ satisfies the following local product structure:
    - for every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in $\mathcal{A}$ about p such that $U \approx \mathbb{R}^3$ and the restriction of $\mathcal{F}$ to U is the union of planes given by $z =$ constant.

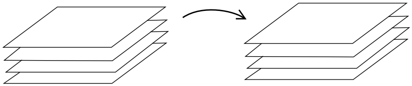


Figure: Local patches of a foliation.

Foliation of a strip

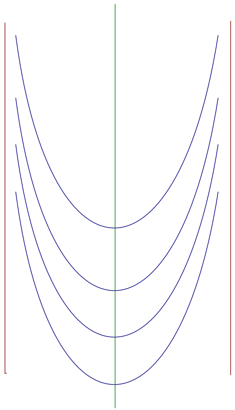


Figure: Translates of a curve which asymptote to the lines $x = \pm 1$

Foliation of a solid cylinder

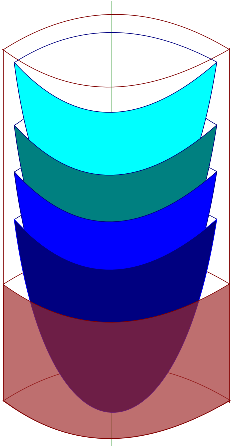


Figure: Rotating the curves about the Y-axis gives planar leaves, along with one cylinder leaf.

Reeb foliation of a solid torus

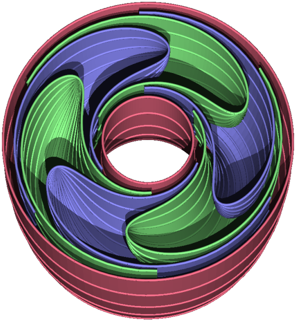


Figure: Taking quotient-space of integer-translations in Y-direction

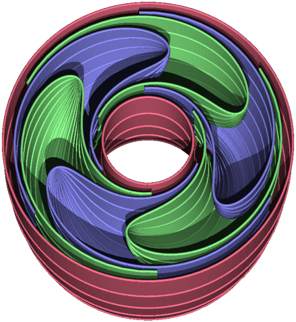


Figure: Reeb foliation of a solid torus

Theorem (Lickorish, Novikov - Zeischang)

Every closed 3-manifold has a codimension one foliation.

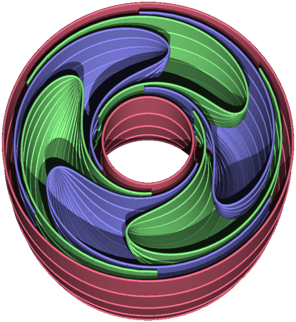


Figure: Reeb foliation of a solid torus

Theorem (Lickorish, Novikov - Zeischang)

Every closed 3-manifold has a codimension one foliation.

Remark: The foliation obtained from this general construction always has Reeb components.

Definition

A *taut foliation* $\mathcal{F}$ of M is a codimension one foliation such that there exists an embedded closed curve in M that intersects each leaf of $\mathcal{F}$ transversely.

Definition

A *taut foliation* $\mathcal{F}$ of M is a codimension one foliation such that there exists an embedded closed curve in M that intersects each leaf of $\mathcal{F}$ transversely.

Remark

Taut foliations are Reebless, i.e, they do not have Reeb components and so do not come from the general construction of foliations for 3-manifolds.

Definition

A *taut foliation* $\mathcal{F}$ of M is a codimension one foliation such that there exists an embedded closed curve in M that intersects each leaf of $\mathcal{F}$ transversely.

Remark

Taut foliations are Reebless, i.e, they do not have Reeb components and so do not come from the general construction of foliations for 3-manifolds.

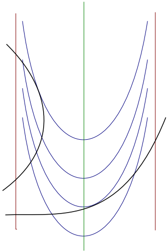


Figure: A proper arc cannot be transverse to leaves of a Reeb foliation.

Definition

A *taut foliation* $\mathcal{F}$ of M is a codimension one foliation such that there exists an embedded closed curve in M that intersects each leaf of $\mathcal{F}$ transversely.

Remark

Taut foliations are Reebless, i.e, they do not have Reeb components and so do not come from the general construction of foliations for 3-manifolds.

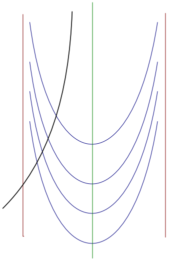


Figure: An arc transverse to leaves of a Reeb foliation.

Definition

A *taut foliation* $\mathcal{F}$ of M is a codimension one foliation such that there exists an embedded closed curve in M that intersects each leaf of $\mathcal{F}$ transversely.

Remark

Taut foliations are Reebless, i.e, they do not have Reeb components and so do not come from the general construction of foliations for 3-manifolds.

Theorem (Novikov)

A foliation of an atoroidal 3-manifold is taut if and only if it has no Reeb components.

Question

What are the topological/geometric consequences of having a taut foliation?

Question

What are the topological/geometric consequences of having a taut foliation?

Theorem (Palmeira, Rosenberg, Haefliger)

If M is a closed, orientable 3-manifold that has a taut foliation with no sphere leaves then M is covered by $\mathbb{R}^3$, M is irreducible and has infinite fundamental group.

Question

What are the topological/geometric consequences of having a taut foliation?

Theorem (Palmeira, Rosenberg, Haefliger)

If M is a closed, orientable 3-manifold that has a taut foliation with no sphere leaves then M is covered by $\mathbb{R}^3$, M is irreducible and has infinite fundamental group.

Theorem (Sullivan)

Let $\mathcal{F}$ be a co-orientable $\mathcal{C}^2$ foliation of M. The following are equivalent:

1. *$\mathcal{F}$ is taut.*
2. *M admits a volume preserving flow transverse to $\mathcal{F}$, for some volume form.*
3. *There is a closed 2-form θ on M which is positive on $T\mathcal{F}$.*
4. *There is a Riemannian metric on M for which leaves of $\mathcal{F}$ are minimal surfaces.*

Question

When does a 3-manifold have a taut foliation?

Question

When does a 3-manifold have a taut foliation?

Theorem (Thurston, Gabai)

Let M be a compact connected irreducible orientable 3-manifold whose boundary is a (possibly empty) union of tori. A properly embedded homologically essential surface Σ is a leaf of a taut foliation of M if and only if it minimizes $-\chi(\Sigma)$ amongst all proper embedded surfaces with no spherical components in its homology class.

Question

When does a 3-manifold have a taut foliation?

Theorem (Thurston, Gabai)

Let M be a compact connected irreducible orientable 3-manifold whose boundary is a (possibly empty) union of tori. A properly embedded homologically essential surface Σ is a leaf of a taut foliation of M if and only if it minimizes $-\chi(\Sigma)$ amongst all proper embedded surfaces with no spherical components in its homology class.

Theorem (Calliat-Gilbert, Matignon; Eisenbud, Hirsch, Neumann, Jankins, Naimi, Roberts etc)

For Seifert fibered rational homology spheres, existence of $\mathcal{C}^2$-taut foliation can be determined in terms of the Seifert invariants.

Question

When does a 3-manifold have a taut foliation?

Theorem (Thurston, Gabai)

Let M be a compact connected irreducible orientable 3-manifold whose boundary is a (possibly empty) union of tori. A properly embedded homologically essential surface Σ is a leaf of a taut foliation of M if and only if it minimizes $-\chi(\Sigma)$ amongst all proper embedded surfaces with no spherical components in its homology class.

Theorem (Calliat-Gilbert, Matignon; Eisenbud, Hirsch, Neumann, Jankins, Naimi, Roberts etc)

For Seifert fibered rational homology spheres, existence of $\mathcal{C}^2$-taut foliation can be determined in terms of the Seifert invariants.

Question

(Open) When do hyperbolic rational homology spheres have taut foliations?

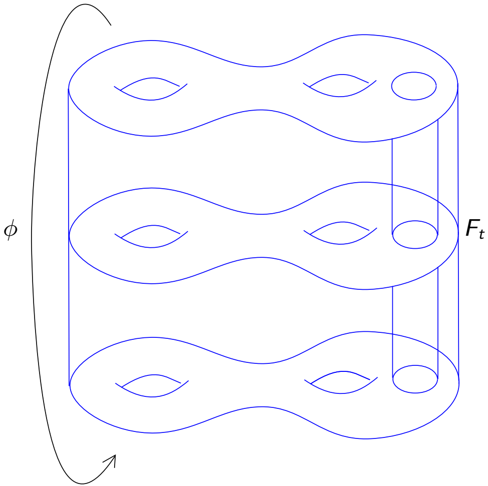


Figure: Surface bundle over a circle, $M_\phi = F \times I/(x,1) \sim (\phi(x),0)$ where ϕ is a homeomorphism of F that fixes each boundary component.

Taut foliation of mapping torus of a surface

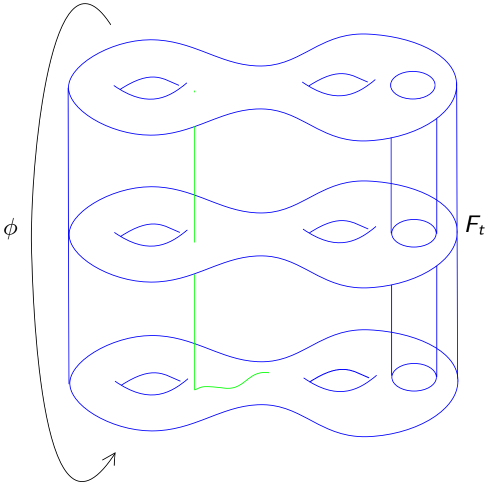


Figure: The foliation of M_ϕ by fibers is a taut foliation.

Taut foliation of mapping torus of a surface

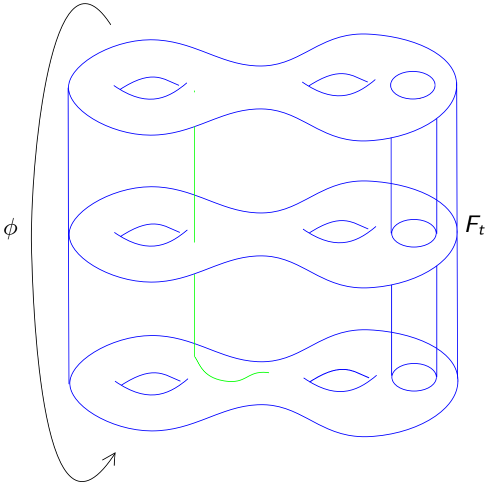


Figure: The foliation of M_ϕ by fibers is a taut foliation.

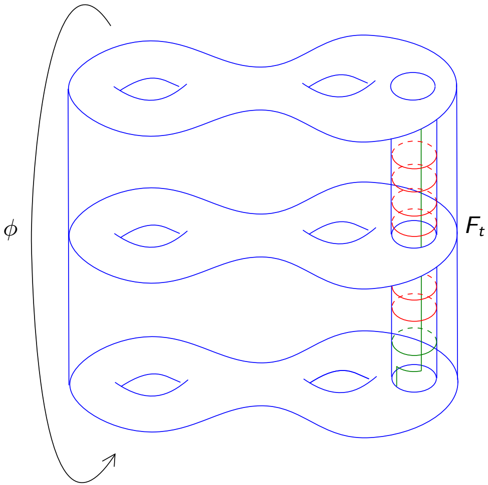


Figure: The foliation of M_ϕ by fibers gives a foliation of the boundary torii by curves of slope 0.

Taut foliation of mapping torus of a surface

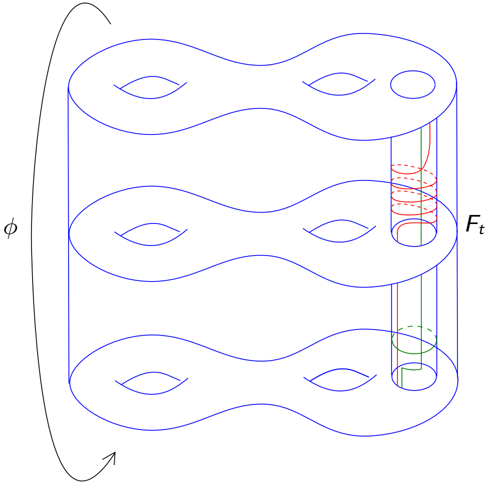


Figure: A foliation of the boundary torii by parallel curves of slope $\frac{1}{5}$.

Theorem (K, Roberts)

Given an orientable, fibered compact 3-manifold, a fibration with fiber surface of positive genus can be perturbed to yield transversely oriented taut foliations realizing a neighborhood of rational boundary multislopes about the boundary multislope of the fibration.

Theorem (K, Roberts)

Given an orientable, fibered compact 3-manifold, a fibration with fiber surface of positive genus can be perturbed to yield transversely oriented taut foliations realizing a neighborhood of rational boundary multislopes about the boundary multislope of the fibration.

Corollary

For a surface-bundle M_ϕ with fibers having k components, there is an open neigborhood $\mathcal{U}$ of $0 \in \mathbb{Q}^k$ such that for each point $(m^1, ..., m^k) \in \mathcal{U}$, the closed manifold obtained by a Dehn filling M_ϕ along the multicurve $(m^1, ..., m^k)$ also has a transversely oriented taut foliation.

Theorem (K, Roberts)

Given an orientable, fibered compact 3-manifold, a fibration with fiber surface of positive genus can be perturbed to yield transversely oriented taut foliations realizing a neighborhood of rational boundary multislopes about the boundary multislope of the fibration.

Corollary

For a surface-bundle M_ϕ *with fibers having k components, there is an open neigborhood* $\mathcal{U}$ *of* $0 \in \mathbb{Q}^k$ *such that for each point* $(m^1, ..., m^k) \in \mathcal{U}$*, the closed manifold obtained by a Dehn filling* M_ϕ *along the multicurve* $(m^1, ..., m^k)$ *also has a transversely oriented taut foliation.*

Theorem (Alexander)

Existence of an open-book decomposition: Any closed orientable 3-manifold can be realized by Dehn filling a surface bundle over a circle.

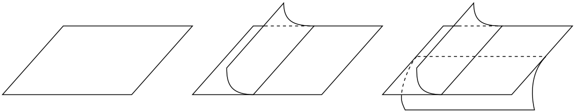


Figure: Local model of a branched surface

Definition

A *branched surface* is a 2-complex B in a 3-manifold M, locally modeled on the spaces shown above.

Lamination carried by a branched surface

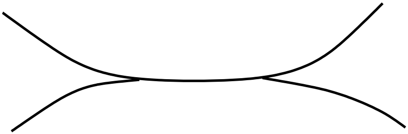


Figure: One-dimensional branch surface B, called a *train-track*.

Lamination carried by a branched surface

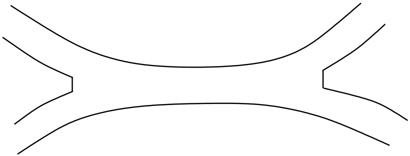


Figure: Neighbourhood $N(B)$

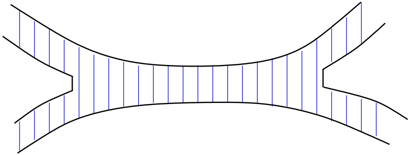


Figure: Fibered neighbourhood $N(B)$

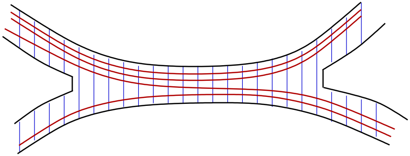


Figure: Lamination λ carried by B

Definition

A lamination λ carried by B is a closed disjoint union of surfaces in $N(B)$, transverse to the I-fibration.

Definition

A branched surface B in a closed 3-manifold M is called a *laminar* branched surface if it satisfies the following conditions:

1. $\partial_h N(B)$ is incompressible in $M \setminus int(N(B))$, no component of $\partial_h N(B)$ is a sphere and $M \setminus B$ is irreducible.
2. There is no monogon in $M \setminus int(N(B))$; i.e., no disk $D \subset M \setminus int(N(B))$ with $\partial D = D \cap N(B) = \alpha \cup \beta$, where $\alpha \subset \partial_v N(B)$ is in an interval fiber of $\partial_v N(B)$ and $\beta \subset \partial_h N(B)$
3. There is no Reeb component; i.e., B does not carry a torus that bounds a solid torus in M.
4. B has no trivial bubbles.
5. B has no sink disk or half sink disk.

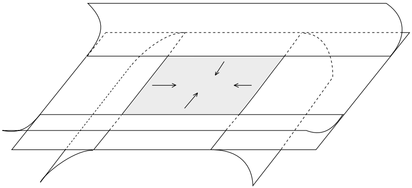


Figure: A sinkdisk

Definition

Let L be the branching locus of B and let X denote the union of double points of L. A *sink disk* is a disk branch sector D of B for which the branch direction of each component of $(L \setminus X) \cap \overline{D}$ points into D. A *half sink disk* is a sink disk which has nonempty intersection with ∂M.

Theorem (Tao Li)

Let M be an irreducible and orientable 3-manifold whose boundary is a union of incompressible tori. Suppose B is a laminar branched surface and $\partial M \setminus \partial B$ is a union of bigons. Then, for any multislope $(s_1, ..., s_k) \in (\mathbb{Q} \cup \{\infty\})^k$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $\hat{M}(s_1, ..., s_k)$, then B fully carries a lamination $\lambda_{(s_1,...,s_k)}$ whose boundary consists of the multislope $(s_1, ..., s_k)$ and $\lambda_{(s_1,...,s_k)}$ can be extended to an essential lamination in $\hat{M}(s_1, ..., s_k)$.

Outline of Proof

There exists a branched surface B in M_ϕ such that:

1. B is laminar.
2. B does not carry any compact surface (other than F).
3. There exists a neighbourhood $\mathcal{U}$ of $0 \in \mathbb{Q}^k$ such that for any $(m^1, ..., m^k) \in \mathcal{U}$ there are closed curves carried by the train track ∂B in the boundary torii, with slopes $(m^1, ..., m^k)$.
4. Furthermore, $M \setminus N(B)$ is a union of product regions $S \times I$, for some components S of $\partial_h N(B)$.

Question

What is the precise interval around slope 0*, in terms of the pseudo-anosov monodromy, which is realised by taut foliations?*

Question

What is the precise interval around slope 0, in terms of the pseudo-anosov monodromy, which is realised by taut foliations?

Theorem (Roberts, 2000)

Suppose M is a surface-bundle over a circle with fiber F, pseudo-anosov monodromy ϕ and a single boundary component. Fix the canonical coordinate system on ∂M determined by the given fibering. Let γ denote a closed orbit of the suspension flow of ϕ restricted to ∂F and let $\lambda = \partial F$. Then, one of the following is true:

1. $|\gamma \cap \lambda| = 1$*, and M contains taut foliations realizing all boundary slopes in $(-\infty, \infty)$; in this case $\widehat{M}(r)$ contains a taut foliation for all rational $r \in \mathbb{Q}$*
2. γ *as positive slope, and M contains taut foliations realizing all boundary slopes in $(-\infty, 1)$; in this case, $\widehat{M}(r)$ contains a taut foliation for all rational $r \in (-\infty, 1)$*
3. γ *as negative slope, and M contains taut foliations realizing all boundary slopes in $(-1, \infty)$; in this case, $\widehat{M}(r)$ contains a taut foliation for all rational $r \in (-1, \infty)$*

Question

Find the open interval explicitly in terms of the pseudo-anosov map, such that every rational point in it is realised by a taut foliation.

Question

Find the open interval explicitly in terms of the pseudo-anosov map, such that every rational point in it is realised by a taut foliation.

Remark

Note, the naive generalization of taking product of open intervals for boundaries does not work, as can be seen by the Baldwin-Etnyre examples.

Theorem (Gabai)

Let k be a non-trivial knot in S^3. Let S be a minimal genus Seifert surface for k in S^3. There exists a taut foliation $\mathcal{F}$ of $M = S^3 \setminus int(N(k))$ such that S is a leaf of $\mathcal{F}$. In particular, there is a foliation whose restriction to the boundary is a collection of circles of slope 0.

Theorem (Gabai)

Let k be a non-trivial knot in S^3. Let S be a minimal genus Seifert surface for k in S^3. There exists a taut foliation $\mathcal{F}$ of $M = S^3 \setminus int(N(k))$ such that S is a leaf of $\mathcal{F}$. In particular, there is a foliation whose restriction to the boundary is a collection of circles of slope 0.

Theorem (Li - Roberts, 2014)

Let k be a non-trivial knot in S^3. Then there is an interval $(-a, b)$ with $a, b > 0$ such that for any rational slope $s \in (-a, b)$, $M = S^3 \setminus int(N(k))$ has a taut foliation whose restriction to the boundary torus ∂M is a collection of circles of slope s. Moreover, by attaching disks along the boundary circles, the foliation can be extended to a taut foliation in $\widehat{M}(s)$, where $\widehat{M}(s)$ is the manifold obtained by performing Dehn surgery to k with surgery slope s.

Current project: Generalise result to link complements

Conjecture

Let L be a non-trivial link in S^3. Then there is an open set $\mathcal{U}$ containing 0 such that for any rational multi-slope $(s_1, \dots, s_k) \in \mathcal{U}$, $M = S^3 \setminus int(N(L))$ has a taut foliation whose restriction to the boundary torii ∂M is a collection of circles of slope s_i. Moreover, by attaching disks along the boundary circles, the foliation can be extended to a taut foliation in $\widehat{M}(s_i)$, where $\widehat{M}(s_i)$ is the manifolds obtained by performing Dehn surgery to L with surgery slopes s_i.

Thankyou

References:

1. Taut foliations in surface bundles with multiple punctures; Tejas Kalelkar and Rachel Roberts, Pacific J. Math. 273 (2015), no. 2, 257275. 57M50 (57R30)

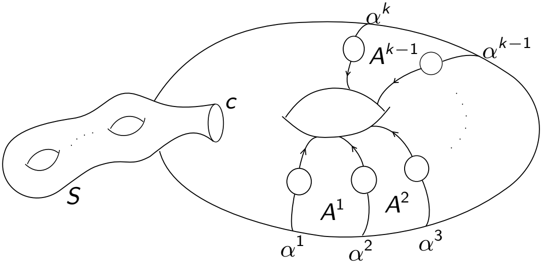


Figure: A Parallel tuple $\{\alpha^i\}$ on the surface F

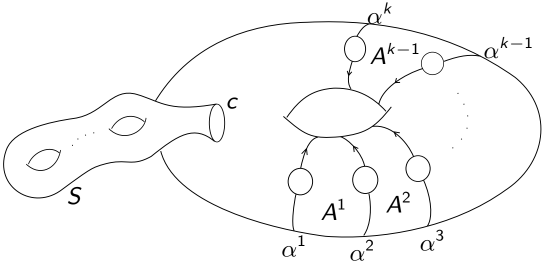


Figure: A Parallel tuple $\{\alpha^i\}$ on the surface F

Definition

Let $(\alpha^1, ..., \alpha^k)$ be a tuple of simple arcs properly embedded in F with $\partial\alpha^j \subset T^j$. Such a tuple will be called *parallel* if $F \setminus \{\alpha^1, ..., \alpha^k\}$ has k components, $k-1$ of which are annuli $\{A^j\}$ with $\partial A^j \supset \{\alpha^j, \alpha^{j+1}\}$ and one component S of genus $g-1$ with $\partial S \supset \{\alpha^1, \alpha^k\}$. Furthermore all α^j are oriented in parallel, i.e., orientation of ∂A^j agrees with $\{\alpha^j, -\alpha^{j+1}\}$ and orientation of ∂S agrees with $\{\alpha^k, -\alpha^1\}$. Note that, in particular, each α^j is non-separating.

Basic definitions: Good Pair of Parallel Tuples

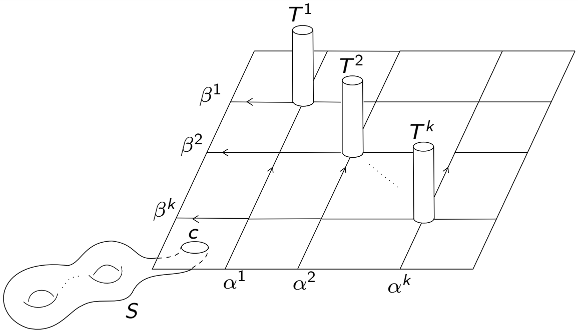


Figure: Neighbourhood of F with a good pair $((\alpha^j), (\beta^j))$

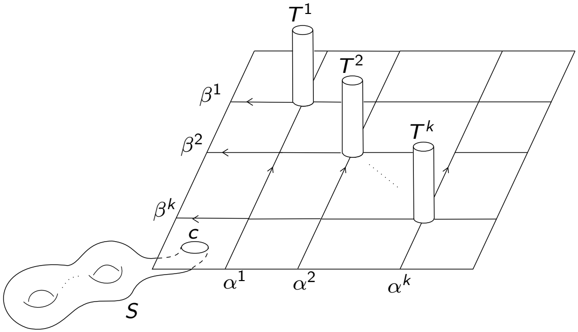


Figure: Neighbourhood of F with a good pair $((\alpha^j),(\beta^j))$

Definition

A pair of tuples $(\alpha^i)_{i=1\dots k}$ and $(\beta^j)_{j=1\dots k}$ will be called *good* if both are parallel tuples and α^i intersects β^j exactly once when $i \neq j$ while α^i is disjoint from β^j when $i = j$.

A sequence of parallel tuples $\sigma = ((\alpha_0^1, \alpha_0^2, ..., \alpha_0^k), (\alpha_1^1, \alpha_1^2, ..., \alpha_1^k), ..., (\alpha_n^1, \alpha_n^2, ..., \alpha_n^k))$ will be called good if for each $0 \leq i < n$, $0 \leq j \leq k$, the pair $((\alpha_i^j), (\alpha_{i+1}^j))$ is good.

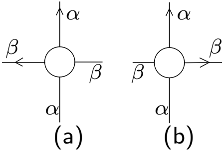


Figure: A pair of arcs in position (a) is called negatively oriented, while a pair in position (b) is called positively oriented

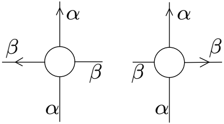
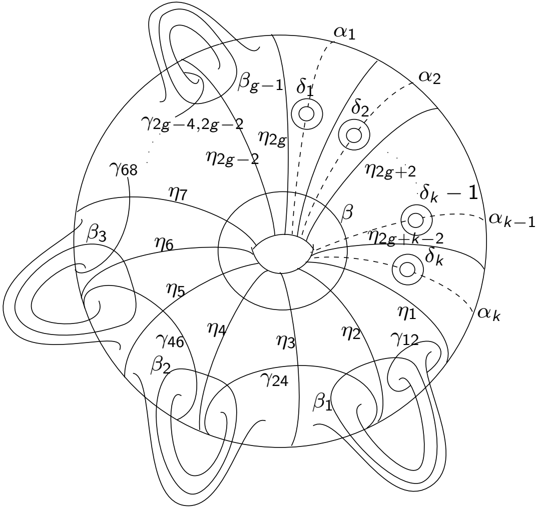


Figure: A pair of arcs in position (a) is called negatively oriented, while a pair in position (b) is called positively oriented

Definition

We say a good pair $((\alpha^j), (\beta^j))$ is *positively oriented* if for each $j \in \{1, ..., k\}$ a neighbourhood of the j-th boundary component in F is as shown in (b) above. Analogously define *negatively oriented*.
We say a good sequence $\sigma = ((\alpha^j_0), (\alpha^j_1), ..., (\alpha^j_n))$ is positively oriented if each pair $((\alpha^j_i), (\alpha^j_{i+1}))$ is positively oriented. Similarly define negatively oriented sequence.



Theorem (Gervais)

Dehn twists along the curves shown in the figure above generate the Mapping Class Group of F relative ∂F.

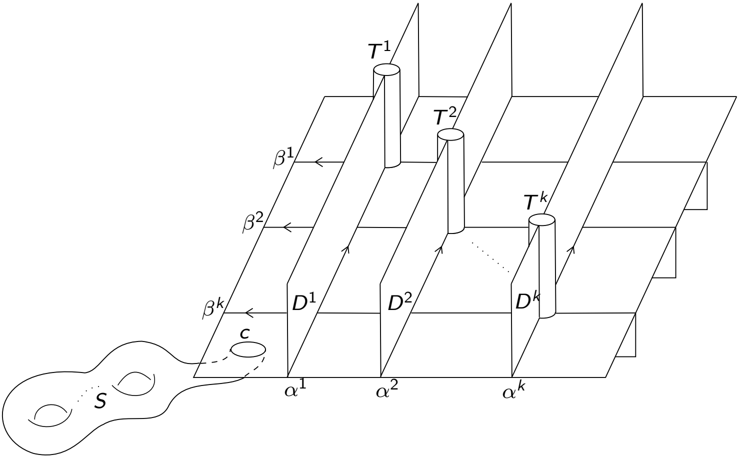


Figure: Neighbourhood of F with a good positively oriented pair $((\alpha^j),(\beta^j))$ in the oriented spine

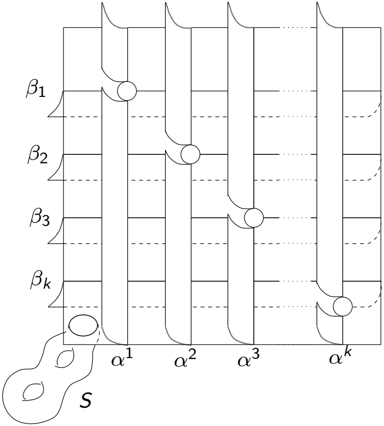


Figure: Neighbourhood of F in the associated branched surface B

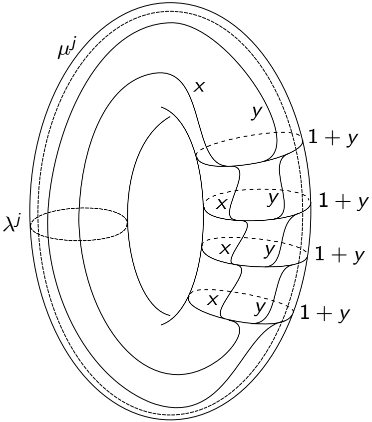


Figure: The weighted boundary train track when $n = 4$

The boundary train track $\tau^j = B \cap T^j$ carries all slopes realizable by $\frac{x-y}{n(1+y)}$ for some $x, y > 0$. Therefore, τ^j carries all slopes in $(-\frac{1}{n}, \infty)$.