On the Positivity and Vanishing of the Coefficients of Normal Hilbert Polynomials

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Integral closure of Ideals

Let R be a commutative ring, I an ideal of R. An element $a \in R$ is called integral over I, if a satisfies an equation:

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

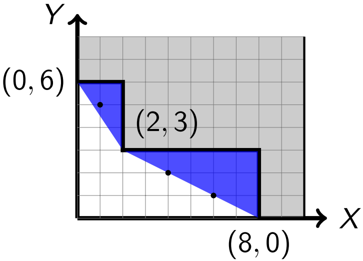
where $a_i \in I^i$ for $i = 1, 2, \ldots, n$. The integral closure $\overline{I}$ of I, is the ideal

$$\overline{I} = \{a \in R \mid a \text{ is integral over } I\}.$$

An ideal I is called complete if $\overline{I} = I$.
O. Zariski, *Polynomial ideals defined by infinitely near base points*, American Journal of Mathematics (1938), 151-204. In this paper, he studied integral closures of ideals in $k[x, y]$, where k is an algebraically closed field of characteristic zero.
These results were generalized to two dimensional regular local rings in Zariski-Samuel, Volume II.

Integral closure and convex hull

The integral closure of a monomial ideal has a nice description in convex geometry.
Let $R = k[x, y]$ and $I = (x^8, x^2y^3, y^6)$.



The monomials corresponding to the lattice points in the blue area are the monomials which when added to I generate the integral closure of I.
Therefore $\overline{I} = I + (x^6y, x^4y^2, xy^5)$.

The Newton polyhedron of a monomial ideal

Let $R = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field. For a subset X of R, put

$$\Gamma(X) = \{\alpha \in \mathbb{N}^n \mid x^\alpha \in X\}.$$

The Newton Polyhedron $NP(I)$ of a monomial ideal I of R is defined to be the convex hull in $\mathbb{R}^n$ of $\Gamma(I)$.

Theorem: (B. Teissier, 1975) Let I be a monomial ideal of R. Then the integral closure of I is also a monomial ideal and $\Gamma(\bar{I}) = NP(I) \cap \mathbb{N}^n$.

Theorem: (L. Reid, L.G. Roberts, M. A. Vitulli, 2003) Let I be a monomial ideal in R. Suppose that $I, I^2, \ldots, I^{n-1}$ are complete then $I^r = \overline{I^r}$ for all $r \geq n$.

Theorem: (S. K. Masuti, T. J. Puthenpurakal, J. K. Verma, 2015) Let k be a field, $R = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. Suppose that I, J, K are $\mathfrak{m}$-primary monomial ideals of R such that $I^r J^s K^t$ is complete for all $r + s + t \leq 2$. Then $I^r J^s K^t$ is complete for all $r, s, t \geq 0$.

Normal Hilbert polynomials

For any $\mathfrak{m}$-primary ideal I in an analytically unramified local ring $(R, \mathfrak{m})$ of dimension d, the normal Hilbert function $\overline{H}_I(n) = \lambda(R/\overline{I^n})$ for large n, is given by the normal Hilbert polynomial $\overline{P}_I(x)$:

$$\overline{P}_I(x) = \overline{e}_0(I)\binom{x+d-1}{d} - \overline{e}_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d \overline{e}_d(I),$$

for some integers $\overline{e}_0(I), \overline{e}_1(I), \ldots, \overline{e}_d(I)$ called the normal Hilbert coefficients of I.

Theorem: (Rees, 1981) A 2-dimensional normal, analytically unramified local ring $(R, \mathfrak{m})$ is pseudo-rational if and only if $\overline{e}_2(I) = 0$ for all $\mathfrak{m}$-primary ideals. Moreover, for any $\mathfrak{m}$-primary ideal I, $\overline{H}_I(n) = \overline{P}_I(n)$ for all $n \geq 0$ and

$$\overline{P}_I(n) = e(I)\binom{n+1}{2} - \overline{e}_1(I)n.$$

Graded algebras for the normal filtration of an ideal

Let $\mathcal{F} = \{\overline{I^n}\}$ be the normal filtration of an ideal I and t be an indeterminate. We use three blow up algebras associated to $\mathcal{F}$.

$$\begin{aligned}
\text{Rees algebra of } \mathcal{F} &= \overline{\mathcal{R}}(I) = \bigoplus_{n=0}^{\infty} \overline{I^n}t^n \\
\text{Extended Rees algebra of } \mathcal{F} &= \overline{\mathcal{R}'}(I) = \bigoplus_{n\in\mathbb{Z}} \overline{I^n}t^n \\
\text{Associated graded ring of } \mathcal{F} &= \overline{G}(I) = \bigoplus_{n=0}^{\infty} \overline{I^n}/\overline{I^{n+1}}
\end{aligned}$$

Theorem: (Rees, 1961) Let $(R, \mathfrak{m})$ be an analytically unramified local ring and $\mathcal{F}$ be the normal filtration of an ideal I. Then
(1) $\overline{\mathcal{R}}(I), \overline{\mathcal{R}'}(I)$ are finite modules over $\mathcal{R}(I)$ and $\mathcal{R}'(I)$ respectively.
(2) $\overline{G}(I)$ is a finite module over $G(I)$.
(3) $\dim \mathcal{R}'(\mathcal{F}) - 1 = \dim G(\mathcal{F}) = d$.

Theorem: (Valla, 1979)
$\dim \mathcal{R}(\mathcal{F}) = d + 1 \Leftrightarrow I_1 \not\subseteq \bigcap\{\mathfrak{p} \mid \dim R/\mathfrak{p} = \dim R\}$.

Postulation number and reduction number of $\{\overline{I^n}\}$

Let $(R, \mathfrak{m})$ be a d-dimensional analytically unramified Noetherian local ring and let $\mathcal{F} = \{\overline{I^n}\}$ be the normal filtration of an $\mathfrak{m}$-primary ideal I.

The Postulation number of $\mathcal{F} := \overline{n}(I) = \max\{n \mid \overline{P}_I(n) \neq \overline{H}_I(n)\}$.

An ideal $J \subset I$ is called a reduction of $\mathcal{F}$ if $J\overline{I^n} = \overline{I^{n+1}}$ for all large n. The reduction number of $\mathcal{F}$ with respect to J is defined as

$$\overline{r}_J(I) = \min\{m \mid J\overline{I^n} = \overline{I^{n+1}} \text{ for all } n \geq m\}.$$

Let I be generated by a system of parameters. Then the normal reduction number of I is defined as $\overline{r}(I) = \overline{r}_I(I)$.

Theorem: (S. Huckaba, T. Marley, 1988) Let R be an analytically unramified CM local ring of dimension d. Suppose depth $\overline{G}(I) \geq d - 1$. Then

(1) $\overline{r}(I) = \overline{n}(I) + d$.

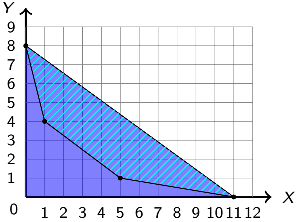
(2) $\overline{e}_k(I) = 0 \iff \overline{r}(I) \leq k - 1$.

Two polytopes associated to monomial ideals

Let $R = k[x_1, x_2, \ldots, x_d]$ be a polynomial ring over a field k. Let $v_1, v_2, \ldots, v_q \in \mathbb{N}^d$ and $I = (x^{v_1}, x^{v_2}, \ldots, x^{v_q})$ be an $\mathfrak{m}$-primary ideal. Then there are natural numbers $a_1, a_2, \ldots, a_d$ such that $v_i = a_i e_i$ for $i = 1, 2, \ldots, d$. Put $a = (1/a_1, 1/a_2, 1/a_3, \ldots, 1/a_d)$.
Let $\langle v_i, a \rangle < 1$ for $i = d+1, \ldots, s$ and $\langle v_i, a \rangle \geq 1$ for $i = s+1, \ldots, q$.

Let $P = \text{conv}\,(v_1, v_2, \ldots, v_s)$, $S = \text{conv}\,(0, v_1, v_2, \ldots, v_d)$ and $Q = \mathbb{Q}_+^d + \text{conv}(v_1, v_2, \ldots, v_q) = \mathbb{Q}_+^d + P$.

Example. Let $R = k[x, y]$ and $I = (x^{11}, y^8, x^5y, xy^4)$.



The coloured area denotes S and the shaded area denotes P.

Normal Hilbert polynomial of a monomial ideal

Theorem: (Ehrhart, 1962) Let P be an integral convex polytope of dimension d. Then the function

$$E_P(n) = |nP \cap \mathbb{Z}^d|$$

is a polynomial function in n of degree d with rational coefficients.

Theorem: (Villarreal, 2008) $\lambda(R/\overline{I^n}) = |\mathbb{N}^d \setminus nQ| = E_S(n) - E_P(n)\ \forall n$ and

$$\overline{P}_I(n) = [\text{vol}(S) - \text{vol}(P)]n^d + \text{lower degree terms.}$$

Theorem: (W. V. Vasconcelos, 2005) $\overline{r}(I) \leq d - 1$.

Theorem: (Villarreal, 2008) $\overline{e}_i(I) \geq 0$, for all i.

Proof: We may assume without loss of generality that k is infinite. The Rees algebra $\overline{\mathcal{R}}(I) = \bigoplus_{n=0}^{\infty} \overline{I^n} t^n$ is a normal semigroup ring.

Normal Hilbert polynomial of monomial ideal

Hence by Hochster's theorem, $\overline{\mathcal{R}}(I)$ is Cohen-Macaulay. This implies that the associated graded ring $\overline{G}(I)$ is Cohen-Macaulay.

Let J be a minimal reduction of I. Then the initial forms of generators of J in degree one component of $G(I)$ form a $\overline{G}(I)$-regular sequence. Therefore, we have the following formula for the Hilbert series

$$\begin{aligned}(1-z)^d H(\overline{G}(I), z) &= H(\overline{G}(I)/J\overline{G}(I), z) \\ &= \lambda\left(\frac{R}{\overline{I}}\right) + \lambda\left(\frac{\overline{I}}{J+\overline{I^2}}\right) z + \cdots + \lambda\left(\frac{\overline{I^{d-1}}}{J\overline{I^{d-2}}+\overline{I^d}}\right) z^{d-1} \\ &= f(z)\end{aligned}$$

Since $\overline{e}_i(I) = \dfrac{f^{(i)}(1)}{i\,!}$, we get $\overline{e}_i(I) \geq 0$.

Note that $\overline{H}_I(n) = \overline{P}_I(n)$ for all $n \geq 0$ and $\overline{e}_d(I) = 0$.

Results of Itoh about $\overline{e}_1(I)$ and $\overline{e}_2(I)$

Theorem: (Itoh, 1988, 1992) Let $(R, \mathfrak{m})$ be an analytically unramified CM local ring of dimension $d \geq 2$ and let I be an ideal generated by a system of parameters. Then

- $\overline{e}_1(I) \geq \lambda(\overline{I}/I) + \lambda(\overline{I^2}/I\overline{I})$. Equality holds if and only if $\overline{r}(I) \leq 2$.
- $\overline{e}_2(I) \geq \overline{e}_1(I) - \lambda(\overline{I}/I)$. Equality holds if and only if $\overline{r}(I) \leq 2$.
- If $\overline{r}(I) \leq 2$, then $\overline{G}(I)$ and $\overline{\mathcal{R}}(I)$ are Cohen-Macaulay. In this case, $\overline{P}_I(n) = \overline{H}_I(n)$ for all $n \geq 1$ and

$$\overline{P}_I(n) = e(I)\binom{n+d-1}{d} - \overline{e}_1(I)\binom{n+d-2}{d-1} + \overline{e}_2(I)\binom{n+d-3}{d-2}.$$

Positivity and vanishing of $\overline{e}_3(I)$

The type of a Cohen-Macaulay local ring $(R, \mathfrak{m})$ is defined to be $\dim \operatorname{Soc}(R/J) = \dim(\frac{J:\mathfrak{m}}{J})$, where J is a system of parameters of R. A Cohen-Macaulay ring R is called Gorenstein if type$(R) = 1$.

Theorem: (Itoh, 1992) Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 3$ and let I be an ideal generated by a system of parameters. Then

(1) $\overline{e}_3(I) \geq 0$.

(2) if $\overline{e}_3(I) = 0$, then $\overline{I^{n+2}} \subseteq I^n$ for all $n \geq 0$.

(3) if R is Gorenstein and $\overline{I} = \mathfrak{m}$, then $\overline{e}_3(I) = 0 \iff \overline{r}(I) \leq 2$.

Conjecture: (Itoh, 1992) Let $(R, \mathfrak{m})$ be an analytically unramified, Gorenstein local ring of dimension $d \geq 3$ and let I be an ideal generated by a system of parameters. Then

$$\overline{e}_3(I) = 0 \iff \overline{r}(I) \leq 2.$$

Recent progress on Itoh's conjecture $\overline{e}_3(I)$

Theorem: (Corso, Polini and Rossi, 2014) Let I be an $\mathfrak{m}$-primary ideal in a Cohen-Macaulay, analytically unramified local ring of dimension $d \geq 3$. Let $\overline{I} = \mathfrak{m}$ and $\text{type}(R) \leq \lambda(\overline{I^2}/\mathfrak{m}I) + 1$. Then

$$\overline{e}_3(I) = 0 \iff \overline{r}(I) \leq 2.$$

Theorem: (M. Kummini, S. K. Masuti, 2015) Let R be a 3-dimensional analytically unramified Cohen-Macaulay local ring and $\overline{I} = \mathfrak{m}$ with $\overline{e}_3(I) = 0$. Then

(1) $\overline{e}_2(\mathfrak{m}) \leq \text{type } R$.

(2) If $\overline{e}_2(\mathfrak{m}) \leq \lambda(\overline{I^2}/\mathfrak{m}I) + 2$, then $\overline{r}(I) \leq 2$.

Theorems of Huneke and Itoh

Theorem: (Huneke - Itoh Intersection Theorem, 1988) Let R be a Noetherian ring and $x_1, x_2, \ldots, x_r$ be an R-sequence with $r \geq 2$. Let $I = (x_1, x_2, \ldots, x_r)$. Then for all $n \geq 1$,

$$I^n \cap \overline{I^{n+1}} = I^n \overline{I}.$$

Definition: Let R be a Noetherian ring and I be an R-ideal. Let $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be an I-admissible filtration, where $I_n = R$ for all $n \leq 0$. For $r \geq 1$, the filtration $\mathcal{F}$ is said to satisfy the condition HI_r if for all $n \geq 0$,

$$I^n \cap I_{n+r} = I^n I_r.$$

Theorem: (—, V. Mukundan, J. K. Verma) Let R be a d-dimensional analytically unramified, Cohen-Macaulay local ring and I be an ideal generated by an R-regular sequence. Let $\mathcal{F} = \{\overline{I^n}\}$ satisfy *the conditions* HI_p for all $p \leq k-2$ and let $k \leq d-1$. Then for all $n \geq k-2$,

$$\ell\left(\frac{R}{\overline{I^{n+1}}}\right) \leq \ell\left(\frac{R}{I}\right)\binom{n+d}{d} - \alpha_1(\mathcal{F})\binom{n+d-1}{d-1} + \cdots + (-1)^{k-1}\alpha_{k-1}(\mathcal{F})\binom{n+d-(k-1)}{d-(k-1)} \qquad (1)$$

where $\alpha_j(\mathcal{F}) = \sum_{i=j-1}^{k-2}\binom{i}{j-1}\ell(\overline{I^{i+1}}/I\overline{I^i})$ for all $j = 1, \ldots, k-1$. The equality holds in the equation (1) if and only if $\bar{r}(I) \leq k-1$. In this case, $\overline{G}(I)$ is Cohen-Macaulay.

The condition HI_r via local cohomology modules

Theorem: (—, V. Mukundan, J. K. Verma) Let R be an equidimensional, universally catenary, and an analytically unramified Noetherian local ring of dimension d. Let I be an ideal generated by an R-regular sequence and $\mathcal{F} = \{\overline{I^n}\}$. Then
(1) if $\operatorname{ht}(I) = 1$, then $\mathcal{F}$ satisfies *the condition* HI_r for all $r \geq 1$.
(2) let $\operatorname{ht}(I) \geq 2$ and for some $r \geq 1$,

$$\mathrm{H}^i_J(\overline{\mathcal{R}'}(I))_j = 0 \quad \forall i, j \text{ such that } i + j = r + 1 \text{ and } 0 \leq i \leq \operatorname{ht}(I),$$

where $J = (t^{-1}, It)$. Then $\mathcal{F}$ satisfies *the condition* HI_r.

Main Theorem

Theorem: (—, V. Mukundan, J. K. Verma) Let $(R, \mathfrak{m})$ be a d-dimensional analytically unramified, Cohen-Macaulay local ring with $d \geq 3$, and I be an R-ideal generated by an R-regular sequence such that $\overline{I} = \mathfrak{m}$.

- Suppose that for some $2 \leq k \leq d$, $\mathrm{H}^i_J(\overline{\mathcal{R}'}(I))_j = 0$ for all i, j such that $3 \leq i + j \leq k - 1$ and $2 \leq i \leq d$.
- $\ell\left(\dfrac{\overline{I^{k-1}}}{I\overline{I^{k-2}}}\right) \geq \text{type}(R) - 1.$

Then $\overline{e}_k(I) = 0$ if and only if $\overline{r}(I) \leq k - 1$. In this case, $\overline{G}(I)$ is Cohen-Macaulay.

Example

Let $R = k[[x_0, x_1, \dots, x_d]]/(x_0^n + \cdots + x_d^n)$, where char $k = 0$, $d \geq 3$ and $n \leq d$. Then R is a d-dimensional analytically unramified CM local ring. $\mathfrak{m} = (x_0, \dots, x_d)$ be the maximal homogeneous ideal of R.

$G(\mathfrak{m}) = k[x_0, \dots, x_d]/(x_0^n + \cdots + x_d^n)$ is a d-dimensional CM domain.

Hence, $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all n.

Let I be a parameter ideal of R such that $\overline{I} = \mathfrak{m}$. Then $\mathcal{F} = \{\overline{I^n}\} = \{\mathfrak{m}^n\}$.

$$H(\overline{G}(I), t) = H(G(\mathfrak{m}), t) = \frac{(1 - t^n)}{(1 - t)^{d+1}} = \frac{1 + t + t^2 + \cdots + t^{n-1}}{(1 - t)^d}.$$

Then $\overline{e}_n(I) = e_n(\mathfrak{m}) = 0$. Moreover, $\overline{e}_k = 0$, for all $n \leq k \leq d$.

Also, $\overline{n}(I) = n - 1 - d$. Since $\overline{G}(I)$ is CM, we get $\overline{r}(I) = \overline{n}(I) + d = n - 1$.

As $\overline{r}(I) < d$, and $\overline{G}(I)$ is CM, we get $\overline{\mathcal{R}'}(I)$ is Cohen-Macaulay. Hence $\mathrm{H}^i_J(\overline{\mathcal{R}'}(I))$ is non-zero if and only if $i = d + 1$. Therefore, the filtration $\mathcal{F}$ satisfies *the condition HI_r* for all r.