Quadratic Finite Element Methods for the Obstacle Problem

Thirupathi Gudi

Department of Mathematics

Indian Institute of Science Bangalore

Joint work with Sharat Gaddam

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Department of Mathematics,IIT Bombay

Outline

- The Obstacle Problem
- Conforming Linear and Quadratic Methods
- Error Estimates
- Conclusions

Obstacle Problem

Find $u \in \mathcal{K}$ such that

$$J(u) = \min_{v \in \mathcal{K}} J(v),$$

where

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f v\,dx,$$

$f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$,

$\chi \in H^1(\Omega) \cap C(\bar{\Omega})$ with $\chi \leq g$ on $\partial\Omega$,

$$H^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^d\},$$

$\mathcal{K} := \{v \in H^1(\Omega) : v \geq \chi$ a.e. in $\Omega, v = g$ on $\partial\Omega\}$,

Ω is a polyhedral domain in $\mathbb{R}^d (1 \leq d \leq 3)$.

Obstacle Problem

- The obstacle problem is a prototype model for variational inequalities of the first kind.
- The obstacle problem arises in contact mechanics, fluid flow and finance.
- Numerical approximation of the obstacle problem is interesting due to the existence of free boundary.

Obstacle Problem

The weak formulation of the obstacle problem is to find $u \in \mathcal{K}$ such that

$$a(u, v-u) \geq (f, v-u) \qquad \forall\, v \in \mathcal{K}.$$

Here

$$\mathcal{K} := \{v \in H^1(\Omega) : v \geq \chi \text{ a.e. in } \Omega, v = g \text{ on } \partial\Omega\},$$

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v \, dx,$$

$$(f, v) = \int_\Omega f v \, dx.$$

Obstacle Problem

The weak formulation of the obstacle problem is to find $u \in \mathcal{K}$ such that

$$a(u, v-u) \geq (f, v-u) \qquad \forall\, v \in \mathcal{K}.$$

If $u \in H^2(\Omega)$, then

$$\begin{aligned} -\Delta u \geq f \quad &\text{in} \quad \Omega, \\ -\Delta u = f \quad &\text{in} \quad \{u > \chi\}, \\ (-\Delta u - f, u - \chi) = 0&, \\ [\![\nabla u]\!] = 0 \quad &\text{on} \quad \partial\{u = \chi\}, \\ [\![u]\!] = 0 \quad &\text{on} \quad \partial\{u = \chi\}. \end{aligned}$$

Obstacle Problem

The weak formulation of the obstacle problem is to find $u \in \mathcal{K}$ such that

$$a(u, v - u) \geq (f, v - u) \qquad \forall\, v \in \mathcal{K}.$$

Define $\lambda \in H^{-1}(\Omega)$ by

$$\langle \lambda, v \rangle = a(u, v) - (f, v) \quad \forall v \in H_0^1(\Omega).$$

Then

$$\langle \lambda, v - u \rangle \geq 0 \quad \forall v \in \mathcal{K},$$

$$\langle \lambda, v \rangle \geq 0 \quad \forall v \in H_0^1(\Omega) \text{ with } v \geq 0.$$

Obstacle Problem

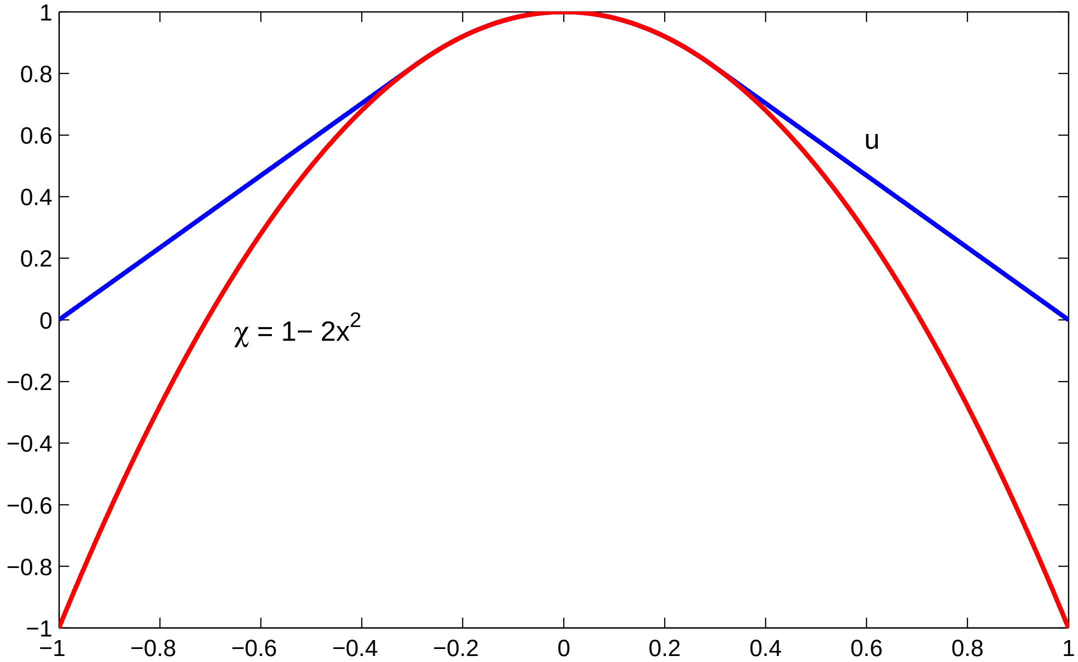
Unlike linear elliptic problems, C^∞ data do not imply a C^∞ solution for obstacle problem.

Regularity Result : If $f \in L^\infty(\Omega) \cap BV(\Omega)$ and $\chi \in C^3(\bar{\Omega})$ and $\partial\Omega$ is sufficiently smooth, then $u \in W^{s,p}(\Omega)$ for all $1 < p < \infty$ and $s < 2 + 1/p$. (Brezis 1971)

Example: Let $\Omega = (-1, 1)$, $f \equiv 0$ and $\chi(x) = 1 - 2x^2$.

$$u(x) := \begin{cases} -4\left(-1 + \frac{1}{\sqrt{2}}\right)(1 + x) & \text{if } x < -1 + \frac{1}{\sqrt{2}}, \\ -4\left(1 - \frac{1}{\sqrt{2}}\right)(x - 1) & \text{if } x > 1 - \frac{1}{\sqrt{2}}, \\ 1 - 2x^2 & \text{otherwise.} \end{cases}$$

Obstacle Problem



FEM and Approximation

Let $\mathcal{T}_h$ be regular triangulation of Ω. For $r \geq 1$, define finite element space

$$V_h := \{v \in C(\bar{\Omega}) : v|_T \in P_r(T),\ \forall T \in \mathcal{T}_h\},$$

where $P_r(T)$ is the space of polynomials of total degree less than or equal to r.

The following approximation holds: Let $v \in H^m(T)$. Then there is some $I_h v \in V_h$ such that

$$\|v - I_h v\|_{H^k(T)} \leq C h_T^{\mu-k} |v|_{H^m(T)}$$

where $\mu = \min\{m, r+1\}$, $0 \leq k \leq \mu$, h_T is the diameter of T and C is a positive constant independent of T and $\mathcal{T}_h$.

Numerical Methods

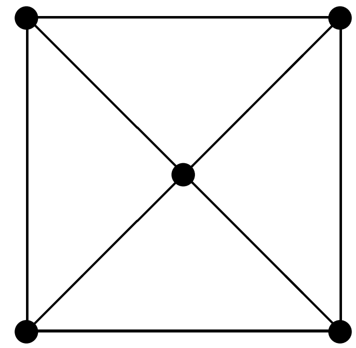
- Falk, 1974: A conforming linear finite element method is studied. Linear order of convergence is derived with obstacle constraints at the vertices. This approach works in both two and three dimensions.
- Brezzi, Hager and Raviart,1977: Conforming quadratic finite element methods were studied. The quadratic FEM in two dimensions incorporate obstacle constraints at the midpoints of the edges. Error estimate of order $h^{3/2-\epsilon}$ is derived.
- Wang 2002: Revisited the analysis in Brezzi et al, 1977, the error analysis relaxes the assumption that the length of the free boundary is finite.

Numerical Methods

- Wang, Han and Cheng, 2010: Studied linear and quadratic DG methods for two dimensional problem.
- Antonietti, Beirao da Veiga, and Verani, 2013. Studied the first order mimetic difference methods.
- Carstensen and Köhler, 2017. Linear CR Nonconforming FEM for two dimensional problem.
- Gustafsson, Stenberg, and Videman, 2017. Study mixed and stabilized methods for two dimensional problem.
- Gaddam and Gudi, 2018. Bubbles enriched quadratic FEM for three dimensional problem.
- Wang et al. 2018. Linear and quadratic virtual element methods for two dimensional problem are studied.

Linear FEM

$\mathcal{T}_h$= regular triangulation of Ω,



$\mathcal{V}_h$=set of vertices in $\mathcal{T}_h$.

Define

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in P_1(T),\ T \in \mathcal{T}_h\},$$
$$\mathcal{K}_h = \{v_h \in V_h : v_h(p) \geq \chi(p),\ \forall p \in \mathcal{V}_h\}.$$

FEM: Find $u_h \in \mathcal{K}_h$ such that (Falk 1974)

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall v_h \in \mathcal{K}_h.$$

A priori Error Analysis

Let $I_h u$ be the Lagrange interpolation of u, i.e.,

$$I_h u(p) = u(p) \text{ for all } p \in \mathcal{V}_h.$$

It is easy to see that $I_h u \geq I_h \chi$ and $u_h \geq I_h \chi$ in Ω.

Define the sets $\mathcal{T}_+$, $\mathcal{T}_0$ and $\mathcal{T}_f$ by

$$\mathcal{T}_+ = \{T \in \mathcal{T}_h : u > \chi \text{ on } T\},$$
$$\mathcal{T}_0 = \{T \in \mathcal{T}_h : u \equiv \chi \text{ on } T\},$$
$$\mathcal{T}_f = \{T \in \mathcal{T}_h : T \notin \mathcal{T}_+ \text{ and } T \notin \mathcal{T}_0\}.$$

Let $u, \chi \in H^2(\Omega)$. Now if $\lambda := -\Delta u - f$, then

$$\lambda \geq 0 \text{ in } \Omega \text{ and } \lambda \equiv 0 \text{ on any } T \in \mathcal{T}_+.$$

A priori Error Analysis

Note that

$$
\begin{aligned}
\|\nabla(u-u_h)\|^2 &= a(u-u_h, u-I_hu) + a(u, I_hu-u_h) \\
&\qquad - a(u_h, I_hu-u_h) \\
&\le a(u-u_h, u-I_hu) + a(u, I_hu-u_h) - (f, I_hu-u_h) \\
&\le a(u-u_h, u-I_hu) + (\lambda, I_hu-u_h) \\
&\lesssim h|u|_{H^2(\Omega)}\|\nabla(u-u_h)\| + (\lambda, I_hu-u_h) \\
&\lesssim h|u|_{H^2(\Omega)}\|\nabla(u-u_h)\| + \sum_{T\in\mathcal{T}_0}\int_T \lambda(I_hu-u_h)\,dx \\
&\quad + \sum_{T\in\mathcal{T}_f}\int_T \lambda(I_hu-u_h)\,dx.
\end{aligned}
$$

A priori Error Analysis

Since for any $T \in \mathcal{T}_0$, there holds $u \equiv \chi$ on T and

$$\int_T \lambda(I_h u - u_h)\, dx = \int_T \lambda(I_h \chi - u_h)\, dx \leq 0.$$

For any $T \in \mathcal{T}_f$,

$$\begin{aligned}
\int_T \lambda(I_h u - u_h)\, dx &= \int_T \lambda(I_h(u-\chi) + I_h\chi - u_h)\, dx \\
&\leq \int_T \lambda I_h(u-\chi)\, dx \\
&= \int_T \lambda(I_h(u-\chi) - (u-\chi))\, dx \\
&\lesssim h^2 \|\lambda\|\, |u-\chi|_{H^2(\Omega)}.
\end{aligned}$$

A priori Error Analysis

Therefore

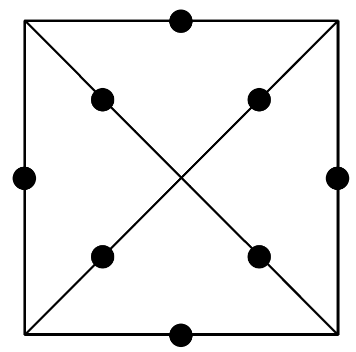
$$\|\nabla(u-u_h)\| \lesssim h\left(|u|_{H^2(\Omega)} + |\chi|_{H^2(\Omega)} + \|f\|\right).$$

(Falk 1974)

Quadratic FEM

$\mathcal{T}_h$= regular triangulation of $\Omega \subset \mathbb{R}^2$,

$\mathcal{V}_h$={vertices of triangles in $\mathcal{T}_h$},



$\mathcal{M}_h$={mid points of the edges in $\mathcal{T}_h$}.

Define

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in P_2(T),\ T \in \mathcal{T}_h\},$$
$$\mathcal{K}_h = \{v_h \in V_h : v_h(p) \geq \chi(p),\ \forall p \in \mathcal{M}_h\}.$$

FEM: Find $u_h \in \mathcal{K}_h$ such that (Brezzi et al. 1977)

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall v_h \in \mathcal{K}_h.$$

Quadratic FEM

The quadrature formula

$$\int_T w\,dx \approx \frac{|T|}{3} \sum_{p \in \mathcal{M}_T} w(p)$$

is exact for quadratic polynomials. Here $\mathcal{M}_T$ is the set of the midpoints of the three edges of T.

This implies if a quadratic polynomial v is nonnegative at the midpoints of a triangle T, then

$$\int_T v\,dx \geq 0.$$

A priori Error Analysis

Let $I_h u$ be the Lagrange interpolation of u, i.e.,

$$I_h u(p) = u(p) \text{ for all } p \in \mathcal{V}_h \cup \mathcal{M}_h.$$

Now $I_h u \ngeq I_h \chi$ and $u_h \ngeq I_h \chi$ in Ω, but $I_h u \in \mathcal{K}_h$. This implies for any $T \in \mathcal{T}_h$ that

$$\int_T (I_h u - I_h \chi)\, dx \geq 0,$$
$$\int_T (u_h - I_h \chi)\, dx \geq 0.$$

A priori Error Analysis

Recall the sets $\mathcal{T}_+$, $\mathcal{T}_0$ and $\mathcal{T}_f$ by

$$\mathcal{T}_+ = \{T \in \mathcal{T}_h : u > \chi \text{ on } T\},$$
$$\mathcal{T}_0 = \{T \in \mathcal{T}_h : u \equiv \chi \text{ on } T\},$$
$$\mathcal{T}_f = \{T \in \mathcal{T}_h : T \notin \mathcal{T}_+ \text{ and } T \notin \mathcal{T}_0\}.$$

Let $f \in H^1(\Omega)$,$\chi \in H^3(\Omega)$.

Then $u \in H^{5/2-\epsilon}(\Omega)$ and $\lambda := -\Delta u - f \in H^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. Further

$$\lambda \geq 0 \text{ in } \Omega \text{ and } \lambda \equiv 0 \text{ on any } T \in \mathcal{T}_+.$$

A priori Error Analysis

Note that with $s = 5/2 - \epsilon$,

$$
\begin{aligned}
\|\nabla(u-u_h)\|^2 &= a(u-u_h, u-I_hu) + a(u, I_hu-u_h) \\
&\qquad - a(u_h, I_hu-u_h) \\
&\le a(u-u_h, u-I_hu) + a(u, I_hu-u_h) - (f, I_hu-u_h) \\
&\le a(u-u_h, u-I_hu) + (\lambda, I_hu-u_h) \\
&\lesssim h^{s-1}|u|_{H^s(\Omega)}\|\nabla(u-u_h)\| + (\lambda, I_hu-u_h) \\
&\lesssim h^{s-1}|u|_{H^s(\Omega)}\|\nabla(u-u_h)\| + \sum_{T\in\mathcal{T}_0}\int_T \lambda(I_hu-u_h)\,dx \\
&\quad + \sum_{T\in\mathcal{T}_f}\int_T \lambda(I_hu-u_h)\,dx.
\end{aligned}
$$

A priori Error Analysis

For any $T \in \mathcal{T}_0$, there holds $u \equiv \chi$ on T and

$$\begin{aligned}\int_T \lambda(I_h u - u_h)\,dx &= \int_T \lambda(I_h\chi - u_h)\,dx \\ &\leq \int_T (\lambda - \bar{\lambda})(I_h\chi - u_h)\,dx \\ &= \int_T (\lambda - \bar{\lambda})(I_h u - u_h)\,dx \\ &= \int_T (\lambda - \bar{\lambda})\big((I_h u - u_h) - \overline{(I_h u - u_h)}\big)\,dx \\ &\lesssim h^{3/2-\epsilon}|\lambda|_{H^{1/2-\epsilon}(\Omega)}\|\nabla(I_h u - u_h)\|,\end{aligned}$$

where $\quad \bar{v}|_T := \dfrac{1}{|T|}\displaystyle\int_T v\,dx$

A priori Error Analysis

For any $T \in \mathcal{T}_f$, similarly

$$\int_T \lambda(I_h u - u_h)\,dx \lesssim h^{3-\epsilon}|\lambda|_{H^{1/2-\epsilon}(\Omega)}\left(|\chi|_{H^3(\Omega)} + |u|_{H^{5/2-\epsilon}(\Omega)}\right) + h^{3/2-\epsilon}|\lambda|_{H^{1/2-\epsilon}(\Omega)}\|\nabla(I_h u - u_h)\|.$$

A priori error estimate: If $f \in H^1(\Omega)$ and $\chi \in H^3(\Omega)$ and $u \in W^{s,p}(\Omega)$ for all $1 < p < \infty$ and $s < 2 + 1/p$, then

$$\|u - u_h\|_1 \leq Ch^{3/2-\epsilon}, \text{ for all } \epsilon > 0.$$

(Brezzi et al. 1977) (Wang 2002)

Numerical Results

Let Ω be the square with corners $\{(-1,0),(0,-1),(1,0),(0,1)\}$. Let $\chi = 1-2r^2$, where $r = \sqrt{x^2+y^2}$. Let $r_0 = (\sqrt{2}-1)/\sqrt{2}$.
The load function f is taken as

$$f(r) := \begin{cases} 4 & \text{if } r < r_0, \\ \\ 4r_0/r & \text{if } r \geq r_0. \end{cases}$$

Then the solution u is given by

$$u(r) := \begin{cases} 1-2r^2 & \text{if } r < r_0, \\ \\ 4r_0(1-r) & \text{if } r \geq r_0. \end{cases}$$

Numerical Results

Order of convergence:

h	$\|\nabla(u - u_h)\|$	order of conv.
$\sqrt{2}/4$	0.206211469561149	–
$\sqrt{2}/8$	0.058342275497902	1.821
$\sqrt{2}/16$	0.025124856349493	1.215
$\sqrt{2}/32$	0.007971135017375	1.656
$\sqrt{2}/64$	0.002583671405642	1.625
$\sqrt{2}/128$	0.000931014496396	1.472
$\sqrt{2}/256$	0.000323678406046	1.524

Quadratic FEM for 3D Problem

Let $\mathcal{T}_h$ be a regular triangulation of $\Omega \subset \mathbb{R}^3$.

For any $T \in \mathcal{T}_h$, let $b_T \in H_0^1(T)$ be a quartic bubble function. Define

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in P_2(T) \oplus b_T,\ T \in \mathcal{T}_h\},$$

$$\mathcal{K}_h = \{v_h \in V_h : \int_T v_h\, dx \geq \int_T \chi\, dx,\ \forall v \in \mathcal{T}_h\}.$$

FEM: Find $u_h \in \mathcal{K}_h$ such that (Gaddam and Gudi, 2016)

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \forall v_h \in \mathcal{K}_h.$$

Quadratic FEM for 3D Problem

Define $I_h u \in V_h$ to be the interpolation of u by

$$I_h u(p) := u(p) \text{ for all vertices of } \mathcal{T}_h,$$
$$I_h u(p) := u(p) \text{ for all midpoints of the edges of } \mathcal{T}_h,$$

and

$$\int_T I_h u \, dx = \int_T u \, dx \text{ for all } T \in \mathcal{T}_h.$$

Degrees of freedom for I_h on each T is 11 which is the same as the dimension of $P_2(T) \oplus b_T$.

Further since $u \geq \chi$ in Ω, we have

$$\int_T I_h u \, dx = \int_T u \, dx \geq \int_T \chi \, dx \text{ for all } T \in \mathcal{T}_h.$$

A Priori Error Estimate

Let $f \in H^1(\Omega)$ and $\chi \in H^3(\Omega)$ and $u \in H^{5/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. Further assume that $u \in W^{s,p}(\Omega)$, where $s < 2 + 1/p$ and $1 < p < \infty$.

Then there holds

$$\|u - u_h\|_1 \leq Ch^{3/2-\epsilon} \text{ for all } \epsilon > 0.$$

Numerical Experiment

Let $\Omega = (0,1)^3$ and the obstacle function be $\chi \equiv 0$. The forcing function f is taken to be

$$f(x,y,z) := \begin{cases} -4(2r^2 + 3(r^2 - r_0^2)) & \text{if } r > r_0, \\ \\ -8r_0^2(1 - r^2 + r_0^2) & \text{if } r \leq r_0, \end{cases}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and $r_0 = 0.7$.

The nonhomogeneous Dirichlet boundary condition is taken in such a way that the solution u is given by

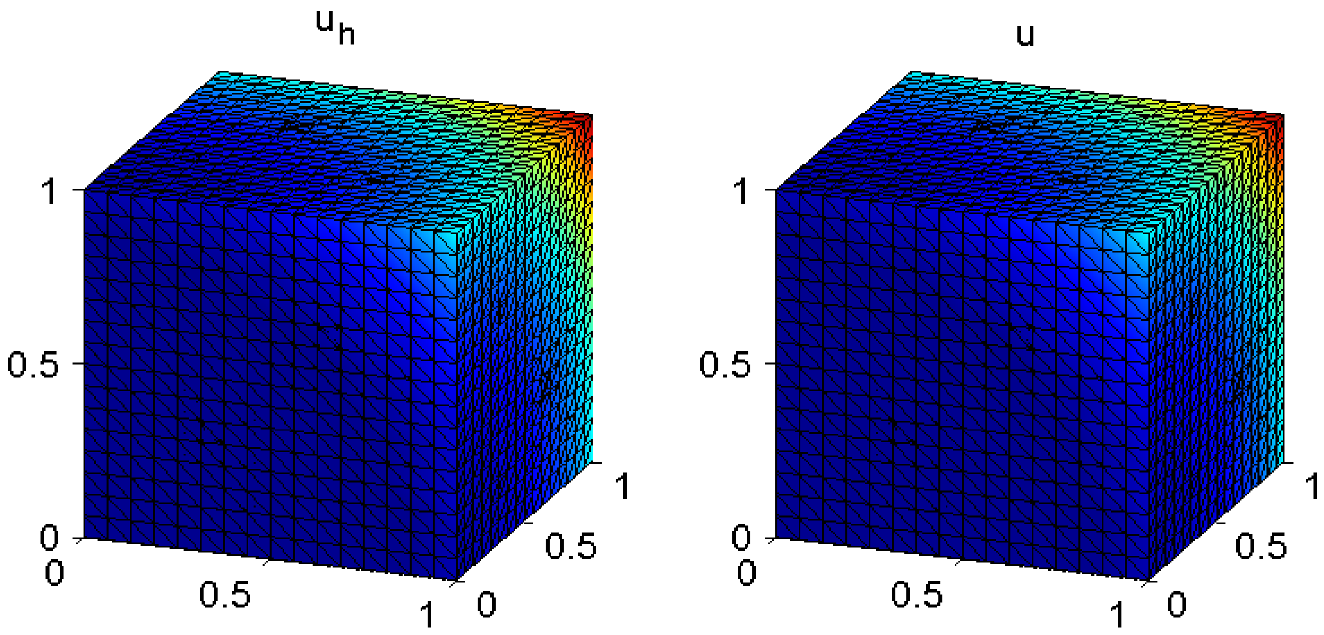
$$u(x,y,z) = (\max(r^2 - r_0^2, 0))^2.$$

Numerical Experiment

Order of convergence:

h	$\|\nabla(u - u_h)\|_{L^2(\Omega)}$	order
0.3467	1.8500e-001	–
0.1733	5.6046e-002	1.3596
0.0867	1.9210e-002	1.4112
0.0433	7.1151e-003	1.3636

Numerical Experiment



Conclusions

A quadratic finite element enriched with element wise bubble functions with integral constraints is designed and shown to be optimal (up to the regularity)

Numerical experiments confirm the theoretical results.

A reliable and efficient (partially) a posteriori error estimator for these methods is also derived.

Reference

- S. Gaddam and T. Gudi. Bubbles enriched quadratic finite element method for the 3D-elliptic obstacle problem. Comput. Meth. Appl. Math., 18 (2018), pp. 223–236