Fixed Point and Best Proximity Point Theorems: Some Open Problems Related to Nonexpansive Mappings

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Let $C$ be a non-empty subset of a normed linear space $X$. A mapping $T : C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$. 

A closed bounded convex subset $C$ of $X$ has fixed point property (FPP) if every nonexpansive mapping on $C$ has a fixed point in $C$. If $C$ is weakly compact convex, then the same property is called weak fixed point property (WFPP). Also, $X$ has FPP (WFPP) if every closed bounded (weakly compact) convex subset of $X$ has FPP (WFPP).
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\[ \| Tx - Ty \| \leq \| x - y \| \]
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A closed bounded convex subset $C$ of $X$ has fixed point property (FPP) if every nonexpansive mapping on $C$ has a fixed point in $C$. If $C$ is weakly compact convex, then the same property is called weak fixed point property (WFPP).

Also, $X$ has FPP (WFPP) if every closed bounded (weakly compact) convex subset of $X$ has FPP (WFPP).
Definition 1 (Brodskii, Milman)

[4, 7] A convex subset $C$ of $X$ has normal structure if for every closed bounded convex subset $K$ of $C$ with $\text{diam}(K) > 0$ there exists a point $x \in K$ such that

\[ r(x, K) = \sup\{\|x - y\| : y \in K\} < \text{diam}(K). \]

Theorem 2 (Kirk)

[10] Every weakly compact convex subset $C$ of $X$ with normal structure has WFPP.

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The classical spaces $\ell_1, c_0, c, \ell_\infty$ do not have FPP. In 1981, Maurey [1, 12] proved that $c_0$ (and $c$) has WFPP. Note that $c_0$ does not have normal structure.

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Question: Does every Banach space has WFPP?
In 1981, Alspach [2] showed that $L_1[0, 1]$ does not have WFPP.

$$K := \{ f \in L_1[0, 1]: \int f = 1, 0 \leq f \leq 2 \}$$

$$(Tf)t = \begin{cases} 
\min\{2f(t), 2\}, & 0 \leq t \leq 1/2 \\
\max\{2f(2t - 1) - 2, 0\}, & 1/2 < t \leq 1.
\end{cases}$$

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Does every reflexive Banach space has FPP? remains open.
In 2009, Benavides [3] showed that every reflexive space can be renormed to satisfy FPP.


Question: Does every super-reflexive space has FPP?- remains open.

In 1981, Maurey [1] proved that every super reflexive space has FPP for isometries.

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Question: Does every renorming of $\ell^2$ has FPP? Also remains open.
In 2013, Jiménez-Melado and Llorens-Fuster [6] proved the following:

**Theorem 3**

*Every equivalent renorming of* $\ell^2$ *of the form*

$$|x| = \max\{|x|_2, p(x)\}, \text{ where } p \text{ is a seminorm on } \ell^2,$$

*has the WFPP if* $p$ *satisfies the following condition:*

*There exists* $k \in \mathbb{N}$ *such that for all* $x_1, \cdots, x_k$ *in* $\ell^2$ *with pairwise disjoint supports we have*

$$p(z) \leq \max\{p(z - x_1), \cdots, p(z - x_k)\}, \text{ for all } z \in \ell^2. \quad (1)$$

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Theorem 4

[8] Let \((X, \| \cdot \|)\) be a Banach space having normal structure. Let \(\{e_n\}\) be a Schauder basis of \(X\). Then every equivalent renorming of \(X\) of the form, 
\[ |x|_\beta = \max\{\|x\|, \beta q(x)\}, \]
where \(q\) is a seminorm on \(X\), has the WFPP, for all \(\beta > 0\), if \(q\) satisfies the following condition:

There exists \(k \in \mathbb{N}\) such that for all \(x_1, \ldots, x_k\) in \(X\) with pairwise disjoint supports with respect to \(\{e_n\}\), we have

\[ q(z) \leq \max\{q(z - x_1), \ldots, q(z - x_k)\}, \text{ for all } z \in X. \]

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Gopal Dutta and P. Veeramani, Some renormings of Banach spaces with the weak fixed point property for nonexpansive mappings, Acta Sci. Math. (Szeged) (Accepted).
Theorem 5

[8] Every Banach space having normal structure and Schauder basis has an equivalent renorming that lacks of asymptotic normal structure but has the WFPP.

[8] Let \( X = \ell^p \), \( 1 < p < \infty \). Define, \( |x|_\beta = \max\{\|x\|_p, \beta \|x\|_\infty\} \), \( \beta \geq 1 \). Then \( |\cdot|_\beta \) is an equivalent remorming of \( \|\cdot\|_p \). We proved that \((\ell^p, |\cdot|_\beta)\) has normal structure if and only if \( \beta < 2^{1/p} \). But it has the WFPP for all \( \beta \geq 1 \).

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Gopal Dutta and P. Veeramani, Some renormings of Banch spaces with the weak fixed point property for nonexpasive mappings, Acta Sci. Math. (Szeged) (Accepted).
**Proximal Normal Structure** A nonempty convex pair \((A, B)\) in a Banach space \(X\) is said to have proximal normal structure if for every closed, bounded, convex proximal pair \((K_1, K_2) \subset (A, B)\) for which \(\text{dist}(K_1, K_2) = \text{dist}(A, B)\) and \(\delta(K_1, K_2) > \text{dist}(K_1, K_2)\), there exists \((x, y) \in K_1 \times K_2\) such that

\[
r_x(K_2) < \delta(K_1, K_2), \quad r_y(K_1) < \delta(K_1, K_2).
\]

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The notion of proximal normal structure introduced by Eldred et. al. to prove:

**Theorem 6 (Eldred, et. al.)**

If \((A, B)\) is a nonempty weakly compact convex pair in a Banach space \(X\) with proximal normal structure and \(T : A \cup B \to A \cup B\) is relatively cyclic nonexpansive (\(\|Tx - Ty\| \leq \|x - y\|\), for all \(x \in A, y \in B\) and \(T(A) \subset B, T(B) \subset A\)), then \(T\) has best proximity point in \(A \cup B\), i.e. there exist \(x \in A, y \in B\) such that

\[\|x - Tx\| = \|y - Ty\| = \text{dist}(A, B).\]

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Theorem 7 (Eldred, et. al.)

If \((A, B)\) is a nonempty weakly compact convex pair in a strictly convex Banach space \(X\) with proximal normal structure and \(T : A \cup B \to A \cup B\) is relatively non cyclic nonexpansive \((||Tx - Ty|| \leq ||x - y||\), for all \(x \in A, y \in B\)) with \(T(A) \subset A, T(B) \subset B\), then \(T\) has best proximity point in \(A \cup B\) i.e. there exist \(x \in A, y \in B\) such that \(Tx = x, Ty = y\) and \(||x - y|| = dist(A, B)\).


Gopal Dutta and P. Veeramani, Some renormings of Banach spaces with the weak fixed point property for nonexpasive mappings, Acta Sci. Math. (Szeged) (Accepted).


THANK YOU