GRADED COMPONENTS OF LOCAL COHOMOLOGY MODULES OF INVARIANT RINGS

Sudeshna Roy

Joint work with Prof. Tony J. Puthenpurakal

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- $Aut(R)$: the group of automorphisms of R.

Introduction

$S = \bigoplus_{n \geq 0} S_n$ standard graded Noetherian ring, $S_+ = \bigoplus_{n > 0} S_n$ it's irrelevant ideal and M a finitely generated graded S-module. Then for all $i \geq 0$,

(1) $H^i_{S_+}(M)_n$ is a finitely generated S_0-module for all $n \in \mathbb{Z}$,

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Example 1 (Brodmann and Sharp[1])

Take $S = A[X, Y]$ where A is any commutative Noetherian ring and $I = (X)$. Then the $A = S_0$ module $H^1_{(X)}(S)_n$ is free but not finitely generated for all $n \in \mathbb{Z}$.

- Negative answer even in the case when S is a polynomial ring.

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• Puthenpurakal (2017) studied $H^i_I(S)_n$ when $S = A[X_1, \ldots, X_m]$, A is a regular ring containing a field K with char $K = 0$ and showed that $H^i_I(S)_n$ exhibits striking good behavior.

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Note. Let R be a regular ring containing a field K and I be an ideal in R. Then

• Huneke and Sharp (1993) showed $H^i_I(R)$ has good properties (e.g., finiteness of (i) injdim, (ii) Bass numbers, (iii) associated primes etc.) if char $K = p > 0$.

• Lyubeznik showed $\mathcal{T}(R)$ has similar good properties in both cases when char $K = 0$ (1993) and char $K = p > 0$ (1997).

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For singular rings analogous results are in general false.

• Hartshorne (1969) gave example of a singular ring R such that $\mu_0(\mathfrak{m}, H^2_I(R))$ is infinite, Singh (2000) and Katzman (2002) gave examples of a singular rings R such that $\operatorname{Ass}_R H^i_I(R)$ is infinite.

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• Betancourt (2012) and Puthenpurakal (2014) showed that $H^i_I(R^G)$ has similar good properties, where R is a regular ring containing a field K with $\operatorname{char} K = 0$, $G \subseteq \operatorname{Aut}(R)$ finite and I is an ideal in R^G.

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► In view of this, we studied some other properties of $H^i_I(S)_n$ when $A = B^G$ where B regular ring containing a field K with char $K = 0$, $G \subseteq \operatorname{Aut}(B)$ finite.

Basic Definitions and Results

Let A be a ring (**not necessarily commutative**) and $G \subseteq Aut(A)$ is finite with $|G|$ is invertible in A.

- The skew-group ring of A (with respect to G) is

$$A * G = \{\sum_{\sigma \in G} a_\sigma \sigma \mid a_\sigma \in A \text{ for all } \sigma\},$$

  with multiplication defined as

$$(a_\sigma \sigma)(a_\tau \tau) = a_\sigma \sigma(a_\tau)\sigma\tau.$$

- An $A * G$ module M is an A-module on which G acts such that for all $\sigma \in G$,

$$\sigma(am) = \sigma(a)\sigma(m) \quad \text{for all } a \in A \text{ and } m \in M.$$

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Definition Let M be an $A * G$-module. Then

$$M^G = \{m \in M \mid \sigma(m) = m \text{ for all } \sigma \in G\}.$$

- Set A^G to be the ring of invariants of G.

§ Graded Lyubeznik functors:

• Let $R = B[X_1, \ldots, X_m]$ be standard graded.

• Y is homogeneous closed subset of $\operatorname{Spec}(R)$ if $Y = V(f_1, \ldots, f_s)$, where f_i's are homogeneous polynomials in R.

• Y is homogeneous locally closed subset of $\operatorname{Spec}(R)$ if $Y = Y'' - Y'$, where $Y' \subset Y''$ are homogeneous closed subsets of $\operatorname{Spec}(R)$.

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Definition A graded Lyubeznik functor is $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \cdots \circ \mathcal{T}_m$ where each $\mathcal{T}_j$ is either $H^i_{Y_j}(-)$ for some homogeneous locally closed subset Y_j of Spec(R) or the kernel, image or cokernel of any arrow appearing in

$$\cdots \to H^i_{Y'_j}(-) \xrightarrow{\phi'_{ij}} H^i_{Y''_j}(-) \xrightarrow{\phi''_{ij}} H^i_{Y_j}(-) \xrightarrow{\phi_{ij}} H^{i+1}_{Y'_j}(-) \to \cdots,$$

where $Y_j = Y''_j - Y'_j$ and $Y'_j \subset Y''_j$ are homogeneous closed subsets of Spec(R).

Standard assumption

- Let A be a regular domain containing a field K with $\operatorname{char} K = 0$.
- G is a finite subgroup of $\operatorname{Aut}(A)$.
- $B = A^G$ the ring of invariants of G.
- $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$ standard graded with $\deg A = 0$, $\deg B = 0$ and $\deg X_i = 1$ for all i.
- Extend the action of G on A to S by fixing X_i's. Note $S^G = R$.

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- Set $M = T(R) = \bigoplus_{n \in \mathbb{Z}} M_n$ where

$$T(-) = H^{i_1}_{I_1}(H^{i_2}_{I_2}(\cdots H^{i_r}_{I_r}(-)\cdots)$$

  for some homogeneous ideals $I_1, \dots, I_r$ in R and $i_1, \dots, i_r \geq 0$.
- Set $N = T'(S) = \bigoplus_{n \in \mathbb{Z}} N_n$ where

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Observation: N_n is an $A * G$-module and $N_n^G = M_n$ for all n.

Bass numbers

Theorem 2 (with standard assumption)

Let P be a prime ideal in B such that B_P is Gorenstein. Fix $j \geq 0$. Then EXACTLY one of the following holds:

(i) $\mu_j(P, M_n) = \infty, \quad \forall n \in \mathbb{Z}.$

(ii) $\mu_j(P, M_n) < \infty, \quad \forall n \in \mathbb{Z}.$

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In this case EXACTLY one of the following holds:

(a) $\mu_j(P, M_n) = 0, \quad \forall n \in \mathbb{Z}$.

(b) $\mu_j(P, M_n) \neq 0, \quad \forall n \in \mathbb{Z}$.

(c) $\mu_j(P, M_n) \neq 0, \quad \forall n \geq 0$ *and* $\mu_j(P, M_n) = 0, \quad \forall n < 0$.

(d) $\mu_j(P, M_n) \neq 0, \quad \forall n \leq -m$ *and* $\mu_j(P, M_n) = 0, \quad \forall n > -m$.

(e) $\mu_j(P, M_n) \neq 0, \quad \forall n \leq -m$, $\mu_j(P, M_n) = 0, \quad \forall n \geq 0$ *and* $\mu_j(P, M_n) = 0$ *for all n with* $-m < n < 0$.

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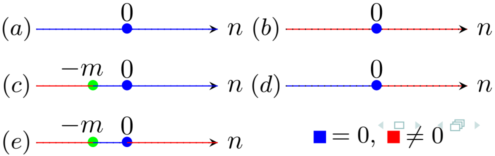

The m-th Weyl algebra over K is the ring

$$A_m(K) = K\langle X_1, \ldots, X_m, \partial_1, \ldots, \partial_m \rangle / \mathfrak{a},$$

where $\mathfrak{a}$ is the two-sided ideal generated by the elements

$$X_i \cdot X_j - X_j \cdot X_i, \quad \partial_i \cdot X_j - X_j \cdot \partial_i - \delta_{i,j}, \quad \partial_i \cdot \partial_j - \partial_j \cdot \partial_i,$$

with $\delta_{i,j}$ is the Kronecker delta.

Consider $A_m(K)$ as graded with $\deg K = 0, \deg X_i = 1, \deg \partial_i = -1$. Let $E = \oplus_{n \in \mathbb{Z}} E_n$ be a graded $A_m(K)$-module. Then E is

- holonomic if E is finitely generated and $\dim E = m$.
- Eulerian (Ma and Zhang) if $\mathcal{E}_m e = ne$ for each $e \in E_n$
- generalized Eulerian (Puthenpurakal) if for each $e \in E_n$, $\exists\ a \in \mathbb{Z}_{>0}$ (depending on e) s.t. $(\mathcal{E}_m - n)^a \cdot e = 0$

where $\mathcal{E}_m := \sum_{i=1}^m X_i \partial_i$ is the Euler operator on $A_m(K)$.

Puthenpurakal proved the followings:

Let $E = \oplus_{n\in\mathbb{Z}} E_n$ be a graded holonomic generalized Eulerian $A_m(K)$-module.

Theorem (Vanishing)

$E_n = 0$ for all $|n| \gg 0 \implies E = 0$.

Theorem (Rigidity)

- *$E_r \neq 0$ for some $r \leq -m \iff E_n \neq 0$ for all $n \leq -m$.*
- *$E_s \neq 0$ for some $s \geq 0 \iff E_n \neq 0$ for all $n \geq 0$.*
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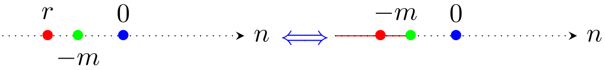

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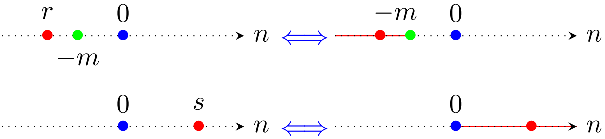

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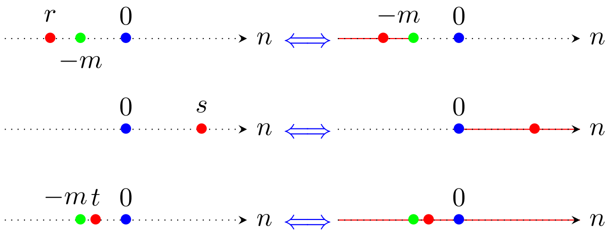

Proof of Theorem 2

- Take $M_n \neq 0$. Since $\mu_j(P, M_n) = \mu_j(PR_P, (M_n)_P)$ so we only prove for $\mathfrak{m}$ considering $(B, \mathfrak{m})$ is Gorenstein local.

Theorem (Tony, 2014)

- *A normal domain, $G \subseteq \mathrm{Aut}(A)$ finite and $|G|$ invertible in A.*
- *$\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ are* **all** *the maximal ideals of A lying above $\mathfrak{m}$, a maximal ideal of A^G.*
- *M an $A * G$-module.*

Then $H^j_{\mathfrak{m}A}(M) = \bigoplus_{l=1}^{r} H^j_{\mathfrak{n}_l}(M) = \bigoplus_{l=1}^{r} E_A(A/\mathfrak{n}_l)^{s_j(n)}$ *for all* $j \geq 0$.

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- $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ all the maximal ideals of A lying over $\mathfrak{m}$. Then $H^j_{\mathfrak{m}A}(N_n) = \bigoplus_{l=1}^r H^j_{\mathfrak{n}_l}(N_n) = \bigoplus_{l=1}^r E_A(A/\mathfrak{n}_l)^{s_j(n)}$ for all $j \geq 0$.

Lemma (with standard assumption)

Let height $P = g$. *Then*

$$\left(H^j_P(N^G_n)\right)_P = H^g_{PB_P}(B_P)^{s_j(n)} \quad \textit{for some } s_j(n) \geq 0.$$

Here $s_j(n)$ is some cardinal (possibly infinite).

Lemma (with standard assumption)

Let P be a prime ideal in B such that B_P is Gorenstein. Then

$$\mu_j(P, M_n) = \mu_0(P, H^j_P(M_n)) \quad \textit{for all } j \geq 0.$$

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- $H^j_{\mathfrak{m}}(M_n) = H^g_{\mathfrak{m}}(B)^{s_j(n)}$ where $\dim B = g$.
- $\mu_j(\mathfrak{m}, M_n) = \mu_0(\mathfrak{m}, H^j_{\mathfrak{m}}(M_n))$ for all $j \geq 0$.
- $(B, \mathfrak{m})$ Gorenstein $\Longrightarrow H^g_{\mathfrak{m}}(B) \cong E_B(B/\mathfrak{m})$ $\Longrightarrow \mu_0(\mathfrak{m}, H^j_{\mathfrak{m}}(M_n)) = s_j(n) = \mu_0(\mathfrak{n}_l, H^j_{\mathfrak{m}A}(N_n))$ for any l.

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  $\implies \mu_j(\mathfrak{m}, M_r) = s_j(r) < \infty \ \forall r$ and satisfies one of $(a), (b), (c), (d), (e)$.

★ Theorem 2 is NOT true in general.

Example 3

- Take a Noetherian local ring $(A, \mathfrak{m})$ with $\mu_0(\mathfrak{m}, H^i_J(A)) = \infty$.
- $R = A[X_1, \dots, X_m]$ be standard graded with $m \geq 1$.
- Set $M = H^i_{JR}(R) = H^i_J(A) \otimes_A R = \bigoplus_{n \in \mathbb{N}} M_n$.

So in this case $\mu_0(\mathfrak{m}, M_n) = 0$ for $n < 0$ (as $M_n = 0$) and $\mu_0(\mathfrak{m}, M_n) = \infty \quad \forall n \geq 0$.

Growth of Bass numbers

- Growth of the function $n \mapsto \mu_j(P, M_n)$ as $n \to -\infty$ and when $n \mapsto \infty$.

Theorem 4 (with standard assumption)

- *P prime ideal in B such that B_P is Gorenstein.*
- *Fix $j \geq 0$.*
- *Suppose $\mu_j(P, M_n) < \infty, \quad \forall n \in \mathbb{Z}$.*

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Then there exist polynomials $f_M^{j,P}(Z), g_M^{j,P}(Z) \in \mathbb{Q}[Z]$ of degree $\leq m-1$ such that

$$f_M^{j,P}(n) = \mu_j(P, M_n) \text{ for all } n \ll 0 \quad AND$$
$$g_M^{j,P}(n) = \mu_j(P, M_n) \text{ for all } n \gg 0.$$

Theorem 5 (with standard assumption)

- *P prime ideal in B such that B_P is Gorenstein.*
- *Fix $j \geq 0$.*
- *Suppose $\mu_j(P, M_c) = 0$ for some c.*

Then

$$f_M^{j,P}(Z) = 0 \quad \textit{or} \quad \deg f_M^{j,P}(Z) = m - 1,$$
$$g_M^{j,P}(Z) = 0 \quad \textit{or} \quad \deg g_M^{j,P}(Z) = m - 1.$$

Theorem 6 (with standard assumption)

- *Assume* $m = 1$.
- *Fix* $j \geq 0$.
- *Let* B *be Cohen-Macaulay but not necessarily Gorenstein and* P *be a prime ideal in* B.

Then $\mu_j(P, M_n) < \infty \quad \forall n \in \mathbb{Z}$.

Theorem 6 (with standard assumption)

- *Assume* $m = 1$.
- *Fix* $j \geq 0$.
- *Let* B *be Cohen-Macaulay but not necessarily Gorenstein and* P *be a prime ideal in* B.

Then $\mu_j(P, M_n) < \infty \quad \forall n \in \mathbb{Z}$.

▶ The above result gives us a sufficient condition under which for any fixed j and prime ideal P in B,

$$\mu_j(P, M_n) < \infty \quad \forall n \in \mathbb{Z}.$$

Theorem 7 (with standard assumption)

- *Assume* $m = 1$*.*
- *P prime ideal in B such that B_P is not Gorenstein.*
- *Fix* $n \in \mathbb{Z}$*.*

Then EXACTLY one of the following holds:

(i) $\mu_j(P, M_n) = 0, \quad \forall j.$

(ii) *there exists c such that*

$$\mu_j(P, M_n) = 0 \text{ for } j < c \quad \text{and}$$
$$\mu_j(P, M_n) > 0, \quad \forall j \geq c.$$

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$$\mu_j(P, M_n) > 0, \quad \forall j \geq c.$$

- Let $M_n \neq 0$ and $\operatorname{injdim}_B M_n < \infty$ for some n. Then

$$\mu_j(P, M_n) \neq 0 \text{ for some } j \implies B_P \text{ is Gorenstein.}$$

Dimension of Supports and injective dimension

Theorem 8 (with standard assumption)
If B is Gorenstein, then the following hold:

(i) $\operatorname{injdim} M_c \leq \dim M_c$ *for all* $c \in \mathbb{Z}$.

(ii) $\operatorname{injdim} M_n = \operatorname{injdim} M_{-m}$ *for all* $n \leq -m$.

(iii) $\operatorname{injdim} M_n = \operatorname{injdim} M_0$ *for all* $n \geq 0$.

(iv) *If* $m \geq 2$ *and* $-m < r, s < 0$*, then*

  (a) $\operatorname{injdim} M_r = \operatorname{injdim} M_s$.

  (b) $\operatorname{injdim} M_r \leq \min\{\operatorname{injdim} M_{-m}, \operatorname{injdim} M_0\}$.

★ Theorem 8 is NOT true if B is not Gorenstein.

Example 9

- $A = \mathbb{C}[[Y_1, \ldots, Y_n]]$ and $G \subseteq Gl_n(\mathbb{C})$ acting linearly with A^G NOT Gorenstein.
- $\mathfrak{m}$ and $\mathfrak{m}^G$ be maximal ideals of A and $A^G = B$ respectively.
- Set $S = A[X_1, \ldots, X_m]$ and $R = B[X_1, \ldots, X_m]$.
- Set $M = H^n_{\mathfrak{m}^G R}(R) = H^n_{\mathfrak{m}^G}(B) \otimes_B R$.

As A^G is NOT Gorenstein we have $\operatorname{injdim} H^n_{\mathfrak{m}^G}(B) = \infty$. It follows that $\operatorname{injdim}_B M_0 = \infty$.

Associate primes

Theorem 10 (with standard assumption)

Assume that either A is local or a smooth affine algebra over a field K of characteristic zero. Then $\bigcup_{n\in\mathbb{Z}} \operatorname{Ass}_B M_n$ *is a finite set.*

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Theorem 10 (with standard assumption)

Assume that either A is local or a smooth affine algebra over a field K of characteristic zero. Then $\bigcup_{n\in\mathbb{Z}} \operatorname{Ass}_B M_n$ is a finite set.

Moreover, if B is Gorenstein then

(1) $\operatorname{Ass}_B M_n = \operatorname{Ass}_B M_{-m}, \quad \forall n \leq -m.$

(2) $\operatorname{Ass}_B M_n = \operatorname{Ass}_B M_0, \quad \forall n \geq 0.$

Theorem 11 (with standard assumption II)

- *J homogeneous ideal in R such that $J \cap B \neq 0$.*
- *B Gorenstein.*
- *$H^i_J(R)_c \neq 0$.*

Then $H^i_J(R)_c$ is NOT finitely generated as a B-module.

► This is one sufficient condition for infinite generation of a component of graded local cohomology module over R.

★ Theorem 11 is NOT true in general.

Example 12

- $(A, \mathfrak{m})$ be a local domain with dimension $d > 0$ such that $H^i_{\mathfrak{m}}(A)$ is finitely generated and non-zero for some $i < d$.
- $R = A[X_1, \dots, X_m]$ be standard graded with $m \geq 1$.
- Set $M = H^i_{\mathfrak{m}R}(R) = H^i_{\mathfrak{m}}(A) \otimes_A R$.

Then $M_0 = H^i_{\mathfrak{m}}(A)$ is non-zero and finitely generated as an A-module.

Latest work

Later on, Puthenpurakal studied graded local cohomology modules in the following case:

- A is a regular ring containing a field K with $\operatorname{char} K = 0$.
- G a finite group with a group homomorphism $\phi : G \to GL_m(A)$.
- $R = A[X_1, \ldots, X_m]$ is standard graded with $\deg A = 0$ and $\deg X_i = 1$ for all i.
- G acts linearly on R fixing A.
- Set $S = R^G$.

Note: S is usually not standard graded.

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Thank you