Fixed Point and Best Proximity Point Theorems: Some Open Problems Related to Nonexpansive Mapppings

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- A closed bounded convex subset C of X has fixed point property (FPP) if every nonexpansive mapping on C has a fixed point in C. If C is weakly compact convex, then the same property is called weak fixed point property (WFPP).

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- Also, X has FPP (WFPP) if every closed bounded (weakly compact) convex subset of X has FPP (WFPP).

Definition 1 (Brodskii, Milman)

[4, 7] A convex subset C of X has normal structure if for every closed bounded convex subset K of C with $diam(K) > 0$ there exists a point $x \in K$ such that

$r(x, K) = \sup\{\|x - y\| : y \in K\} < diam(K)$.

Theorem 2 (Kirk)

[10] Every weakly compact convex subset C of X with normal structure has WFPP.

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[4] M. S. Brodskii and D. P. Mil'man, On the center of a convex set, Doklady Akad. Nauk SSSR (N.S.) 59 (1948), 837-840.

[7] K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[10] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004-1006.

The classical spaces $\ell_1, c_0, c, \ell_\infty$ do not have FPP.

In 1981, Maurey [1, 12] proved that c_0 (and c) has WFPP. Note that c_0 does not have normal structure.

[1] A. G. Aksoy and M. A. Khamsi, Nonstandard methods in fixed point theory, Universitext, Springer-Verlag, New York, 1990.

[12] B. Maurey, Points fixes des contractions de certains faiblement compacts de L1, seminar on Functional Analysis, 1980-1981, Ecole Polytech., Palaiseau, 1981, pp. Exp. No. VIII, 19.

Question: Does every Banach space has WFPP?

In 1981, Alspach [2] showed that $L_1[0,1]$ does not have WFPP.

$$K := \{f \in L_1[0,1] : \int f = 1, 0 \le f \le 2\}$$

$$(Tf)t = \begin{cases} \min\{2f(t), 2\}, & 0 \le t \le 1/2 \\ \max\{2f(2t-1) - 2, 0\}, & 1/2 < t \le 1. \end{cases}$$

[2] D. E. Alspach, A fixed point free nonexpansive map, Proc. Amer. Math. Soc. 82 (1981), no. 3, 423-424.

Question:

Does every reflexive Banach space has FPP?- remains open.

In 2009, Benavides [3] showed that every reflexive space can be renormed to satisfy FPP.

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[3] T. Dominguez Benavides, A renorming of some nonseparable Banach spaces with the fixed point property, J. Math. Anal. Appl. 350 (2009), no. 2, 525-530.

[5] Linares, Carlos A. Hernandez; Japon, Maria A. A, Renorming in some Banach spaces with applications to fixed point theory. J. Funct. Anal. 258 (2010), no. 10, 3452-3468.

[12] B. Maurey, Points fixes des contractions de certains faiblement compacts de L1, seminar on Functional Analysis, 1980-1981, Ecole Polytech., Palaiseau, 1981, pp. Exp. No. VIII, 19.

Question: Does every super-reflexive space has FPP?- remains open.

In 1981, Maurey [1] proved that every super reflexive space has FPP for isometries.

[1] A. G. Aksoy and M. A. Khamsi, Nonstandard methods in fixed point theory, Universitext, Springer-Verlag, New York, 1990.

Question: Does every renorming of ℓ^2 has FPP?- Also remains open.

In 2013, Jiménez-Melado and Llorens-Fuster [6] proved the following:

Theorem 3

Every equivalent renorming of ℓ^2 of the form $|x| = \max\{\|x\|_2, p(x)\}$, where p is a seminorm on ℓ^2, has the WFPP if p satisfies the following condition:
There exists $k \in \mathbb{N}$ such that for all $x_1, \cdots, x_k$ in ℓ^2 with pairwise disjoint supports we have

$$p(z) \leq \max\{p(z - x_1), \cdots, p(z - x_k)\}, \text{ for all } z \in \ell^2. \quad (1)$$

[6] A. Jimenez-Melado and E. Llorens-Fuster, A class of renormings of ℓ^2 with the fixed point property, J. Nonlinear Convex Anal. 14 (2013), no. 2, 351-362.

Theorem 4

[8] Let $(X, \|\cdot\|)$ *be a Banach space having normal structure. Let* $\{e_n\}$ *be a Schauder basis of* X*. Then every equivalent renorming of* X *of the form,* $|x|_\beta = \max\{\|x\|, \beta q(x)\}$*, where* q *is a seminorm on* X*, has the WFPP, for all* $\beta > 0$*, if* q *satisfies the following condition:*
There exists $k \in \mathbb{N}$ *such that for all* $x_1, \ldots, x_k$ *in* X *with pairwise disjoint supports with respect to* $\{e_n\}$*, we have*

$$q(z) \leq \max\{q(z - x_1), \ldots, q(z - x_k)\}, \text{ for all } z \in X.$$

[8] Gopal Dutta and P. Veeramani, Some renormings of Banch spaces with the weak fixed point property for nonexpasive mappings, Acta Sci. Math. (Szeged) (Accepted).

Theorem 5

[8] Every Banach space having normal structure and Schauder basis has an equivalent renorming that lacks of asymptotic normal structure but has the WFPP.

[8] Let $X = \ell^p$, $1 < p < \infty$. Define, $|x|_\beta = \max\{\|x\|_p, \beta\|x\|_\infty\}$, $\beta \geq 1$. Then $|\cdot|_\beta$ is an equivalent remorming of $\|\cdot\|_p$. We proved that $(\ell^p, |\cdot|_\beta)$ has normal structure if and only if $\beta < 2^{1/p}$. But it has the WFPP for all $\beta \geq 1$.

[8] Gopal Dutta and P. Veeramani, Some renormings of Banch spaces with the weak fixed point property for nonexpasive mappings, Acta Sci. Math. (Szeged) (Accepted).

Proximal Normal Structure *A nonempty convex pair* (A, B) *in a Banach space* X *is said to have proximal normal structure if for every closed, bounded, convex proximal pair* $(K_1, K_2) \subset (A, B)$ *for which* $dist(K_1, K_2) = dist(A, B)$ *and* $\delta(K_1, K_2) > dist(K_1, K_2)$*, there exists* $(x, y) \in K_1 \times K_2$ *such that*

$$r_x(K_2) < \delta(K_1, K_2), \;\; r_y(K_1) < \delta(K_1, K_2).$$

[2] Eldred,A.A, W.A.Kirk, and P.Veeramani . Proximal normal structure and relatively nonexpansive mappings. Studia Math. 171(3), 2005, 283-293.

The notion of proximal normal structure introduced by Eldred et. al. to prove:

Theorem 6 (Eldred, et. al.)

If (A, B) is a nonempty weakly compact convex pair in a Banach space X with proximal normal structure and $T : A \cup B \to A \cup B$ is relatively cyclic nonexpansive ($||Tx - Ty|| \leq ||x - y||$, for all $x \in A, y \in B$ and $T(A) \subset B$, $T(B) \subset A$), then T has best proximity point in $A \cup B$, i.e. there exist $x \in A, y \in B$ such that $||x - Tx|| = ||y - Ty|| = dist(A, B)$.

[2]Eldred,A.A, W.A.Kirk, and P.Veeramani . Proximal normal structure and relatively nonexpansive mappings. Studia Math. 171(3), 2005, 283-293.

Theorem 7 (Eldred, et. al.)

If (A, B) is a nonempty weakly compact convex pair in a strictly convex Banach space X with proximal normal structure and $T : A \cup B \to A \cup B$ is relatively non cyclic nonexpansive $(||Tx - Ty|| \leq ||x - y||$, for all $x \in A, y \in B)$ with $T(A) \subset A$, $T(B) \subset B$, then T has best proximity point in $A \cup B$ i.e. there exist $x \in A, y \in B$ such that $Tx = x$, $Ty = y$ and $||x - y|| = dist(A, B)$.

[2] Eldred,A.A, W.A.Kirk, and P.Veeramani . Proximal normal structure and relatively nonexpansive mappings. Studia Math. 171(3), 2005, 283-293.

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