Schematic Harder-Narasimhan Stratification for Families of Principal Bundles

In this talk, I will summarize the results of 3 papers of mine written jointly with Nitin Nitsure. These papers are on the same overall theme. Although our results are valid for families of principal bundles with a reductive algebraic group as structure group, I will for the most part of the talk stick to vector bundles, or equivalently principal GL_n-bundles. Also, in the case of vector bundles, many of these results hold for singular projective varieties as well. I will interchangeably go from vector bundles of rank n and locally free sheaves of rank n. We begin with the following basic definitions :

Definition

Let X be a smooth, projective variety over k and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let E be a vector bundle on X (or equivalently a locally free sheaf).

We define the degree of E (w.r.t $\mathcal{O}_X(1)$) to be

$$\deg(E) = c_1(E).c_1(\mathcal{O}_X(1))^{n-1}$$

Define the slope of E to be the ratio : μ(E)=deg(E)/rank(E)

Definition

A vector bundle E is said to be μ-**semistable (resp. stable)** if for any proper subsheaf $F \subset E$, we have the inequality $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$). μ- (semi)stability is also called Mumford-Takemoto (semi)stability and is intimately connected to the notion of (semi)stability of points for the action for reductive algebraic groups on projective varieties in the sense of Geometric Invariant Theory.

Notice that when the dimension of X is one, the notions of semistability and stability are independent of the choice of the very ample line bundle.

Examples:

1. The trivial bundle of any rank is semistable. It is stable if and only if the rank is 1.

2. If $0 \to E' \to E \to E'' \to 0$ is a short-exact sequence of vector bundles, such that E' and E'' are semistable of slope μ, then E is also semistable of slope μ (but not stable).

3. The tangent bundle of $\mathbb{P}^n$ is stable. Same is true for the Grassmannian.

4. If characteristic k is zero, then tensor products, symmetric and exterior powers of semistable (stable) bundles are again semistable (resp. polystable).

5. If X is a smooth, projective variety over $\mathbb{C}$, bundles coming from irreducible unitary representations of $\pi_1(X)$ are stable with vanishing chern classes. The converse of this is the celebrated theorem of Narasimhan-Seshadri.

Moduli Problem

The following theorem motivates the above definitions.

Theorem

Let X be a smooth, projective curve over a field $k = \bar{k}$. Then there exists a normal, projective variety of dimension $r^2(g-1)+1$ "parametrizing" rank r, degree d, semistable bundles on X.

The open subset of stable bundles is smooth.

Analogues theorems exist in higher dimensions as well (Maruyama, Gieseker, Simpson, Langer ...).

Schematic Harder-Narasimhan stratification

Around 1975, Harder and Narasimhan introduced a filtration of a vector bundle, now known as the Harder-Narasimhan filtration which proved to be of central importance in the subject.

Theorem

Let X be a smooth projective curve over any field k. Let E be a vector bundle on X. Then there exists a unique filtration of E by subbundles:

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$

such that:

1. *The successive quotients $E^i = E_i/E_{i-1}$ are locally-free and semistable.*
2. $\mu(E^i) > \mu(E^{i+1}) \ \forall i$

Similar filtration exists in higher dimensions as well. In higher dimensions, the filtration is by coherent subsheaves with torsion-free successive quotients. It is also unique (but the uniqueness has to be formulated more carefully).

The collection of all n-tuples of pairs $\{(\mathrm{rk}(E_1), \deg(E_1)), \cdots, (\mathrm{rk}(E_n), (\deg(E_n))\}$ is called the Harder-Narasimhan type of E, denoted $HN(E)$. Associated to such a type, there is a polygon called the Harder-Narasimhan polygon. It is the convex polygon with vertices $(\mathrm{rk}(E_i), \deg(E_i))$. There is a natural partial ordering on the set of Harder-Narasimhan types of all possible bundles on a given space. Given two HN types τ_1 and τ_2 corresponding to vector bundles E_1 and E_2, we say $\tau_1 \leq \tau_2$ if the corresponding HN polygon for τ_1 has the same end point as that of τ_2 (starting point is always $(0, 0)$ and the HN polygon for τ_1 lies below the polygon for τ_2.

I will now state our main theorem for vector bundles.

Theorem

(Main Theorem) *Let X/S be a smooth projective family of geometrically irreducible varieties over a locally noetherian scheme S/k and let E be a vector bundle on X. Then for each HN-type τ that occurs in this family, there exists a unique locally closed subscheme $S^{\tau}(E) \subset S$ with the following universal property: Any morphism $f : T \to S$ of k-schemes factors via $S^{\tau}(E) \subset S$ if and only if the pullback $E_T = f^*E$ on X_T admits a relative HN-filtration of type τ on T. Moreover, a relative HN-filtration is unique.*

Sketch of proof:
1. One first proves this theorem for a smooth family of curves. This was already known for vector bundles.
2. By invoking the Mehta-Ramanathan restriction theorem, one then shows that after passing to an open cover of S if needed, one can find a smooth, relative hypersurface $Y \subset X$ such that at any point $s \in S^{\tau}(E)$, the HN-filtration on E_s/X_s restricts to the HN-filtration of $E_s|_{Y_s}$. This is rather subtle and requires a certain double semicontinuity argument.
3. Using this one exhibits the desired closed subscheme $S^{\tau}(E)$ as a closed subscheme of $S^{m\tau}(E|_Y/Y/S)$.

The proof uses lot of the standard "Grothendieckian" Algebraic Geometry - relative Picard Schemes, descent theory, deformation theory, relative duality, semicontinuity theorems, m-regularity etc.

We also prove the analogues theorem for Principal bundles in general. Here the characteristic of the field plays an important role. When proving it for Principal Bundles, basic facts about structure of reductive groups and their representations are also used. There is also some subtlety in defining the right analogue of the relative HN-filtration for Principal bundles.

Corollary

Under the hypothesis of the above theorem, if moreover S is reduced and the HN type is globally constant over S, there exists a relative HN-filtration over S.

This follows from the fact that under the above hypothesis, $S^{\tau}(E) = S$ set-theoretically. Since a reduced scheme does not admit a proper closed subscheme which equals it set-theoretically, it follows that $S^{\tau}(E) = S$ scheme-theoretically as well.

Corollary

Vector bundles of any given Harder-Narasimhan type τ on fibers of X/S form an Artin algebraic stack $Bun^{\tau}_{X/S}$ and as τ varies, these stacks define a stratification of the stack $Bun_{X/S}$ by locally-closed substacks.

This follows from combining two facts; namely that all families of Principal bundles form an Artin stack and the schematic stratification discussed above.