Interpolation from affine hypersurfaces

Vamsi Pritham Pingali

IISc

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- These functions give examples of "coherent states" in quantum optics, this problem apparently plays a role in signal processing, and the higher dimensional version of this problem might help with the minimal model programme (Demailley-Hacon-Paun).

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- $f = hg$ interpolates f_n.
- The problem is with the L^2 bound !

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- They must not be too dense, i.e., $\limsup_{r\to\infty} \sup_z \dfrac{\#D(z,r)}{\pi r^2} < \dfrac{1}{\pi}$.

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- No necessary conditions are known.
- Some sufficient conditions are known (Ortega-Cerda-Schuster-Varolin, Pingali-Varolin).

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- Theorem : (OSV, PV) A uniformly flat smooth algebraic hypersurface is interpolating.
- Actually, the above theorem works for certain singular algebraic hypersurfaces too (PV).

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- The Ohsawa-Takegoshi extension theorem implies that interpolation can be done for algebraic hypersurfaces if $|dT|^2 e^{-\fint_{B(z,r)} \ln(|T|^2)} \geq C$.
- Uniform flatness implies the desired lower bound.

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Thank you