Unitary Invariants for Commuting tuples of Hypercontractions

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(A joint work with Bata Krishna Das and Jaydeb Sarkar)

Invariant Subspaces of $H^2(E)$

Let E be a separable Hilbert space of infinite dimension and $H^2(E)$ be the E-valued Hardy space, that is,

$$H^2(E) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \;\middle|\; \sum_{n=0}^{\infty} \|a_n\|_E^2 < \infty, z \in \mathbb{D} \right\}.$$

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Theorem (Beurling-Lax-Halmos)

A closed subspace $M \subseteq H^2(E)$ is invariant under M_z if and only if there exists a separable Hilbert space E_* and an inner multiplier Θ such that $M = \Theta H^2(E_*)$.

Sz.-Nagy-Foias model

Definition

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be in pure if $\|T^{*n}h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$.

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- The Sz.-Nagy and Foias analytic model: If T is a pure contraction then
$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$$
where $\mathcal{Q} \subseteq H^2(\mathcal{E}_*)$ is an M_z^*-invariant closed subspace.

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where $\mathcal{Q} \subseteq H^2(\mathcal{E}_*)$ is an M_z^*-invariant closed subspace.

Moreover, using Beurling-Lax-Halmos theorem, we have a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued inner multiplier Θ such that

$$\mathcal{Q}^{\perp} = \Theta H^2(\mathcal{E}).$$

Characteristic Function for contraction

For a contraction $T \in \mathcal{B}(\mathcal{H})$, consider the following contractive analytic function from $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$,

$$\Theta_T(z) = [-T + zD_{T^*}(1 - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \quad \text{for } z \in \mathbb{D},$$

where $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$, $\mathcal{D}_T = \overline{\text{ran}} D_T$ and $\mathcal{D}_{T^*} = \overline{\text{ran}} D_{T^*}$. It is known as the Characteristic Function for a contraction T.

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- Modified Sz.-Nagy-Foias model: If T is a pure contraction on a seperable Hilbert space $\mathcal{H}$, then there exist a co-invariant subspace $\mathcal{Q}$ of $H^2(\mathcal{D}_{T^*})$ such that

$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

And also the co-invariant subspace $\mathcal{Q}$ can be expressed in terms of the characteristic function of T, that is,

$$\mathcal{Q}^{\perp} = \Theta_T H^2(\mathcal{D}_T).$$

For an n-tuple of commuting bounded linear operators $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$, we say T is a row contraction if the operator

$$\mathcal{H}^n \to \mathcal{H},\ (h_1, \ldots, h_n) \mapsto \sum_{i=1}^{n} T_i h_i$$

is a contraction. That is, the operator viewed as a row operator $T : \mathcal{H}^n \to \mathcal{H}$ is a contraction.

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$$\sigma_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), X \mapsto \sum_{i=1}^{n} T_i X T_i^*.$$

Using the map σ_T, for each $k \in \mathbb{N}$, we are going to define the defect operators of different orders of T.

m-Hypercontraction Operators

Let us consider the operator for a positive integer k,

$$\Delta_T^{(k)} = (1 - \sigma_T)^k (I_{\mathcal{H}}).$$

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An n-tuple of commuting bounded linear operators T is said to be a *m-hypercontraction* if $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$.

The defect operator of order m is $D_{m,T^*} = (\Delta_T^{(m)})^{\frac{1}{2}}$ and the defect space is $\mathcal{D}_{m,T^*} = \overline{\text{ran}}(\Delta_T^{(m)})^{\frac{1}{2}}$.

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- An m-hypercontraction $T \in \mathcal{B}(\mathcal{H})^n$ is said to be *pure* if

$$\text{SOT} - \lim_{k \to \infty} \sigma_T^k (I_{\mathcal{H}}) = 0.$$

Theorem (Muller and Vasilescu)

Let T be a pure m-hypercontraction on a Hilbert space $\mathcal{H}$. Then there exists a co-invariant subspace $\mathcal{Q}$ of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that $T \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$.

The $\mathcal{E}$-valued weighted Bergman space

$$\mathbb{H}_\ell(\mathbb{B}^n, \mathcal{E}) := \Big\{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\|f_\alpha\|^2}{\rho_\ell(\alpha)} < \infty \Big\},$$

where $\rho_\ell(\alpha) = \frac{(\ell + |\alpha| - 1)!}{\alpha!(\ell - 1)!}$.

Model for m-Hypercontraction operators

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where $\rho_\ell(\alpha) = \frac{(\ell+|\alpha|-1)!}{\alpha!(\ell-1)!}$.

- It is also a reproducing kernel Hilbert space with kernel

$$K_\ell : \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{B}(\mathcal{E}), \quad K_\ell(z,w) = \frac{1_{\mathcal{E}}}{(1 - \langle z, w \rangle)^\ell}.$$

- In particular, if $\ell = 1$, $\mathbb{H}_1(\mathbb{B}^n, \mathcal{E})$ is known as the Drury-Arveson space and we use $H^2_n(\mathcal{E})$ to denote it.

Invariant Subspaces of Weighted Bergman space

Theorem (Sarkar)

A closed subspace of $M \subseteq \mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ is a joint $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$-invariant subspace if and only if there exists a Hilbert space $\mathcal{E}_*$ and a partial isometric multiplier Φ from $H^2_n(\mathcal{E}_*)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ such that $M = \Phi H^2_n(\mathcal{E}_*)$.

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- Combining above two results we have that given a m-hypercontraction T there exists a Hilbert space $\mathcal{E}_*$ and a partial isometric multiplier Φ from $H_n^2(\mathcal{E}_*)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ such that

$$T_i \cong P_{(\Phi H_n^2(\mathcal{E}_*))^\perp} M_{z_i}|_{(\Phi H_n^2(\mathcal{E}_*))^\perp} \qquad (i = 1, \dots, n).$$

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- Is there is any explicit description of that multiplier (analogous to the characteristic function for contraction)?

Construction of the Characteristic function

- For a m-hypercontraction T on $\mathcal{H}$, one can canonically define the operator $C_{m,T} : \mathcal{H} \to l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ such that $\begin{bmatrix} T^* \\ C_{m,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ is an isometry.

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- By adding a suitable Hilbert space $\mathcal{E}$, we can find $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$, $D \in \mathcal{B}\big(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})\big)$ such that

$$U = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$$

  is a unitary.

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Definition (Characteristic triple)

We call those triple $(\mathcal{E}, B, D)$ consists of a Hilbert space $\mathcal{E}$ and operators $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$, $D \in \mathcal{B}\big(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})\big)$ is a characteristic triple.

Construction of the Characteristic function

For a characteristic triple $(\mathcal{E}, B, D)$ of T, let us define an operator-valued analytic function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$ by

$$\Phi(z) = \sum_{\alpha\in\mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_\alpha z^\alpha + D_{m,T^*}(1 - ZT^*)^{-m} ZB \qquad \big(z \in \mathbb{B}^n\big).$$

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Theorem

Let T be a pure m-hypercontraction on $\mathcal{H}$. Suppose $(\mathcal{E}, B, D)$ is a characteristic triple of T. Then the above defined function Φ defines a partial isometric multiplier from $H^2_n(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that $\mathcal{Q}^\perp = \Phi H^2_n(\mathcal{E})$, where $\mathcal{Q}$ is the model space for T.

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Theorem

For an another characteristic triple $(\mathcal{E}_1, B_1, D_1)$, consider the partial isometric multiplier Φ_1 from $H_n^2(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ defined analogously. Then there exists a unitary $I \otimes U : H_n^2(\mathcal{E}_1) \to H_n^2(\mathcal{E})$ such that $M_{\Phi_1} = M_\Phi(I \otimes U)$.

Unitary Equivalence

Two row contractions $T = (T_1, \cdots, T_n)$ and $R = (R_1, \ldots, R_n)$ on $\mathcal{H}$ are said to be unitary equivalent if there exist a unitary U on $\mathcal{H}$ such that $T_i = UR_iU^*$, for all $i = 1, \ldots, n$.

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Definition

The characteristic functions Φ_T and Φ_R of two pure m-hypercontractions T and R are said to coincide if there exists two unitary $\Gamma : (\text{Ker} M_{\Phi_R})^\perp \to (\text{Ker} M_{\Phi_T})^\perp$ and $\tau : \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,R^*}$ such that

$$M_{\Phi_R}|_{(\text{Ker} M_{\Phi_R})^\perp} = (I \otimes \tau) M_{\Phi_T} \Gamma.$$

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$$M_{\Phi_R}|_{(\mathrm{Ker} M_{\Phi_R})^{\perp}} = (I \otimes \tau) M_{\Phi_T} \Gamma.$$

Theorem

Two pure m-hypercontractions are unitarily equivalent if and only if their characteristic functions coincide.

Factorization w.r.t invariant subspace

- We are going to produce a necessary and sufficient condition for the existence of an joint invariant subspace of a pure m-hypercontraction in terms of a factorization the characteristic function.

Factorization w.r.t invariant subspace

- We are going to produce a necessary and sufficient condition for the existence of an joint invariant subspace of a pure *m*-hypercontraction in terms of a factorization the characteristic function.

Theorem

Let $T = (T_1, \ldots, T_n)$ be a pure m-hypercontraction on $\mathcal{H}$. Then T has a joint invariant subspace if and only if there exists two Hilbert spaces $\mathcal{E}_T$ and $\mathcal{E}$, and two multipliers Φ_1 from $H^2_n(\mathcal{E}_T)$ to $H^2_n(\mathcal{E}_2)$ and Φ_2 from $H^2_n(\mathcal{E}_2)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that the characteristic function of T has a factorization, that is,

$$\Phi_T = \Phi_2 \Phi_1.$$

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