SINGULAR SOLUTION

PART I

GENERALITIES

Question No. 1

Where does the word ‘singular’ come from?

Answer to Q.1

Single $\longrightarrow$ singular

Similarly derived words:

(i) Angle $\longrightarrow$ angular

(ii) Table $\longrightarrow$ tabular

(iii) Particle $\longrightarrow$ particular

(iv) Vehicle $\longrightarrow$ vehicular

Question No. 2

What are the euphemisms for singular?

Answer to Q.2

(A) Irregular

(B) Abnormal

(C) Degenerate

(D) Accidental

(E) Exceptional/unique

(F) Critical/unstable

(G) Not in general position

(H) Of measure zero

(I) Undesirable (Kelley)

Question No. 3

Where does singularity originate in analysis?

Answer to Q. 3

From the number 0.
It is the only number which has no reciprocal.

A regular matrix reamins regular under small perturbations.

Question No. 4

Which is the true meaning of ‘singular’?

(A) Isolated

(B) Not having company

Question No. 5

Are singularities bad?

Answer to Q.5

Far from it.

Interesting things happen mostly at singularities.

Complex analysis will be dull if there were no singulaities.

There is also resolution of algebraic singularities.

Nullity of a matrix is a quantitative measure of its degree of singularity.

And if this is not enough, in differential equations,

there are regular singular points and singular singular points!

SINGULAR SOLUTION
PART II

OF DIFFERENTIAL EQUATION

A solution which satisfies a differential equation but is not a member of the family of curves represented by it is called a singular solution, because it cannot be obtained by giving any value to the arbitrary constants in the general solution. Geometrically, the singular solution of the d.e. obtained from a 1-parameter family of curves represents a curve which has the property that at every point on it, it touches a member of the original family of curves.

Sometimes, such a curve is called an envelope of that family.

Consider, for example, the family of all circles of radii 1 with centres lying on the x-axis. This is a 1-parameter family. A typical member of it has an equation of the form

$$(x-a)^2 + y^2 = 1 \tag{1}$$

where a is an arbitrary constant. Differentiating (1) gives $x - a = -yy'$. Putting this into (1) again, we get

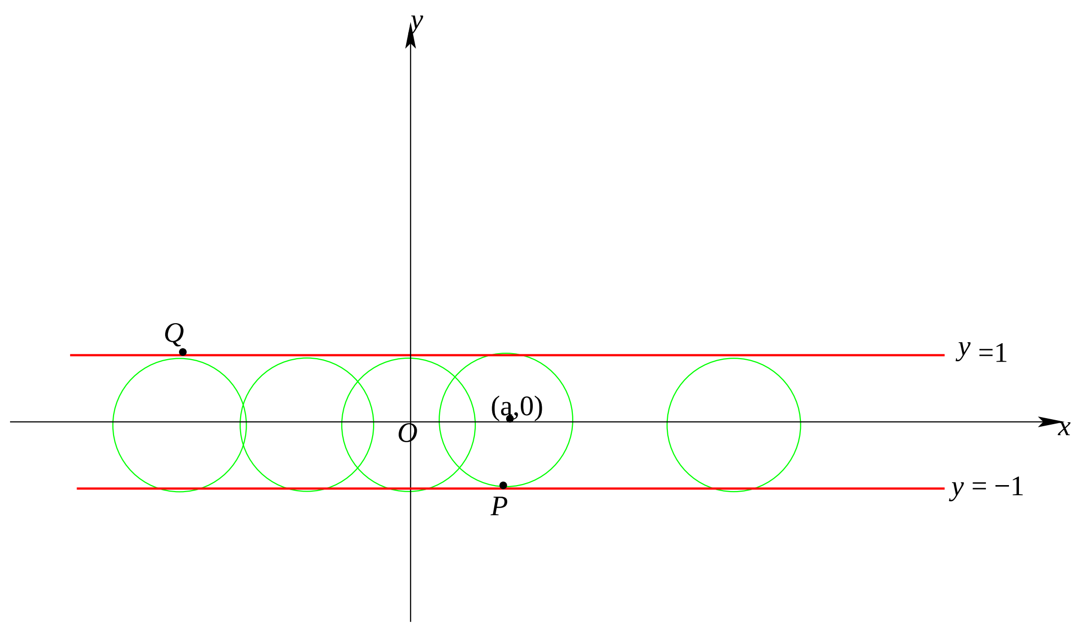
$$y^2(1 + y'^2) = 1 \tag{2}$$

as the differential equation of the family (1).

Let us now see what happens if we solve (2). Rewriting it as $\frac{dy}{dx} = \pm\frac{\sqrt{1-y^2}}{y}$ and separating

the variables we get $\sqrt{1-y^2} = c \pm x$ as the general solution. Curves given by this are of the form $(c+x)^2+y^2 = 1$ or of the form $(c-x)^2+y^2 = 1$. Together, they constitute the family of all circles of radii 1 with centres lying on the x-axis. So we got back the 1-parameter family of circles that we started with. But the strange thing is that (1) has two more solutions which are not of this form. They are given by $y = 1$ and $y = -1$. For, in either case y' vanishes identically and so (2) is satisfied!

The mysterious appearance of these two extra solutions of (2) can be explained geometrically. In the figure below a few members of the family of curves (1) are sketched. The two solutions $y = \pm 1$ represent two envelopes of this family.



Let L be the line $y = -1$, P be any point on L and C be the (unique) member of the family (1) passing through P. Then C is a circle of radius 1 which has P as its lowest point. Hence the value of $\frac{dy}{dx}$ at P as a point on the circle C, is 0. But this is also the value of $\frac{dy}{dx}$ at P as a point of L since the line L has slope 0 at all points. Thus we see that the line L, although not a member of the family represented by (1), has the property that at every point P on it, it shares with C not only the coordinates x and y of P (as happens when any two curves intersect) but also the slope $\frac{dy}{dx}$. So any equation which involves only x, y and $\frac{dy}{dx}$ and which is satisfied by every member of the family represented by (1) will also be satisfied by the line L. Since (2) is one such equation, it

is hardly surprising that $y = -1$ is a solution of it. An identical explanation holds for the line $y = 1$, except that a typical point Q on it is the highest point of the circle passing through it.

Another example

Consider the family of straight lines of the form

$$y = cx - c^2 \tag{3}$$

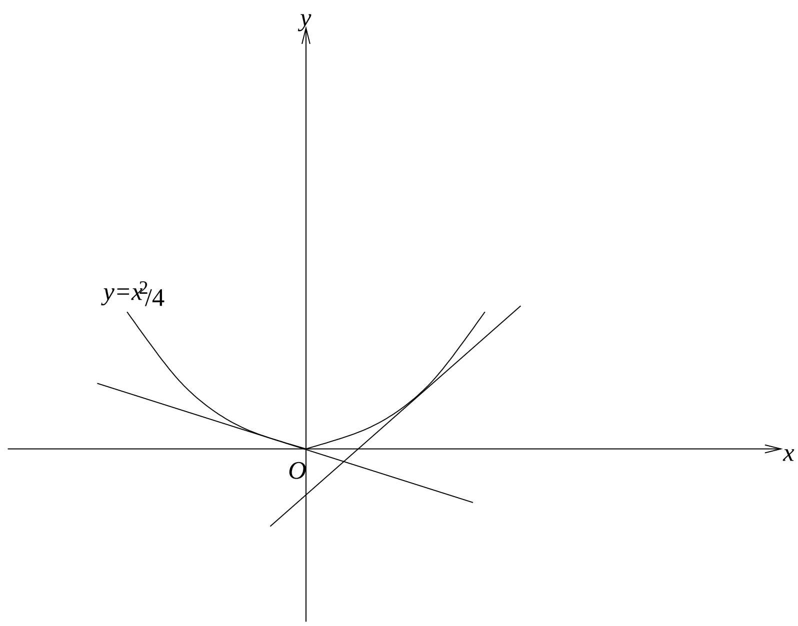
where c is a parameter. The d.e. which represents this family is

$$y'^2 - xy' + y = 0 \tag{4}$$

and has

$$y = \frac{x^2}{4} \tag{5}$$

as a singular solution. It is a parabola.

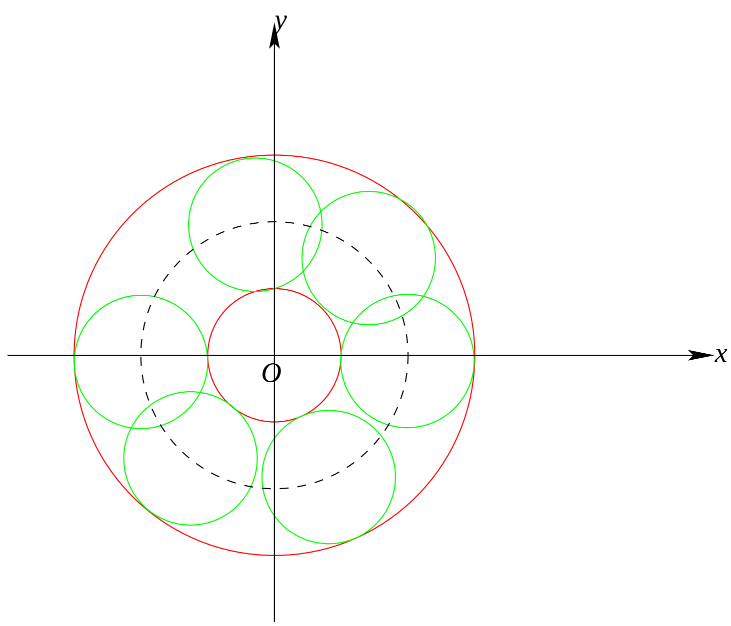


One More Example

Consider the family of circles of radii 1 and having centres on the circle

$$x^2 + y^2 = 4 \tag{6}$$

(shown as the dotted circle in the figure below).



The d.e. representing this family is obtained by eliminating θ form the two equations

$$\begin{aligned} (x - 2\cos\theta)^2 + (y - 2\sin\theta)^2 &= 1 & (7)\\ \text{and} \quad (x - 2\cos\theta) + (y - 2\sin\theta)y' &= 0 & (8) \end{aligned}$$

Here there are two singular solutions,

$$x^2 + y^2 = 9 \tag{9}$$

and

$$x^2 + y^2 = 1 \tag{10}$$

The second solution represents a circle which is congruent to every member of the family of the circles.

Still it is not a member of that family!

SINGULAR SOLUTION
PART III

TOLERATING THE SINGULAR

HAPPY MEMORIES OF CAMPUS

(A) Liberal Atmosphere

(B) Quiet, scenic surroundings

(C) On-campus accommodation

(D) Excellent Utilities

(E) Cultural, recreational facilities

HAPPIER MEMORIES OF JOB

(A) No muster!

(B) Light workload

(C) No bossing

(D) Job security

(E) Job satisfaction

SINGULAR MEMORIES

(A) Not a member of any group

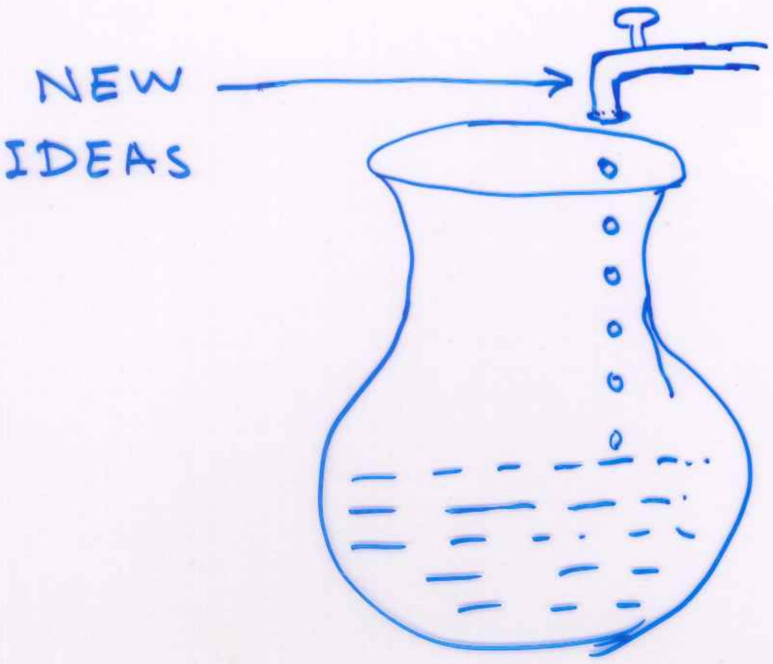
(B) Refused to subscribe to 'Publish OR perish'

(C) Equated research with innovation

(D) Wrote an article 'Publish And Perish'

(E) Got several singular awards

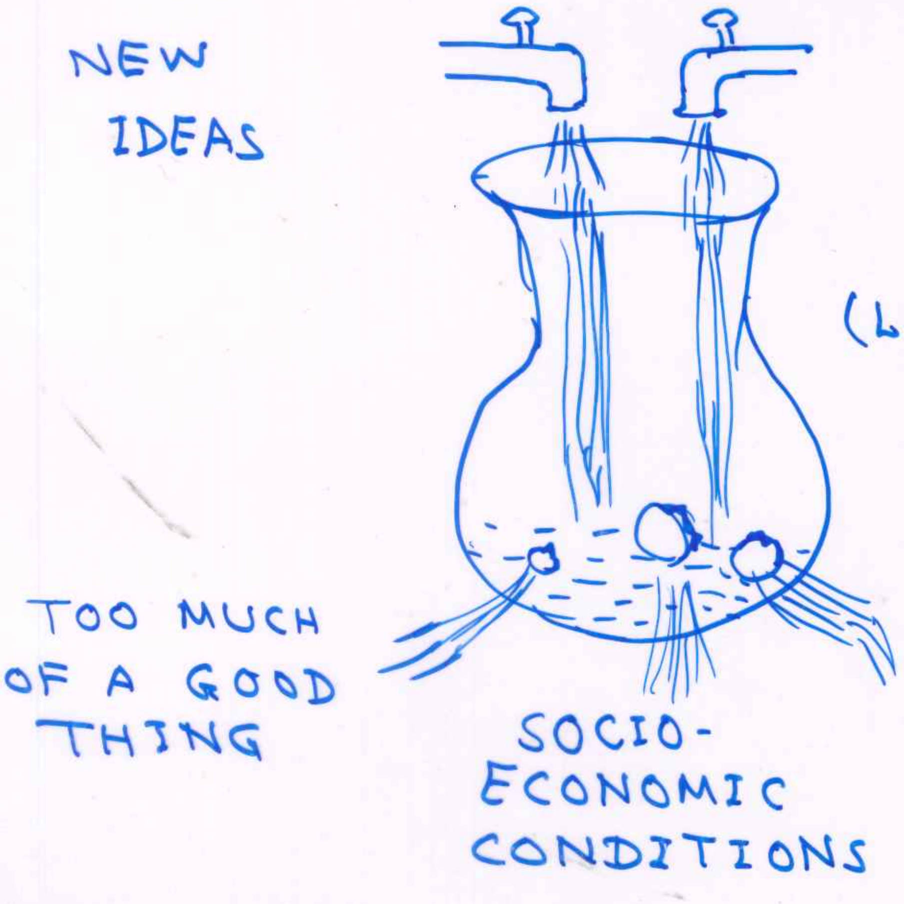
THEME OF THE TALK



PUBLISH OR (ELSE YOU) PERISH

(ENCOURAGE NEW IDEAS, AVOID STAGNATION)

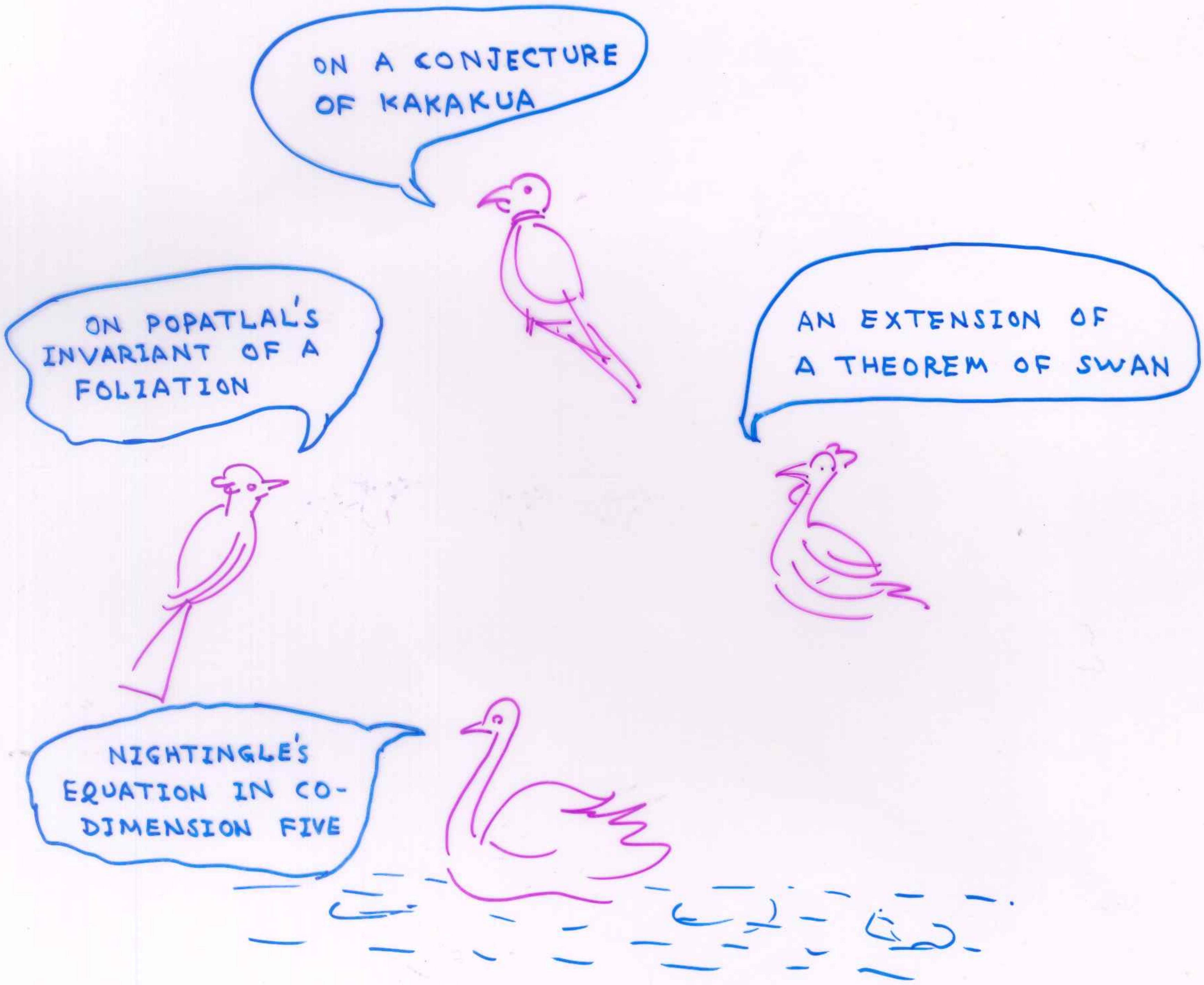
?



PUBLISH AND (LET MATHEMATICS) PERISH

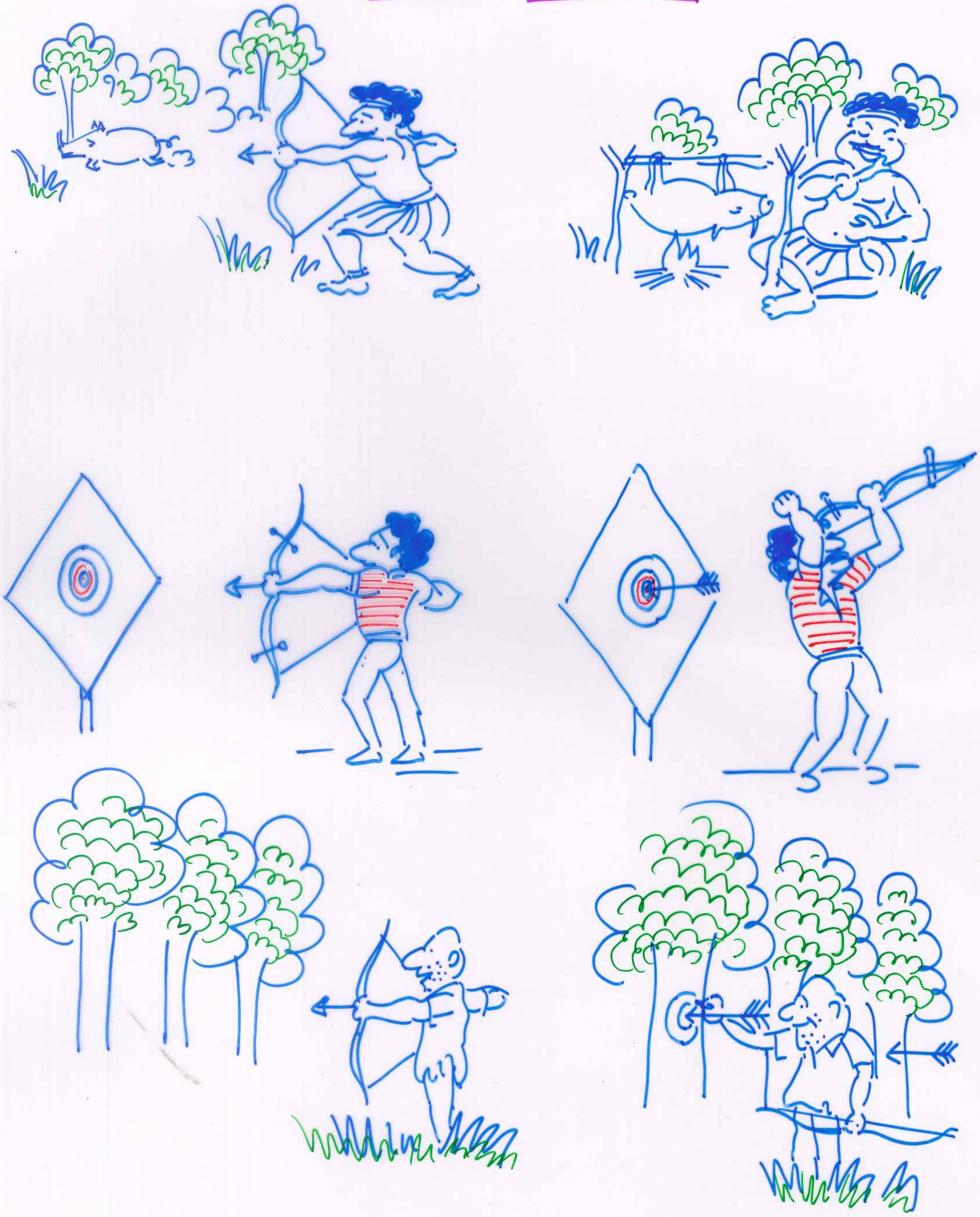
PECULIAR NATURE OF MATHEMATICS

BROTHERHOOD OF MATHEMATICS:



BIRDS OF THE SAME FEATHER!

THREE ARCHERS



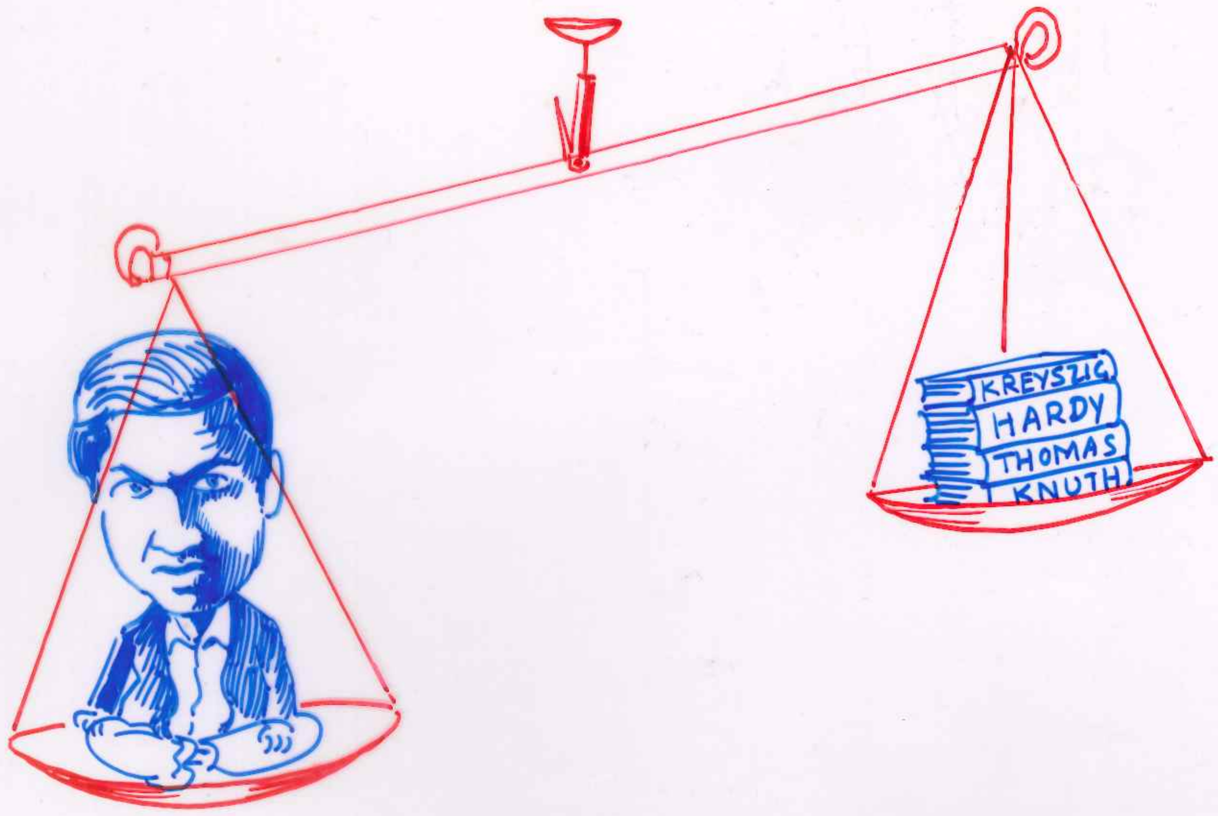
WHY DO I CLIMB THE MOUNTAIN?

BUSINESSMAN: BECAUSE THERE IS GOLD UP THERE.

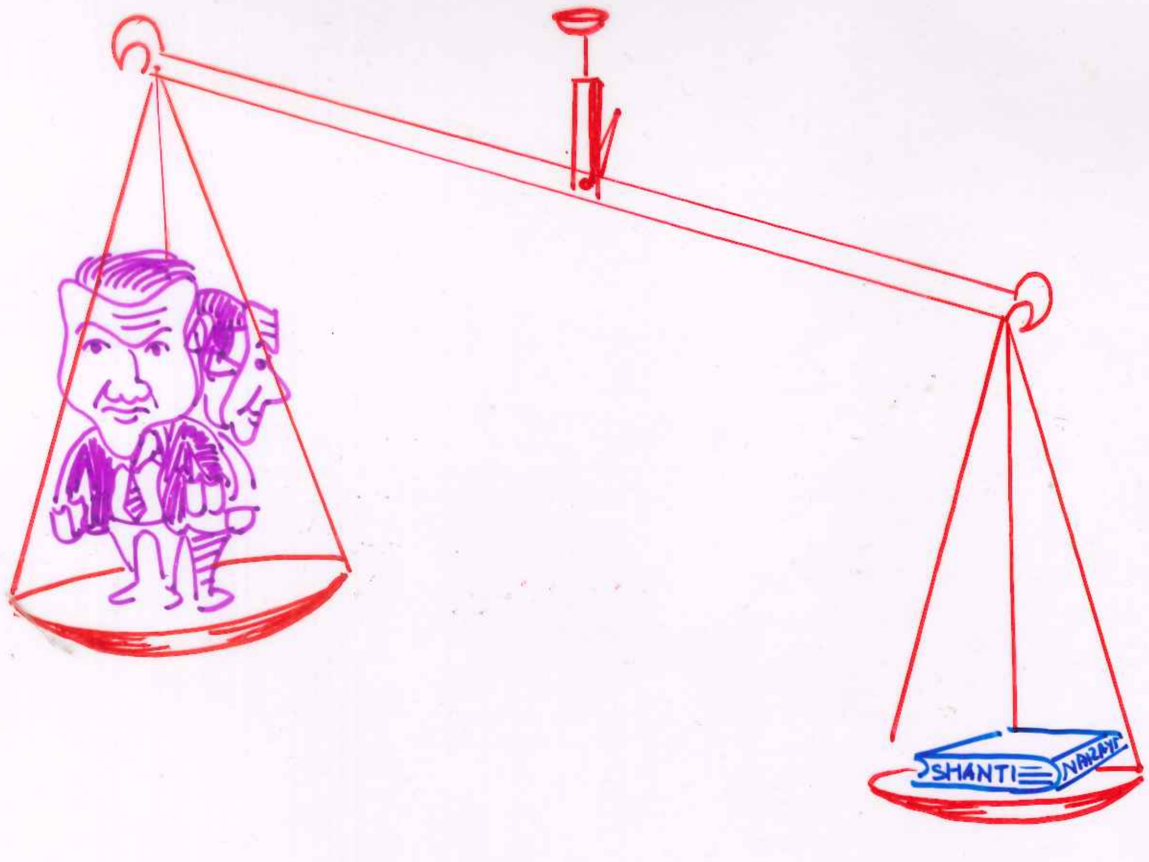
MOUNTAINEER: BECAUSE THE MOUNTAIN IS THERE!

MATHEMATICIAN:
BECAUSE I HAVE ERECTED THE MOUNTAIN THERE!

MAGIC OF MATHEMATICS



THE BALANCE OF MATHEMATICS:



If a man does not keep pace

with his companions,

perhaps it is because

he hears a different drummer.

Let him step to the music

which he hears.

Thoreau (1817-62)

SINGULARLY GRATEFUL TO IITB

THANKS