Isometric Dilation and Von Neumann Inequality

Sibaprasad Barik

Mathematics Department, IIT Bombay

Diamond Jubilee Symposium
Mathematics Department, IIT Bombay

January 4, 2019

(Joint work with B.K. Das, K. Haria and J. Sarkar)

Outline of the talk.

1. Isometric dilations.
2. Isometric dilations for finite rank tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.
3. Isometric dilations for arbitrary tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.
4. Von Neumann inequality for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

Introduction.

- Normal operators (T is normal if $TT^* = T^*T$) are well-understood by spectral theory.
- In many situation dilation enables us to reduce problems of tuple of commuting non-normal operators to simpler problems involving tuple of commuting normal operators.
- One such problem is characterization of tuples of commuting contractions which satisfies the von Neumann inequality.
- If a tuple of commuting contractions has an isometric dilation then it satisfies the von Neumann inequality.

Isometic dilations.

Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a contraction.

- $V : \mathcal{K} \to \mathcal{K}\,(\mathcal{K} \supseteq \mathcal{H})$ is a *dilation* of T if

$$T^k = P_{\mathcal{H}} V^k|_{\mathcal{H}} \quad \forall\, k \in \mathbb{Z}_+.$$

  i.e,

$$V^k = \begin{bmatrix} T^k & * \\ * & * \end{bmatrix} \quad \forall\ k \in \mathbb{Z}_+$$

  with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K} \ominus \mathcal{H})$.

- If V is an isometry (respectively unitary) and satisfies the above relation then V is an isometric (respectively unitary) dilation of T.

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$$\mathcal{T}^n(\mathcal{H}) = \{(T_1, \ldots, T_n) : T_i \in \mathcal{B}(\mathcal{H}), \|T_i\| \leq 1, T_i T_j = T_j T_i, 1 \leq i, j \leq n\}.$$

- $V \in \mathcal{T}^n(\mathcal{K})$ $(\mathcal{K} \supseteq \mathcal{H})$ is a *dilation* of $T \in \mathcal{T}^n(\mathcal{H})$ if

$$T^{\boldsymbol{k}} = P_{\mathcal{H}} V^{\boldsymbol{k}}|_{\mathcal{H}} \quad \forall\, \boldsymbol{k} \in \mathbb{Z}_+^n.$$

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Existence of isometric dilation.

Theorem (Sz.-Nagy and Foias)

A contraction on a Hilbert space always possesses an isometric dilation.

Theorem (Ando)

Every pair of commuting contractions on a Hilbert space has an isometric dilation.

- Counter example of Parrot shows that, for $n \geqslant 3$, an n-tuple of commuting contractions does not possess an isometric dilation in general.

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Hardy space.

- $H^2(\mathbb{D})$: Space of all $f = \sum_{k \in \mathbb{Z}_+} a_k z^k$ ($a_k \in \mathbb{C}$) on $\mathbb{D}$ such that

$$\sum_{k \in \mathbb{Z}_+} |a_k|^2 < \infty.$$

- $H^2_{\mathcal{E}}(\mathbb{D}^n)$: Space of all $f = \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$ ($a_{\boldsymbol{k}} \in \mathcal{E}$) on $\mathbb{D}^n$ such that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \|a_{\boldsymbol{k}}\|^2 < \infty.$$

- For all $i = 1, \ldots, n$, $M_{z_i} : H^2_{\mathcal{E}}(\mathbb{D}^n) \to H^2_{\mathcal{E}}(\mathbb{D}^n)$, called *Hardy shifts on* $H^2_{\mathcal{E}}(\mathbb{D}^n)$, are defined by

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}) \quad (\boldsymbol{w} \in \mathbb{D}^n).$$

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Szegö positivity.

- Szegö kernel on $\mathbb{D}^n$, denoted as $\mathbb{S}_n$ is

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} (1 - z_i \bar{w}_i)^{-1} \quad (\text{for all } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

- $T \in \mathcal{T}^n(\mathcal{H})$ satisfies *Szegö positivity* if

$$\mathbb{S}_n^{-1}(T, T^*) := \sum_{F \subset \{1,\ldots,n\}} (-1)^{|F|} T_F T_F^* \geqslant 0.$$

For n=2: $T = (T_1, T_2)$ and

$$\mathbb{S}_n^{-1}(T, T^*) := I - T_1 T_1^* - T_2 T_2^* + T_1 T_2 T_1^* T_2^* \geqslant 0.$$

- $T \in \mathcal{T}^n(\mathcal{H})$ is *pure* if for all $i = 1, \ldots, n$ and for all $h \in \mathcal{H}$, $\|T_i^{*m}(h)\| \to 0$ as $m \to \infty$.

Theorem (R. E. Curto and F. H. Vasilescu)

Let $T \in \mathcal{T}^n(\mathcal{H})$ be a pure tuple satisfying Szegö positivity. Let

$$\mathcal{D}_T = \overline{ran}\, \mathbb{S}_n^{-1}(T, T^*).$$

Then T dilates to $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{D}_T}(\mathbb{D}^n)$.

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Tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

Let $T \in \mathcal{T}^n(\mathcal{H})$. For each $i \in \{1, \ldots, n\}$, we denote

$$\hat{T}_i := (T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n) \in \mathcal{T}^{(n-1)}(\mathcal{H}).$$

Our class : For fixed $1 \leqslant p < q \leqslant n$,

$$\mathcal{T}^n_{p,q}(\mathcal{H}) = \{T \in \mathcal{T}^n(\mathcal{H}) : \hat{T}_p, \hat{T}_q \text{ satisfy Szegö positivity and } \hat{T}_p \text{ is pure}\}.$$

- Our class contains the class

$$\mathcal{P}^n_{p,q}(\mathcal{H}) = \{T \in \mathcal{T}^n(\mathcal{H}) : \|T_i\| < 1 \,\forall i \;\&\; \hat{T}_p, \hat{T}_q \text{ satisfy Szegö positivity}\}.$$

  introduced by Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman.
- $(M_{z_1}, \ldots, M_{z_n})$ belongs to $\mathcal{T}^n_{p,q}(H^2(\mathbb{D}^n))$ but not in $\mathcal{P}^n_{p,q}(H^2(\mathbb{D}^n))$.

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Isometric dilation of Finite rank tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

We say $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ is of *finite rank* if for all $i = p, q$, $\dim \mathcal{D}_{\hat{T}_i} < \infty$.

Theorem (B-, Das, Haria and Sarkar)

If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ is a finite rank tuple, then T dilates to the n-tuple of commuting isometries

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_p}, \ldots, M_{z_{n-1}}) \quad \text{on } H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$$

where

$$\Phi_p(\boldsymbol{z}) = \varphi(z_{q-1}) \quad \forall\, \boldsymbol{z} \in \mathbb{D}^{n-1}$$

for some inner function $\varphi \in H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_p})}(\mathbb{D})$.

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Isometric dilation for tuples in $\mathcal{T}^n_{p,q}(\mathcal{H})$.

Theorem (B-, Das, Haria and Sarkar)

Let $\mathcal{H}$ be a Hilbert space, and let $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$. Then, there exists some Hilbert space $\mathcal{E}$ such that T dilates to the isometric tuple

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_{p+1}}, \ldots, M_{z_{q-1}}, M_{\Phi_q}, M_{z_q}, \ldots, M_{z_{n-1}}) \quad \text{on } H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$$

where Φ_p and Φ_q in $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^{n-1})$ are inner polynomials in z_p of degree at most one and

$$\Phi_p(\boldsymbol{z})\Phi_q(\boldsymbol{z}) = \Phi_q(\boldsymbol{z})\Phi_p(\boldsymbol{z}) = z_p I_{\mathcal{E}} \quad (\text{ for all } \boldsymbol{z} \in \mathbb{D}^{n-1}).$$

Von Neumann inequality.

We say $T \in \mathcal{T}^n(\mathcal{H})$ satisfies *von Neumann inequality* if

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\boldsymbol{z} \in \mathbb{D}^n} |p(\boldsymbol{z})|$$

for all $p \in \mathbb{C}[z_1, \ldots, z_n]$.

- A single contraction or a pair of commuting contractions always satisfies the von Neumann inequality.
- Von Neumann inequality does not hold in general for n-tuple of commuting contractions where $n \geqslant 3$.

Theorem (B-, Das, Haria and Sarkar)

If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$, then for all $p \in \mathbb{C}[z_1, \ldots, z_n]$, the following holds:

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Refined von Neumann inequality for finite rank tuples.

Theorem (B-, Das, Haria and Sarkar)

If $T \in \mathcal{T}^n_{p,q}(\mathcal{H})$ is a finite rank operator, then there exists an algebraic variety V in $\overline{\mathbb{D}}^n$ such that for all $p \in \mathbb{C}[z_1, \dots, z_n]$, the following holds:

$$\|p(T)\| \leq \sup_{\boldsymbol{z} \in V} |p(\boldsymbol{z})|.$$

If, in addition, T_p is a pure contraction, then there exists a distinguished variety V' in $\mathbb{D}^2$ such that

$$V = V' \times \mathbb{D}^{n-2} \subseteq \mathbb{D}^n.$$

Note : A *distinguished variety* is an algebraic variety V such that

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : p(z_1, z_2) = 0\} \quad \text{and} \quad \overline{V} \cap \partial\mathbb{D}^2 = \overline{V} \cap (\partial\mathbb{D} \times \partial\mathbb{D}),$$

where $p \in \mathbb{C}[z_1, z_2]$ and $\overline{V}$ is the closure of V in $\overline{\mathbb{D}^2}$.

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THANK YOU !