Quantum invariants for Knots and Links

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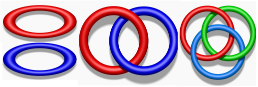
Knots and Links

Definition

Links are smooth, 1 dimensional, closed submanifolds of $\mathbb{R}^3$ *or* S^3. *If a link is connected it is a knot.*



Classifying Knots and Links

Definition

Two links L_1 and L_2 are said to be equivalent (ambient isotopic) if there exists an orientation preserving diffeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$..

Problem: Classify links on the basis of ambient isotopy.

Definition

Any quantity/structure/polynomial that helps us in distinguishing knots or links is known as a knot invariant.

Examples: bridge number, knot group, Jones polynomial.

Quantum invariants

Definition

A family of knot invariants that are defined with the help of Topological Quantum Field Theory (TQFT) are collectively known as Quantum invariants.

Example: Jones polynomial, Colored Jones polynomial, Khovanov Homology etc.

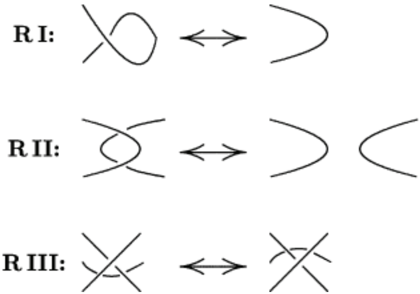
Main Task: Find effective ‘Link invariants’

Two successful approaches:

1. Combinatorial (represent a link using a ‘diagram’), Thanks to P.G.Tait (1867) and Reidemister (1927).

2. Using braids (represent a link by closure of a braid), Thanks to Alexandar (1923) and Markov (1925).

Combinatorial approach- **Reidemeister Moves**:



Braid Closure— **Markov Moves.**

- Type 1: $\alpha \in B_n$ can be changed to $\gamma\alpha\gamma^{-1}$ for all $\gamma \in B_n$
- Type 2: $\alpha \in B_n$ can be changed to $\alpha\sigma_n$ or $\alpha\sigma_n^{-1}$

The Jones polynomial $V_K(t)$

The original definition by V.F.R. Jones (1984) (Fields medal in 1990).

- Let K be a knot which is closure of an $n-$braid β. Then

$$V_K(t) = (\frac{-(t+1)}{\sqrt{t}})^{n-1} tr(r_t(\beta)).$$

- $r_t : B_n \to A_n$ is a representation of B_n into $n-$ dimensional von-Neumann Algebra A_n over $\mathbb{C}$.

For a fixed $t \in \mathbb{C}, t \neq 0, -1$

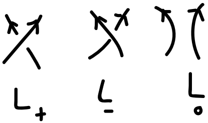
- A_n is generated by 1 and $n-1$ projections $e_1, e_2, \ldots, e_{n-1}$ satisfying
  (i) $e_i^2 = e_i$, $e_i^* = e_i$(ii)$e_i e_j = e_j e_i$ for $|i-j| > 1$ (iii) $e_i e_{i\pm 1} e_i = \frac{t}{(t+1)^2} e_i$ ($*$ means conjugate-transpose)
- $r_t(\sigma_i) = g_i$ where $g_i) = \sqrt(t)(te_i - (1-e_i))$.
- $tr : A_n \to \mathbb{C}$ is determined by $tr(1) = 1$ and (i) $tr(ab) = tr(ba)$ (ii) $tr(we_n) = \frac{t}{(1+t)^2} tr(w)$ if $w \in A_n$ (iii) $tr(aa^*) > 0$ for $a \neq 0$
- $A_0 = \mathbb{C}$
- A finite dimensional von-Neumann algebra is product of matrix algebras.

Skein Relation

- For any skein related triples (L_+, L_-, L_0) V_L satisfies

$$tV_{L_+} - t^{-1}V_{L_-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0} = 0$$

where



- Using the above relation along with the normalization $V_{Unknot}(t) = 1$ one can compute $V_L(t)$ for any link.

General Skein Invariant Theory

- Skein relations were first observed by John Conway (1969) as an inductive step to compute his polynomial invariant known as **Conway polynomial**.
- A group of mathematicians defined a two variable polynomial invariant called HOMFLYPT which is an element of $\mathbb{Z}[l, l^{-1}, m, m^{-1}]$ satisfying a skein relation.

Universal Property of a Skein Invariant

- It can be proved that a skein invariant P taking values in a Commutative ring R with 1 is uniquely determined by three invertible elements a_+, a_- and a_0 if $a_+P_{L_+} + a_-P_{L_-} + a_0P_{L_0} = 0.$
- This triggered the Maths Community to re-look into the Jones Polynomial.

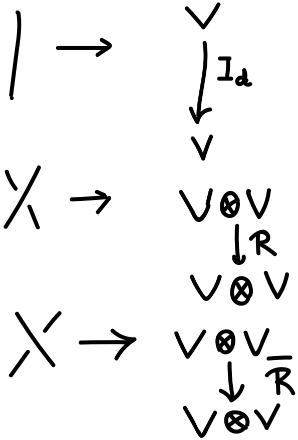
State Sum Models for Jones Polynomial

- Louis Kauffman defined his bracket polynomial in 1987, which was turned into an invariant and with a change of variable became an element of $\mathbb{Z}[t, t^{-1}]$ and satisfied the same skein relation, hence can be treated as another definition for the Jones polynomial.
- In 1989 Witten observed a connection of Jones polynomial with Quantum Field Theory.
- Witten's theory is an example of a TQFT.

TQFT

- An n dimensional TQFT is a monoidal functor from **n-Cob** to $Vect_K$ satisfying certain conditions.
- 1 dimensional TQFT is simply Vector spaces and linear transformations.
- Braids can be regarded as morphisms in **1-Cob**.
- Thus a 1 **dimensional TQFT** will associate an $n-$ braid to an element of $End(V^{\otimes n})$, giving rise to map $B_n \to End(V^{\otimes n})$. This may not be a representation though.
- Thus a choice of V and a suitable TQFT is important to get a representation first then combining it with a trace function from $End(V^{\otimes n})$ to K may become a link invariant with some modifications.

Representations of B_n using 1 dim TQFT



Yang -Baxter Equation

- If the previous association is a representation for B_n it must respect the braid relation: Hence we must have

$$(R \otimes I) \circ (I \otimes R) \circ (R \otimes I) = (I \otimes R) \circ (R \otimes I) \circ (I \otimes R)$$

each being maps $V \otimes V \otimes V \to V \otimes V \otimes V$. $\qquad (**)$

- $(**)$ is known as Yang-Baxter Equation. A linear map $R : V \otimes V \to V \otimes V$ satisfying the Yang Baxter Equation is called an $R-$ matrix or an Yang-Baxter operator.
- Thus we need to find R matrices to define such representations of B_n which can be used to define link invariants.

Finding R matrices

- Finite dimensional irreducible representations of Hopf algebras naturally give rise to R matrices.
- A $2-$ dimensional irreducible representation of $U_q(sl(2))$ gave rise to an $R-$ matrix (with $q^2 = t$)

$$M = \begin{bmatrix} t^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & t^{\frac{1}{2}} - t^{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & t^{\frac{1}{2}} \end{bmatrix}$$

which is used in defining the Jones polynomial. (Another definition for Jones Polynomial)

The quantum group $U_q(sl(2))$

- $U_q(sl(2))$ is generated by E, F, K, K^{-1} subject to: $KK^{-1} = K^{-1}K = 1$, $KEK^{-1} = q^2E$, $KFK^{-1} = q^{-2}F$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$.
- It has a Hopf algebra structure: $\Delta(E) = E \otimes 1 + K \otimes E$, $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$, $\Delta(K) = K \otimes K$. $\eta(E) = \eta(F) = 0$, $\eta(K) = 1$. $S(E) = -K^{-1}E$, $S(F) = -FK$, $S(K) = K^{-1}$.

Colored Jones Polynomial

Let V_N denote an N dimensional irreducible representation of $U_q(sl(2))$, having basis $\{v_1, v_2, \ldots, v_N\}$. Then R matrix is given by $R : V \otimes V \to V \otimes V$ defined as:

$R(v_i \otimes v_j) = qv_j \otimes v_i$ if $i = j$

$= v_i \otimes v_j$ if $i < j$

$= v_i \otimes v_j + (q - q^{-1})v_j \otimes v_i$ if $i > j$.

This defines a link invariant called **Nth Colored Jones polynomial denoted by** $J_N(K, q)$**.** Here $J_2(K, q) = V_K(t^{\frac{1}{2}})$.

Significance of Colored Jones Polynomial

- In 1980s, William Thurston's seminal work established a strong connection between hyperbolic geometry and knot theory, namely that most knot complements are hyperbolic. Thurston introduced tools from hyperbolic geometry to study knots that led to new geometric invariants, especially hyperbolic volume.
- As knots are determined by their complements, it immediately created a curiosity that there must be connection between topological invariants and the geometric invariants of knots.

Volume Conjecture

- $$Vol(S^3 \setminus K) = 2\pi Lim_{N\to\infty} \frac{Log|(J_N(K, exp(2\pi \frac{\sqrt{(-1)}}{N})|}{N}.$$

  This is an open problem. If the volume conjecture is true it will give interesting relations between quantum topology and hyperbolic geometry.
- Till now, among all the Hyperbolic knots, Volume Conjecture is verified only for Figure eight knot.
- Even to verify for other knots or class of knots we need a closed formula for $J_N(K, q)$.
- Note that there is no skein relation known for $J_N(K, q)$.

More on Colored Jones Polynomial

- Recall: The knot polynomial obtained by ‘coloring’ the (zero framed) knot K with the irreducible representation V_N is the $N-$ colored Jones polynomial $J_N(K)$. We get $J_N(Unknot) = [n] = \frac{t^{N/2} - t^{-N/2}}{t^{1/2} - t^{-1/2}}$.
- $J_2(K) = (t^{1/2} + t^{-1/2})V_K$.
- We may also color K by non-irreducible representation say $V_2^{\otimes N}$ and obtain polynomials say $J(K, V_2^{\otimes N})$.
- For a zero framed knot K let K^N denote the link obtained by replacing K with N parallel copies , then $J(K, V_2^{\otimes N}) = J_2(K^N)$.
- Using the representation theory of $U_q(sl(2))$ we have $V_2 \otimes V_N = V_{N+1} \oplus V_{N-1}$. Using this one gets $J_{N+1}(K) = \Sigma_{j=0}^{j=N/2} (-1)^j \binom{N-j}{j} J_2(K^{N-2j})$. This helps in developing State Sum models.

Colored Jones Polynomial for few knots

- $J_N(3_1) = q^{N-1}\Sigma_{n=0}^{\infty} q^{nN}(1-q^{N-1})(1-q^{N-2})\cdots(1-q^{N-n})$.

- $J_N(4_1) = q^{1-N}\Sigma_{n=0}^{N-1}\Sigma_{k=0}^{n}\binom{n}{k}_q q^{n+k(k+1)}\{\Pi_{j=1}^{n}(1-q^{j-N})\}\{\Pi_{i=1}^{n-k}(1-q^{k+i-N}\}$.

  Here $\binom{n}{k}_q = \Pi_{i=0}^{k-1}\frac{1-q^{n-i}}{1-q^{i+1}}$. (q-binomial coefficients)

State Sum Models

- Using the known models for Jones polynomial.
- V.Huynh and T.T.Le (2007) gave a formula for braid closure using inverse of the q-determinant of an almost quantum matrix.
- C. Armond (2014) proposed a **Jump down** model using braid walks and found a state sum model by summing some quantities over all **Simple Walks**. These quantities do not commute and hence involves very complex $q-$ multinomial coefficients.
- Garoufalidis and Lobel (2006) had proposed a **jump up** model from a knot diagram by associating a di-graph with it where the summation is done over flows of the di-graph.

Our work

- We show that the states in both the models are in one to one correspondence.
- We extend the jump up model for links as well.
- We simplify the model in a more computable form and find a Multi-Sum formula for Weave knots $W(3,l)$.
- As Weave knots are Hyperbolic knots, there is a hope to verify Volume Conjecture (Sounds an ambitious plan!!)

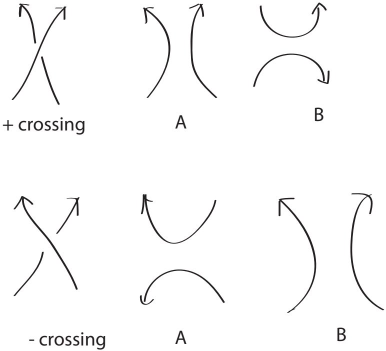
Khovanov Homology: 2 dim TQFT invariant

- Khovanov Homology categorifies the Jones polynomial.
- It is defined using 2 dim TQFT.

Khovanov Homology

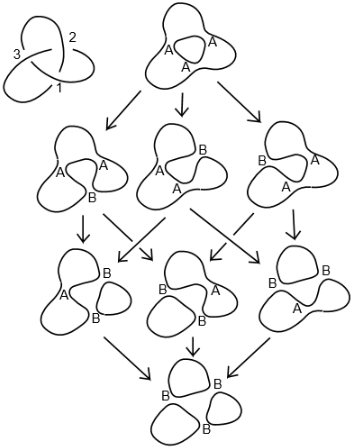
Some background :

Smoothing:



Resolution cube

States: Outcome of a choice of A or B smoothing at each crossing. There are 2^n states for a diagram with n crossings.



2 dim TQFT

- $2-$Cob category is a monoidal category: objects- 1 dimensional, closed, smooth manifolds; morphisms are cobordisms between them. Product is disjoint union.
- We have another monoidal category $\mathcal{M}_R$, category of R modules, R any ring.
- Any monoidal functor F from $2-$Cob to $\mathcal{M}_R$ is a TQFT.
- we can apply a TQFT on the cube category of the diagram.

Khovanov Complex

We choose a TQFT that assigns a graded module $A = A_+ \oplus A_-$, A_+ is generated by $\{1\}$ has grading $+1$, A_- is generated by $\{x\}$ with grading -1. Define the chain group C_i to be the graded R modules generated by all the states with number of B smoothing =i. The boundary maps are defined using combination of following two maps:

Multiplication: $m : A \otimes A \to A$ defined as $m(1 \otimes 1) = 1$, $m(1 \otimes x) = m(x \otimes 1) = x$, $m(x \otimes x) = 0$.

Comultiplication $\Delta : A \to A \otimes A$ defined as $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$.

This gives rise to a co-chain complex, where boundary map is of bi-degree $(1, 0)$. $(i, j)th$ homology of this complex is denoted by $KH^{i,j}(L)$. **It is invariant under all three Reidemeister Moves and hence is a Link invariant.**

Khovanov Homology is Unknot detector

Theorem

Let K be a knot (1 component link). K is ambient isotopic to the unknot iff $KH^{i,j} \approx K$ for $(i,j) = (0,1),(0,-1)$ AND $\{0\}$ otherwise.

Our work

We have computed the ranks of Khovanov Homology groups for Weave knots $W(3, n)$.

THANK YOU