Recent Advances in Knot Theory

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Knots in the three-sphere

Definition

A *knot* is an embedding of a circle S^1 in the 3-sphere S^3, and two knots $K_1,\ K_2$ are considered equivalent if there is an orientation preserving diffeomorphism $f : S^3 \to S^3$, such that $f(K_1) = K_2$.

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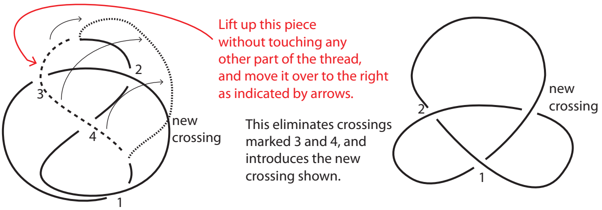

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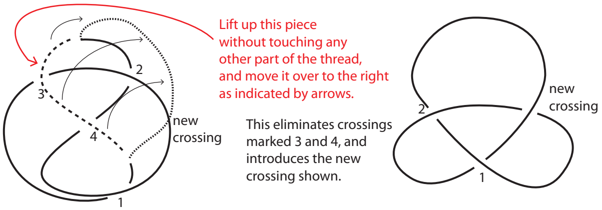
Remark

Diagram moves may help show equivalence between knots.



Knot theory

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- Effects of certain enzymes on DNA
- Structures of neural networks
- Altering chemical and physical properties of compounds through synthesis of topologically different molecules
- Understanding 3- and 4-dimensional manifolds

Classical knot invariants

- Minimal number of crossings
- Unknotting number
- Polynomial invariants: Alexander, Conway, Jones, 2-variable
- Exterior of the knot in S^3
- Fundamental group of the exterior/complement
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  Closed $M^3 \rightsquigarrow$ link diagram with surgery instructions
- Compact orientable four-manifolds with boundary can be built from the 4-ball B^4 by attaching $1-$, $2-$ and $3-$handles; $3-$handles are attached uniquely; a $2-$handle $D^2 \times D^2$ is attached along an attaching $S^1 \times D^2$, and gluing instructions are framings.

  The boundary of a B^4 with $2-$handles is a surgery on S^3 along a link.

  Closed $4-$manifolds are obtained by capping the tops with 4-balls.

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- $HF^\circ(M, \mathfrak{s})$ have a relative $\mathbb{Z}_d$–grading gr where $d = \gcd\{\langle c_1(\mathfrak{s}), h\rangle \mid h \in H_2(M; \mathbb{Z})\}$.

  If (the first Chern class of) $\mathfrak{s}$ is torsion, the gr lifts to an absolute $\mathbb{Q}$–grading $\widetilde{gr}$.

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- HF homology groups fit into a TQFT framework when 3-manifolds M_1 and M_2 cobound a spin^c 4-manifold

Heegaard Floer Correction Terms

The HF groups are related by means of long exact sequences. For example, $HF^{\pm}(M, \mathfrak{s})$ and $HF^{\infty}(M, \mathfrak{s})$ fit into the sequence

$$\ldots \to HF^{-}(M, \mathfrak{s}) \to HF^{\infty}(M, \mathfrak{s}) \xrightarrow{\pi} HF^{+}(M, \mathfrak{s}) \to HF^{-}(M, \mathfrak{s}) \to \ldots$$

If $\mathfrak{s}$ is torsion then the maps preserve the absolute grading $\widetilde{gr}$ except for $HF^{+}(M, \mathfrak{s}) \to HF^{-}(M, \mathfrak{s})$ which drops degree by 1.

Definition

The *correction term* $d(M, \mathfrak{s})$ for a torsion Spin^c-structure $\mathfrak{s} \in Spin^c(M)$ is defined as

$$d(M, \mathfrak{s}) = \min\{\widetilde{gr}(\pi(x)) \,|\, x \in HF^{\infty}(M, \mathfrak{s})\},$$

where π is from the above LES.

Applications of HF

Theorem

If a rational homology 3-sphere M bounds a rational homology 4-ball X, then $|H^2(M;\mathbb{Z})| = n^2$ for some n and $\exists \mathcal{P} \leq H^2(M;\mathbb{Z})$ of order n such that

$$d(M, \mathfrak{s}) = 0 \qquad \forall\, \mathfrak{s} \in \mathcal{P}$$

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- Detecting the unknot, other low crossing knots
- Fibering a 3-manifold over S^1
- Characterization of Seifert fibrations admitting tight contact structures
- Seifert genus
- Thurston norms
- Simpler proofs in case of older results (ex. Milnor conjecture)

Periodic Knots

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A *periodic knot of period* $p \geq 2$ is a knot $K \subset S^3$ for which there exists a orientation preserving diffeomorphism $f : S^3 \to S^3$ of order p, such that $f(K) = K$ and the fixed point set of f is $Fix(f) \cong S^1$.

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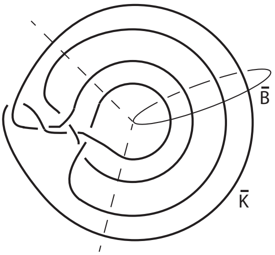
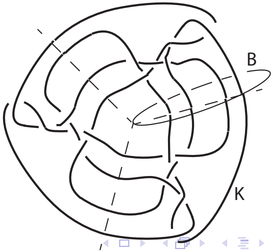
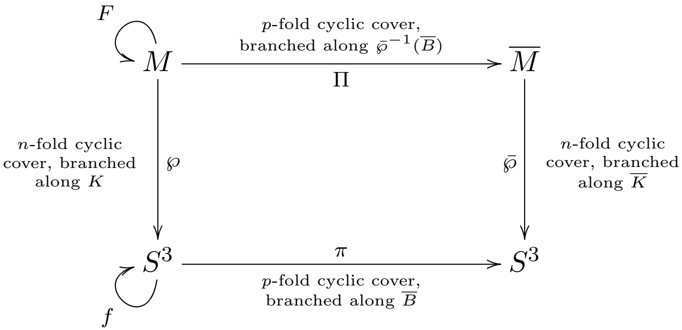




Classical obstructions to periodicity



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- Davis condition on homology: A lift F of the periodic map induces F_*, a $\mathbb{Z}_p$ action on $H_1(M)$; for a prime $\ell \not| p$, we have $\mathrm{Fix}\left(F\big|_{H_1(M)_\ell}\right) \cong H_1(\overline{M})_\ell$, where H_l is the ℓ primary subgroup of H.

  (Davis-N) Let $m = m_p(\ell) \in \mathbb{N}$ be the smallest number such that $\ell^m \equiv \pm 1 \pmod p$. Then there exist integers $t, a_1, \ldots, a_t \geq 0$ such that

$$H_1(M)_\ell / H_1(\overline{M})_\ell \cong \mathbb{Z}_\ell^{2ma_1} \oplus \mathbb{Z}_{\ell^2}^{2ma_2} \oplus \cdots \oplus \mathbb{Z}_{\ell^t}^{2ma_t}.$$

Heegaard Floer obstructions to periodicity

- Kristen Hendricks used link Floer homology on $K \cup B$ for a 2-periodic knot K and its axis B, to obtain a spectral sequence with E^1 page the link Floer homology of $K \cup B$, and which converges to the link Floer homology of $\overline{K} \cup \overline{B}$.

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- Jabuka-Naik used the order p lift to the 2-fold branched cover and examined its interaction with the Heegaard Floer homology of M.
- Notice that any diffeomorphism $g : M \to M$ induces a pull-back map $g^* : Spin^c(M) \to Spin^c(M)$.

Heegaard Floer obstructions to periodicity

Theorem (Jabuka-N)

Let M be an oriented, closed 3-manifold, $\mathfrak{s} \in Spin^c(M)$ a $Spin^c$-structure on M, and $F : M \to M$ an orientation preserving diffeomorphism. Then there are induced isomorphisms $F^\circ : HF^\circ(M, F^*(\mathfrak{s})) \to HF^\circ(M, \mathfrak{s})$ of relatively $\mathbb{Z}/\mathfrak{d}(\mathfrak{s})\mathbb{Z}$-graded $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H^1(M;\mathbb{Z})$-modules, for any $\circ \in \{\infty, \pm, \widehat{\quad}\}$.
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- The isomorphisms F° fit into the commutative diagrams

$$\begin{array}{ccccc} HF^-_{(d)}(M, F^*(\mathfrak{s})) & \longrightarrow & HF^\infty_{(d)}(M, F^*(\mathfrak{s})) & \longrightarrow & HF^+_{(d)}(M, F^*(\mathfrak{s})) \\ \downarrow F^- & & \downarrow F^\infty & & \downarrow F^+ \\ HF^-_{(d)}(M, \mathfrak{s}) & \longrightarrow & HF^\infty_{(d)}(M, s) & \longrightarrow & HF^+_{(d)}(M, s) \end{array}$$

Heegaard Floer obstructions to periodicity

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$$\begin{array}{ccccc} \widehat{HF}_{(d)}(M, F^*(\mathfrak{s})) & \longrightarrow & HF^+_{(d)}(M, F^*(\mathfrak{s})) & \xrightarrow{U} & HF^+_{(d-2)}(M, F^*(\mathfrak{s})) \\ \downarrow \hat{F} & & \downarrow F^+ & & \downarrow F^+ \\ \widehat{HF}_{(d)}(M, \mathfrak{s}) & \longrightarrow & HF^+_{(d)}(M, s) & \xrightarrow{U} & HF^+_{(d-2)}(M, s) \end{array}$$

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Corollary

Under the assumptions of the previous theorem, and with M a rational homology 3-sphere, we obtain

$$d(M, F^*(\mathfrak{s})) = d(M, \mathfrak{s}), \quad \forall \mathfrak{s} \in Spin^c(M).$$

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- Going forward, we shall identify $Spin^c(M)$ with $H_1(M; \mathbb{Z})$

Theorem (Jabuka-N)

Let K be a p-periodic knot with associated order p diffeomorphism $f : S^3 \to S^3$. Let $\wp : M \to S^3$ be the n-fold cyclic cover with branching set K and let $F : M \to M$ be induced by f. Let ℓ be a prime not dividing p, and assume that $H_1(\overline{M}; \mathbb{Z})_\ell = 0$ (as happens if $\Delta_{\overline{K}}(t) = 1$).

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Corollary

With the assumptions as above, there is a free $\mathbb{Z}_p$-action on $H_1(M; \mathbb{Z})_\ell$ such that $d(M, F^*(\mathfrak{s})) = d(M, \mathfrak{s})$ for all $\mathfrak{s} \in H_1(M; \mathbb{Z})_\ell$.

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- In particular, the correction terms of M have values that come in multiples of p.

Example

Consider the knot $K = 12a_{100}$ from the knot tables. This knot satisfies

- the Murasugi condition with $p = 3$, $\Delta_{\overline{K}} = 1$ and $\lambda = 2$.
- the Davis-Naik condition with $p = 3$ since

$$5^1 \equiv -1 (\text{mod } 3) \quad \text{and} \quad H_1(M;\mathbb{Z})_5 \cong \mathbb{Z}_5^2.$$

The "classical obstructions" do not prevent it from being of period 3.
However, the correction terms of its 2-fold cyclic branched cover (and their multiplicities) corresponding to spin^c-structures $\mathfrak{s} \in H_1(M;\mathbb{Z})_5$, are

$d(M,\mathfrak{s})$	$-\frac{4}{5}$	$-\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{4}{5}$
Multiplicity of $d(M,\mathfrak{s})$	2	6	6	6	4

Application to branched covers

See how the previous argument applies to a more general case:

Theorem

Let q be a prime and let M be the q-fold cyclic cover of S^3 branched along a knot K, and assume that $H_1(M;\mathbb{Z})_2 = 0$. Then for every prime ℓ, the isomorphism type of each Heegaard Floer group $HF^\circ(M,\mathfrak{s})$ with $\mathfrak{s} \in H_1(M;\mathbb{Z})_\ell - \{0\}$ and $\circ \in \{\infty, -, +, \widehat{\ }\}$ occurs with a multiplicity that is divisible by q. Likewise, the correction terms $d(M,\mathfrak{s})$ with $\mathfrak{s} \in H_1(M;\mathbb{Z})_\ell - \{0\}$ also occur with multiplicities divisible by q.

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Example

In particular, the 2-fold cyclic cover M of S^3 branched over the knot $K = 12a_{100}$ cannot be a q-fold branched cover of S^3 over **any** knot, for $q > 2$.

Heegaard Floer obstructions to periodicity

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Theorem (Jabuka-N)

Let K be a p-periodic knot with p a prime and let ℓ be a prime different from p. Then there exists a subgroup H of $H_1(M;\mathbb{Z})_\ell$ (and $H \cong H_1(\overline{M};\mathbb{Z})_\ell$) such that if $n_1, \ldots, n_k$ are the multiplicities of the various correction terms $d(M, \mathfrak{s})$ with $\mathfrak{s} \in H_1(M;\mathbb{Z})_\ell$, and $m_1, \ldots, m_k$ are their mod p reductions, then

$$m_1 + \cdots + m_k \leq |H|.$$

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- Accordingly, p-periodicity of K can be obstructed by showing that no such subgroup H exists.

Heegaard Floer obstructions to periodicity

Example

Let $K_1 = 7_4$ and $K_2 = 9_2$ and $K = K_1 \# K_1 \# K_2$. Note that

$$\Delta_{K_i} = 4 - 7t + 4t^2, \quad i = 1, 2,$$
$$H_1(M_i; \mathbb{Z}) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_3, \quad M_i = \text{2-fold branched cover of } K_i \ .$$

Thus K passes the "classical"(algebraic) conditions for 3-periodicity with $\Delta_{\overline{K}}(t) = 4 - 7t + 4t^2$ and $\lambda = 1$ Let $M = M_1 \# M_1 \# M_2$, then the correction terms $d(M, \mathfrak{s})$ with $\mathfrak{s} \in H_1(M; \mathbb{Z})_5 \cong \mathbb{Z}_5^3$, and their multiplicities are

$-\frac{29}{10}$	$-\frac{5}{2}$	$-\frac{17}{10}$	$-\frac{13}{10}$	$-\frac{9}{10}$	$-\frac{1}{2}$	$-\frac{1}{10}$	$\frac{3}{10}$	$\frac{7}{10}$	$\frac{11}{10}$	$\frac{3}{2}$
8	8	20	24	8	16	20	10	6	4	1

The sum of the mod 3 multiplicities gives

$$2+2+2+2+1+2+1+1+1 = 14 > 5 = |H| = |H_1(\overline{M}; \mathbb{Z})_5| = |\mathbb{Z}_5|.$$

Thank you