On the Space of Norms[2]

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Abstract

On the direct sum $U \oplus V$ of two normed linear spaces, we discuss the problem of finding inequivalent norms which agree with the given norms on the subspaces U, V.

Introduction

Throughout this talk we shall be only dealing with vector spaces over the reals. Let me begin with an elementary and standard result.

Theorem

On a finite dimensional vector space any two norms are equivalent.

Recall that two norms, $\| - \|_a$ and $\| - \|_b$ on a vector space V are said to be equivalent iff there exist positive constants, c_1, c_2 such that

$$c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_a$$

for all $x \in V$. It is easily checked that this is the same as saying that the two norms induce the same topology on the vector space.

Question I:

Let $(U, \|-\|_u), (V, \|-\|_v)$ be any two normed linear spaces. Let $\|-\|_a, \|-\|_b$ be any two norms on $U \oplus V$ such that

$$\|(u,0)\|_a = \|u\|_u = \|(u,0)\|_b, u \in U;$$

and

$$\|(0,v)\|_a = \|v\|_v = \|(0,v)\|_b, v \in V.$$

Are the two norms $\|-\|_a, \|-\|_b$ equivalent?
Before proceeding further with this question let us recall another relevant result which is also quite a standard one.

Theorem

Let U and V be any two normed linear spaces as above. Then for any $1 \leq p \leq \infty$ the following formula gives a norm $\| - \|_p$ on $U \oplus V$ inducing the product topology and agreeing with the given norms on the subspaces $U \times 0$ and $0 \times V$:

$$\|(u, v)\|_p = \begin{cases} (\|u\|_u^p + \|v\|_v\|^p)^{1/p} & \text{if} \quad 1 \leq p < \infty \\ \mathrm{Max}\,\{\|u\|_u, \|v\|_v\} & \text{if} \quad p = \infty. \end{cases} \tag{1}$$

(See, for example,
B. V. Limaye, *Functional Analysis,* Wiley Eastern Ltd, New Delhi,1989.

Product Norm

Definition

Any norm on $U \oplus V$ which induces the product topology and coincides with the given norms on U and V is called a product norm.

Product Norm

Theorem

Let $\|-\|$ be a norm which coincides with the given norms on U, V.
(i) *Then the identity map* $Id : (U \oplus V, \|-\|_1) \to (U \oplus V, \|-\|)$ *is continuous.*
(ii) *Further $\|-\|$ is a product norm iff the projection maps*

$$\pi_U : (U \oplus V, \|-\|) \to (U, \|-\|_u) \;\&\; \pi_V : (U \oplus V, \|-\|) \to (V, \|-\|_v)$$

are continuous.
(iii) In case (ii) holds, both U, V will be closed subspaces of $(U \oplus V, \|-\|)$.

Because of (iii) above, there is a need to modify Q. I

Product Norm: Question II

Let $(U, \| - \|_u), (V, \| - \|_v)$ be any two normed linear spaces. Let $\| - \|_a, \| - \|_b$ be any two norms on $U \oplus V$ such that

$$\|(u, 0)\|_a = \|u\|_u = \|(u, 0)\|_b, u \in U;$$

and

$$\|(0, v)\|_a = \|v\|_v = \|(0, v)\|_b, v \in V.$$

Assume that $U \times 0$ and $0 \times V$ are closed subspaces of $U \oplus V$ under both the norms. Are the two norms $\| - \|_a, \| - \|_b$ equivalent?

As we have seen, the answer is YES if U and V are finite dimensional. Here is a pleasant surprise.

Theorem

Let $(U, \| - \|_u), (V, \| - \|_v)$ be any normed linear spaces. Let $\| - \|_a, \| - \|_b$ be any two norms on $U \oplus V$ such that

$$\|(u, 0)\|_a = \|u\|_u = \|(u, 0)\|_b, u \in U$$

and

$$\|(0, v)\|_a = \|v\|_v = \|(0, v)\|_b, v \in V.$$

Suppose further that U is a closed subspace and V is finite dimensional. Then the two norms on $U \oplus V$ are equivalent.

A construction

Let U, V be copies of the $\ell_1(\mathbb{N})$-space, viz., the space of infinite sequences (a_n) with $\sum_n |a_n| < \infty$ with the norm

$$\|(a_n)\|_1 = \sum_n |a_n|.$$

Let $\{u_n\}, \{v_n\}$ be the standard basis for U, V respectively. A general element of $U \oplus V$ looks like

$$(a, b) = \left(\sum_n a_n u_n, \sum_n b_n v_n \right),$$

where $\sum_n |a_n| < \infty$ and $\sum_n |b_n| < \infty$.

A construction

For each sequence $s : \mathbb{N} \to (-1, 1)$ and $(a, b) \in U \oplus V$, define

$$\|(a, b)\|_s := \sum_n \sqrt{a_n^2 + b_n^2 + 2s_n a_n b_n}.$$

Since the RHS is $\leq \sum_n (|a_n| + |b_n|) < \infty$, this makes sense.

A construction

Notice that for each fixed n, restricted to the 2-dimensional subspace A_n spanned by $\{u_n, v_n\}$ the above function is the norm induced by the inner product corresponding to the positive definite symmetric matrix

$$\begin{pmatrix} 1 & s_n \\ s_n & 1 \end{pmatrix}.$$

and therefore we have the triangle inequality

$$\|(a_n u_n, b_n v_n) + (c_n u_n, d_n u_n)\| \leq \|(a_n u_n, b_n v_n)\| + \|(c_n u_n, d_n v_n)\|$$

for all n.

A construction

It follows that $\| - \|_s$ defines a norm on $U \oplus V$ which when restricted to U and V is nothing but the respective ℓ_1-norm. The first problem now is to determine the equivalence classes of these s-norms.

A construction

Lemma

For $\alpha, \beta \in \mathbb{R}$ we have

$$\sqrt{\frac{1-t}{2}}(|\alpha| + |\beta|) \leq \sqrt{\alpha^2 + \beta^2 + 2s\alpha\beta}, \text{ for all } 0 \leq s \leq t < 1. \quad (2)$$

A construction

Theorem

The norm $\| - \|_s$ is equivalent to ℓ_1-norm $\| - \|_1$ iff 1 is not a limit point of the set $\{s_n\}$.

Proof: Suppose 1 is a limit point of $\{s_n\}$. Choose a subsequence $\{n_i\}$ such that $s_{n_i} \longrightarrow 1$. Consider the sequence $\{(u_{n_i}, -v_{n_i})\}$. We have

$$\|(u_{n_i}, -v_{n_i})\| = \sqrt{1 + 1 - 2s_{n_i}} \to 0.$$

Moreover,

$$\|(u_{n_i}, -v_{n_i})\|_1 = 2.$$

This just means that $\| - \|_s$ is not equivalent to the ℓ_1-norm.

Now suppose 1 is not a limit point. Then there exists $0 < t < 1$ such that $s_n < t$ for all n. From the above lemma, it follows that

$$\sqrt{\frac{1-t}{2}}\|z\|_1 \leq \|z\|_s \leq \|z\|_1, \quad z \in U \oplus V.$$

This completes the proof of the theorem. ♠

A Construction

Theorem

Let $s, t : \mathbb{N} \to (-1, 1)$ be any two sequences. For each $n \in \mathbb{N}$, put

$$f_n(\lambda) = \frac{1 + t_n\lambda}{1 + s_n\lambda}, \quad \lambda \in (-1, 1).$$

The norms $\| - \|_s, \| - \|_t$ on $U \oplus V$ are equivalent iff the family $\{f_n\}_n$ is uniformly bounded above and uniformly bounded away from 0, i.e., there exists $c_1, c_2 > 0$ such that

$$c_1^2 \leq f_n(\lambda) \leq c_2^2, \quad \lambda \in (-1, 1), \quad n \in \mathbb{N}.$$

A Construction

Proof: Put $z = (a, b), a = (a_n), b = (b_n),$ Then $\| - \|_s$ is equivalent to $\| - \|_t$ iff there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} & c_1\left(\sum_n \sqrt{a_n^2 + b_n^2 + 2s_n a_n b_n}\right) \\ \leq & \sum_n \sqrt{a_n^2 + b_n^2 + t_n a_n b_n} \\ \leq & c_2\left(\sum_n \sqrt{a_n^2 + b_n^2 + 2s_n a_n b_n}\right) \end{aligned}$$

whenever $\sum_n |a_n| < \infty$ and $\sum_n |b_n| < \infty$.

A Construction

Specialized to the case, when $a_n = 0 = b_n$ except for $n = m$, this implies

$$\begin{array}{rl} & c_1\sqrt{a_m^2 + b_m^2 + 2s_m a_m b_m} \\ \leq & \sqrt{a_m^2 + b_m^2 + 2t_m a_m b_m} \\ \leq & c_2\sqrt{a_m^2 + b_m^2 + 2s_m a_m b_m} \end{array}$$

for all $m \in \mathbb{N}$ and for all $a_m, b_m \in \mathbb{R}$. The converse is obvious.

Putting $a_m = r\cos\theta, b_m = r\sin\theta$ this last condition is equivalent to

$$c_1^2(1 + s_m \sin 2\theta) \leq 1 + t_m \sin 2\theta \leq c_2^2(1 + s_m \sin 2\theta)$$

for all m. Putting $\sin 2\theta = \lambda$, this in turn is equivalent to

$$c_1^2 \leq \frac{1 + t_m\lambda}{1 + s_m\lambda} \leq c_2^2$$

for all $\lambda \in (-1, 1)$ and for all $m \in \mathbb{N}$. ♠

A Construction

We can now prove:

Theorem

There are uncountably many inequivalent norms on $U \oplus V$ which when restricted to $U \times 0, 0 \times V$ give the ℓ_1-norm.

The Space of Norms

Let V be any vector space over $\mathbb{R}$ of dimension bigger than 1. We fix the topology on V which is coinduced by the standard Euclidean topology on each finite dimensional subspace of V, viz., a subset $F \subset V$ is closed in V iff $F \cap U$ is closed in U for every finite dimensional subspace U of V, where U is given the Euclidean topology.

The Space of Norms

Let $C(V)$ denote the space of all continuous real valued functions on V with the topology of uniform convergence on compact sets. Let $\mathcal{N}(V)$ denote the subspace of $C(V)$ consisting of all norm functions on V. Let $\overline{\mathcal{N}}(V)$ denote the space of all equivalence classes of norms on V with the quotient topology. We make a beginning in understanding this space. We shall use the notation $\mathcal{N}_n$ to denote $\mathcal{N}(V)$ when $V = \mathbb{R}^n$.

The space of norms

Theorem

$\mathcal{N}(V)$ and $\overline{\mathcal{N}}(V)$ are both contractible.

Lemma

(**Jensen**) For $1 \leq p < q \leq \infty$ we have $\|a\|_q \leq \|a\|_p$ for all $a \in \ell_1(\mathbb{N})$.

Definition

Let η_1, η_2 be any two norms on V. We say η_2 is *finer than* η_1 if there exists $c_1 > 0$ such that $c_1\eta_1 \leq \eta_2$. We use the symbol $\eta_1 \preceq \eta_2$.

Thus two equivalent norms are finer than each other.

Theorem

On $\ell_1(\mathbb{N})$ we have $\|-\|_q \prec \|-\|_p$ whenever $p < q$, i.e., $\|-\|_p$ are all inequivalent norms on $\ell_1(\mathbb{N}), 1 \leq p \leq \infty$.

Theorem

The space $\mathcal{N}(V)$ is a 'conical' subspace of $C(V)$, i.e,
(a) *if $\eta \in \mathcal{N}(V)$, and $\lambda > 0$ then $\lambda\eta \in \mathcal{N}(V)$*
(b) *if $\eta_i \in \overline{\mathcal{N}},\ 0 < r_i < 1, i = 1, 2 \ldots, n$ such that $\sum r_i = 1$ then the 'strict convex combination' $\eta = \sum_i r_i\eta_i \in \overline{\mathcal{N}}$. Moreover,*
(i) *a sequence in V is convergent wrt to η iff it is convergent wrt each η_i.*
(ii) *η is finer than all η_i.* (iii) *If one of the η_i is finer than all other η_j then η is equivalent to η_i.*
(iv) *Any two (strict) convex combinations of a finitely many norms are equivalent norms.*

Proof: This is routine checking. ♠

Lemma

On $\mathbb{R}^2$, the set of norms $\{\| - \|_p \ : \ 1 \leq p \leq \infty\}$ is independent in the vector space $C(\mathbb{R}^2)$.

As an easy corollary we obtain:

Theorem

The space $\mathcal{N}(V)$ is of uncountable dimension.

Theorem

If $\eta_1 \prec \eta_2 \in \mathcal{N}(V)$ then in $\overline{\mathcal{N}}(V)$, $[\eta_1]$ is in the closure of $[\eta_2]$. In particular, $\overline{\mathcal{N}}(V)$ is not a T_1 space when V is infinite dimensional.

Proof: The line segment $((1-t)\eta_1 + t\eta_2$ will intersect every neighbourhood of η_1 in $\mathcal{N}(V)$. Therefore every neighbourhood of $[\eta_1]$ in $\overline{\mathcal{N}}(V)$ contains points of the form $[(1-t)\eta_1 + t\eta_2]$ for $t < 1$. But these are all equal to $[\eta_2]$, by theorem 10. ♠

Some Open Questions

We have just begun the study of the space of norms on an infinite dimensional space. There are several questions that come to ones mind. Some of them may be easy and some other formidable. Here are a couple of them.

Some Open Questions

(A) Note that any norm function $\eta : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ is completely determined by its values on the unit sphere $\eta' : \mathbb{S}^1 \to \mathbb{R}^{\geq 0}$. Obtain a characterization of η' in terms of its Fourier coefficients.
(B) Is the space of s-norms on $U \oplus V$ dense in $\overline{\mathcal{N}}(U \oplus V)$?
(C) Is the space of p-norms dense in $\overline{\mathcal{N}}(V)$?

THANK YOU FOR YOUR ATTENTION