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# ON THE DECOMPOSITION OF A REPRESENTATION OF $GL(3)$ RESTRICTED TO $GL(2)$ OVER A $P$-ADIC FIELD

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It is a theorem of J. Bernstein that, given irreducible admissible representations $V$ of $GL(n)$ and $W$ of $GL(n-1)$, the space of $GL(n-1)$-equivariant maps from $V$ to $W$ has dimension at most one where $GL(n-1)$ sits in $GL(n)$ as the subgroup $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Here and in the rest of the paper, $GL(n)$ will denote the group of $k$-rational points of the corresponding algebraic group over a fixed nonarchimedean local field $k$. In this paper we give a complete classification of those irreducible admissible representations $V$ of $GL(3)$, and $W$ of $GL(2)$, such that there is a nonzero $GL(2)$-equivariant map of $V$ onto $W$, and make a conjecture for the general $n$.

This paper is being written essentially as an exercise in dealing with the extension problems which arise in the Mackey theory about restriction of an induced representation to a subgroup. However, as the reader will see, we have been lucky here in the sense that certain extensions which might have created a problem have been taken care of by some other considerations.

I would like to thank Y. Flicker for posing the question of characterisation of $GL(3)$-representations with $GL(2)$-invariant forms, i.e., the case corresponding to $W = \mathbf{C}$ (and more generally for $GL(n)$). He himself obtained a complete classification of those *unitary* representations of $GL(n)$ which have $GL(n-1)$-invariant form; see [F1].

**1. Notation and other preliminaries.** For a representation $\rho$ of $GL(n)$ and a character $\chi$ of $GL(1)$, we let $\rho \cdot \chi$ denote the representation of $GL(n)$ obtained by twisting $\rho$ by $\chi$: $\rho \cdot \chi(x) = \chi(\det x)\rho(x)$. In particular, a character of $k^*$ will also be thought of as a character of $GL(n)$. We let $\nu$ denote the character $\nu(x) = |\det x|$ of $GL(n)$ or of any of its subgroups.

For any locally compact group $G$, $\delta_G$ will denote the positive square root of its modulus function.

We will always use normalised induction in this paper (and so $\mathrm{ind}_H^G \pi$ will have the "extra factor" $\delta_G/\delta_H$). Let us note that $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$, where $P = MN$ denotes the standard maximal parabolic of $GL(3)$ of type $(2, 1)$, and $\rho \otimes \chi$ is a representation of $M = GL(2) \times GL(1)$, is the space of functions $f$ on $GL(3)$ with values in the representation space of $\rho$ such that $f(p \cdot g) = \delta_P^{-1}(m)\chi(m)\rho(m)f(g)$ where $p = mn$, and $\delta_P(m) = |\det A|^{-1/2}|d|$ for $m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix}$, where $A$ is a $2 \times 2$ matrix.

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We will denote by $St_n$ the Steinberg representation of $PGL(n)$, and $L$ will denote the irreducible representation of $PGL(3)$ which sits in the exact sequence (and is a Langlands quotient)

$$0 \to St_3 \to \mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{1/2}) \otimes \nu^{-1}] \to L \to 0.$$

This representation $L$ also sits in the exact sequence

$$0 \to \mathbf{C} \to \mathrm{Ind}_P^{GL(3)}[\nu^{-1/2} \otimes \nu] \to L \to 0.$$

We will be using the normalised Jacquet functor in this paper. For $P = MN$ any group such that $M \cap N = \{e\}$, $N$ a normal subgroup of $P$ which is union of its compact open subgroups, and $\theta$ a character of $N$ which is left invariant under the inner conjugation action of $M$ on $N$, the Jacquet functor associates to any smooth representation $V$ of $P$, the representation $r_{N,\theta}(V)$ of $P$ (or $M$)

$$r_{N,\theta}(V) = \frac{\delta_P}{\delta_M} \cdot \frac{V}{\{n \cdot v - \theta(n)v \mid n \in N, v \in V\}}.$$

If $\theta = 1$, we denote $r_{N,\theta}$ simply by $r_N$.

We let $P_0$ denote the subgroup of $GL(n)$ with last row equal to $(0, 0, \ldots, 0, 1)$ and let $U(n)$ be the group of upper triangular matrices in $GL(n)$ with 1 on the diagonal. We fix a nontrivial character $\psi$ of $k$ and let $\psi_n$ be the character of $U(n)$, given by $\psi_n((z_{ij})) = \psi(\sum_{k=1}^{k=n-1} z_{k,k+1})$.

For a representation $\rho$ of $GL(n)$, we let $\rho^{(i)}$ denote its $i$th derivative which is a representation of $GL(n-i)$. Recall that if $R_{n-i} = GL(n-i)V_i$ is the subgroup of $GL(n)$ consisting of $\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$, with $g \in GL(n-i)$, $v \in M(n-i, i)$, $z \in U(i)$, and if the character $\psi_i$ on $U(i)$ is extended to $V_i$ by extending it trivially across $M(n-i, i)$, then $\rho^{(i)} = r_{V_i, \psi_i}(\rho)$.

For $\rho_1$ an admissible representation of $GL(n_1)$ and $\rho_2$ of $GL(n_2)$, we let $I(\rho_1 \times \rho_2)$ denote the representation of $GL(n_1 + n_2)$ induced from the representation $\rho_1 \otimes \rho_2$ of the parabolic with Levi subgroup $GL(n_1) \times GL(n_2)$. It is a theorem of Bernstein and Zelevinsky, see [BZ2], that there is a composition series of $I(\rho_1 \times \rho_2)^{(k)}$ whose successive quotients are $I(\rho_1^{(i)} \times \rho_2^{(k-i)})$, $0 \leqslant i \leqslant k$ (or, in their terminology, the representation $I(\rho_1 \times \rho_2)^{(k)}$ is glued from the representations $I(\rho_1^{(i)} \times \rho_2^{(k-i)})$). If $\mu$ is a one-dimensional representation of $GL(n)$, then $\mu^{(i)} = 0$ if $i \neq 0, 1$, and $\mu^{(1)} = \nu^{-1/2}\mu$; also, $St_2^{(1)} = \nu^{1/2}$. Using these results and the two exact sequences above in which the representation $L$ appears, it follows that $St_3^{(1)} = \nu^{1/2} St_2$, and $St_3^{(2)} = \nu$.

We will repeatedly use a theorem of Gelfand and Kazhdan [GK] according to which the outer automorphism $A \to {}^tA^{-1}$ takes an irreducible admissible representation of $GL(3)$ to its dual (and more generally for $GL(n)$). Since this automorphism preserves $GL(2)$, an irreducible admissible representation $W$ of $GL(2)$ is a quotient of an irreducible admissible representation $V$ of $GL(3)$ if and only if $\tilde{W}$ is a quotient of $\tilde{V}$.

We let $W_k' = W_k \times SL(2, \mathbf{C})$ be the Weil-Deligne group of $k$ where $W_k$ is the Weil group of $k$. For an irreducible admissible representation $\pi$ of $GL(n)$, we let $\sigma_\pi$ denote the $n$-dimensional representation of $W_k'$ associated to $\pi$ by the Langlands correspondence (which is known at least for $n = 2$ and 3).

**2. Theorem on invariant forms.** The aim of this section is to prove the following theorem characterising irreducible admissible representations of $GL(3)$ having a $GL(2)$-invariant form.

THEOREM 1. *An irreducible admissible representation $\pi$ of $GL(3)$ has a $GL(2)$-invariant linear form if and only if the 3-dimensional representation $\sigma_\pi$ of $W_k'$ contains a copy of the trivial representation of $W_k'$ such that the 2-dimensional quotient representation corresponds to an infinite-dimensional (equivalently generic) representation of $GL(2)$.*

It follows from this theorem that, if $\pi$ has a $GL(2)$-invariant linear form, then its standard $L$-function has a pole at $s = 0$. However, the converse is not true.

It is easy to see from the classification theorem of irreducible admissible representations of $GL(3)$, see [JPS1], that Theorem 1 is equivalent to the following theorem (which is what we will be proving below).

THEOREM 2. *The following is a complete list of irreducible admissible representations of $GL(3)$ over a nonarchimedean local field which have a $GL(2)$-invariant linear form*:

(1) *the trivial representation*;

(2) *irreducible representations of $GL(3)$ of the form $\mathrm{Ind}_P^{GL(3)}[\rho \otimes 1]$ where $\rho$ is an arbitrary infinite-dimensional representation of $GL(2)$*;

(3) *irreducible representations of $GL(3)$ of the form $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ where $\rho = \nu^{\pm 1/2}$, and $\chi$ is an arbitrary representation of $GL(1)$*;

(4) *the representations $L \otimes \nu$, and its dual.*

*Proof.* Clearly, a one-dimensional representation of $GL(3)$ has a $GL(2)$-invariant form if and only if it is the trivial representation, and by a theorem of Gelfand and Kazhdan [GK], a supercuspidal representation of $GL(3)$ does not have a $GL(2)$-invariant form. In fact, Gelfand and Kazhdan prove that the restriction of an irreducible supercuspidal representation of $GL(n)$ to $P_0$ is $\mathrm{ind}_{U(n)}^{P_0}\psi_n$. From this it follows that the restriction of an irreducible supercuspidal representation of $GL(n)$ to $GL(n-1)$ is $\mathrm{ind}_{U(n-1)}^{GL(n-1)}\psi_{n-1}$. By Frobenius reciprocity, this representation does not have a $GL(n-1)$-invariant form (for $n > 2$).

We therefore assume in the rest of the proof that the representation is neither supercuspidal nor one-dimensional. Since any nonsupercuspidal representation of $GL(3)$ is a quotient of a representation of $GL(3)$ induced from the parabolic $P = MN$, $M = GL(2) \times GL(1)$ from an irreducible admissible representation of $GL(2) \times GL(1)$ of the form $\rho \otimes \chi$ where $\rho$ is a representation of $GL(2)$ and $\chi$ of $GL(1)$, we investigate below which of these have a $GL(2)$-invariant form.

To study the restriction of $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ to $GL(2)$, we must study the orbit structure of the $GL(2)$ action on $GL(3)/P$ which is the space of 2-dimensional subspaces $U$ of a fixed 3-dimensional space $\langle e_1, e_2, e_3\rangle$.

The orbits for the action of $GL(2)$ on $GL(3)/P$ are as follows:

(1): $X$ = the orbit passing through $U = \langle e_1, e_2\rangle$. This orbit consists of a single point, and the stabiliser is all of $GL(2)$.

(2): $Y$ = the orbit passing through $U = \langle e_2, e_3\rangle$. This is a closed orbit. The stabiliser of $U$ in $GL(2)$ is $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ which corresponds to the subgroup $\begin{bmatrix} d & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$ in $P$ (in the basis $\{e_2, e_3, e_1\}$).

(3): $Z$ = the orbit passing through $U = \langle e_1, e_2 + e_3\rangle$. This is the unique open orbit, and the stabiliser of $Z$ in $GL(2)$ is $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ which corresponds to the subgroup $\begin{bmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $P$ (in the basis $\{e_1, e_2 + e_3, e_3\}$).

By the Mackey theory, the restriction of the representation $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ to $GL(2)$ sits in the exact sequence of $GL(2)$-modules

$$0 \to \mathrm{ind}_{B_0}^{GL(2)} \rho|_{B_0} \to \mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi] \to \rho \cdot \nu^{1/2} \oplus \mathrm{ind}_{\bar{B}}^{GL(2)} \rho_\chi \to 0, \tag{2.1}$$

where $B_0$ is the subgroup $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ of $GL(2)$, $\bar{B}$ is the full lower triangular subgroup, and $\rho_\chi$ is the representation of the group of lower triangular matrices on the same representation space as $\rho$ but acting by the operators

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \to \rho\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \chi(a) \cdot |a|^{-1/2}.$$

From the exact sequence (2.1) of $GL(2)$-modules, we see that a $GL(2)$-invariant linear form on $\rho \cdot \nu^{1/2}$, or on $\mathrm{ind}_{\bar{B}}^{GL(2)} \rho_\chi$ gives a $GL(2)$-invariant linear form on $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$. However, a $GL(2)$-invariant linear form on $\mathrm{ind}_{B_0}^{GL(2)} \rho|_{B_0}$ may or may not extend to give a $GL(2)$-invariant linear form on $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$.

Clearly, $\rho \cdot \nu^{1/2}$ has a $GL(2)$-invariant linear form if and only if $\rho = \nu^{-1/2}$, and therefore the representation $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ has a $GL(2)$-invariant linear form if $\rho = \nu^{-1/2}$. But since an irreducible admissible representation of $GL(3)$ has a $GL(2)$-invariant form if and only if its dual has one, we can therefore refine this to say that $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ has a $GL(2)$-invariant linear form if $\rho = \nu^{\pm 1/2}$.

Before we analyse when the representation $\mathrm{ind}_{\bar{B}}^{GL(2)} \rho_\chi$ has a $GL(2)$-invariant linear form, we recall the following lemma; see Lemmas 8 and 9 in [W].

LEMMA 1. *For an irreducible infinite-dimensional representation $\pi$ of $GL(2)$ and a character $\mu$ of the diagonal torus which restricts to the central character of $\pi$ on the centre, there is a linear form on $\pi$ which transforms by the character $\mu$ under the diagonal torus.*

By Frobenius reciprocity,

$$\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi, \mathbf{C}] = \mathrm{Hom}_{\bar{B}}[\rho_\chi, \bar{\delta}^{1/2}],$$

where $\bar{\delta}$ is the character $\bar{\delta}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = |d|/|a|$.

Therefore by the above lemma, $\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi$ has a $GL(2)$-invariant linear form if and only if $\chi$ is the trivial character and either $\rho$ is infinite-dimensional or is the one-dimensional representation $\rho = \nu^{1/2}$. Therefore the representation $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ has a $GL(2)$-invariant linear form if $\chi$ is the trivial character and $\rho$ is infinite-dimensional. (The representation $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ for $\rho = \nu^{1/2}$ and $\chi = 1$ has already been shown to have a $GL(2)$-invariant form.)

We now analyse which of the $GL(2)$-representations $\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}$ have a $GL(2)$-invariant form. By Frobenius reciprocity,

$$\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}, \mathbf{C}] = \mathrm{Hom}_{T_0}[r_N(\rho), \mathbf{C}],$$

where $T_0 = \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} | a \in k^*\right\}$. In particular, if $\rho$ is supercuspidal, $\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}$ will have no $GL(2)$-invariant form. For an irreducible principal series $\rho$ induced from the character $(\chi_1, \chi_2)$ of the upper triangular subgroup, the Jacquet module $r_N(\rho)$ is the two-dimensional representation of $T_0$ consisting of the characters $\chi_1, \chi_2$. Therefore unless $\chi_1$ or $\chi_2$ is the trivial character, $\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}$ will have no $GL(2)$-invariant form. (It is irrelevant for us here and in the rest of the paper that the Jacquet module is not necessarily semisimple for the toral action.) If $\rho = St_2 \cdot \mu$, the twist of the Steinberg representation $St_2$ of $GL(2)$ by the character $\mu$ of $k^*$, $r_N(\rho) = \mu \otimes \nu^{1/2}$. Therefore $\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}$, for $\rho = St_2 \cdot \mu$, will have a $GL(2)$-invariant form if and only if $\mu = \nu^{-1/2}$. However, we claim that for $\chi \neq 1$, none of these invariant forms extend to $\mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{-1/2}) \otimes \chi]$ (if this representation is irreducible). Indeed, if $\mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{-1/2}) \otimes \chi]$ had a $GL(2)$-invariant form, then so would its dual $\mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{1/2}) \otimes \chi^{-1}]$, but from the analysis done above, $\mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{1/2}) \otimes \chi^{-1}]$ does not have a $GL(2)$-invariant form.

We now prove that none of the twists of the Steinberg representation $St_3 \cdot \mu$ have a $GL(2)$-invariant form. For this, we will have to recall a little bit of the Bernstein-Zelevinsky theory about smooth representations of $P_0$, the subgroup of $GL(n)$ with last row equal to $(0, 0, \ldots, 0, 1)$. Here is the proposition from their theory that we shall need.

PROPOSITION 1. *Any smooth representation $V$ of $P_0$ has a natural filtration of $P_0$ modules $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$, such that*

$$V_{i+1}/V_i \cong \mathrm{ind}_{R_i}^{P_0}[\nu^{1/2}V^{(n-i)} \otimes \psi_{n-i}] \qquad \textit{for } i = 0, 1, \ldots, n-1,$$

*where* $R_i$ *is the subgroup of* $GL(n)$ *consisting of* $\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$, *with* $g \in GL(i)$, $v \in M(i, n-i)$, $z \in U(n-i) \subset GL(n-i)$.

Returning now to our representation $V$ of $GL(3)$, since $P_0 = GL(2) \cdot U(3)$, it is clear that $V_1$ which is a certain number of copies of $\mathrm{ind}_{U(3)}^{P_0} \psi_3$ does not have a $GL(2)$-invariant linear form. Since $P_0 = GL(2) \cdot R_1$ with $GL(2) \cap R_1 = B_0$, the subgroup $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ of $GL(2)$, we have $V_2/V_1 \cong \mathrm{ind}_{R_1}^{P_0}[\nu^{1/2} V^{(2)} \otimes \psi_2] = \mathrm{ind}_{B_0}^{GL(2)}[\nu^{1/2} V^{(2)}]$. (Observe that $\delta_{R_1} = \delta_{P_0} \cdot \delta_{B_0}$ on $B_0$.) Therefore as the second derivative of $St(3) \cdot \mu$ is $\nu \cdot \mu$ (as a representation of $k^*$), $\mathrm{Hom}_{GL(2)}[V_2/V_1, \mathbf{C}] = \mathrm{Hom}_{B_0}[\nu^{3/2} \cdot \mu, \nu^{1/2}]$, where $\nu$ is the obvious character of $B_0$. Therefore $V_2/V_1$ has a $GL(2)$-invariant linear form if and only if $\mu = \nu^{-1}$. Since the first derivative of $St_3$ is $St_2 \cdot \nu^{1/2}$, $V_3/V_2 = St_2 \cdot \nu$. Therefore if there is a nonzero $GL(2)$-invariant linear form on $St_3 \cdot \mu$, it must be nonzero on $V_2$, and $\mu$ must be $\nu^{-1}$. But again if $St_3 \cdot \nu^{-1}$ has a $GL(2)$-invariant form, so will its dual $St_3 \cdot \nu$, which does not have a $GL(2)$-invariant form. So we deduce that no twist of the Steinberg representation has a $GL(2)$-invariant form.

Finally, we show that $L \cdot \nu$ has a $GL(2)$-invariant form. For this let us look at the exact sequence

$$0 \to St_3 \cdot \nu \to \mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{3/2}) \otimes 1] \to L \cdot \nu \to 0.$$

The representation $St_3 \cdot \nu$ has just been shown not to have any $GL(2)$-invariant form, whereas we have constructed a $GL(2)$-invariant form on $\mathrm{Ind}_P^{GL(3)}[(St_2 \cdot \nu^{3/2}) \otimes 1]$ which must therefore come from $L \cdot \nu$.

We have now considered all the irreducible admissible representations of $GL(3)$ as follows from their classification theorem, see [JPS1], and therefore the proof of the theorem is complete.

*Remark* 1. Exactly the same proof gives that a representation of $GL(n)$ has a $GL(n-1)$-invariant linear form if it is induced from a parabolic $P = MN$ with a representation $\rho_1 \otimes \rho_2$ of $M = GL(2) \times GL(n-2)$ with $\rho_1$ infinite-dimensional and $\rho_2 = 1$. The exact determination of the representations of $GL(n)$ with $GL(n-1)$-invariant forms seems more difficult in the case $n > 3$ by the method adapted here, but we expect that a similar result is true in general.

CONJECTURE 1. *An irreducible admissible representation* $\pi$ *of* $GL(n)$, *for* $n > 2$, *has a* $GL(n-1)$-*invariant form if and only if the* $n$-*dimensional representation of* $W_k'$ *associated to* $\pi$ *by the Langlands correspondence has a subrepresentation of dimension* $n-2$ *corresponding to the trivial representation of* $GL(n-2)$ *such that the* 2-*dimensional quotient representation of* $W_k'$ *corresponds to an infinite-dimensional representation of* $GL(2)$.

*Remark* 2. We expect that this conjecture is true in the archimedean case too when we take Harish-Chandra module of representations. From the branching laws about the decomposition of a finite-dimensional representation of $GL(n)$ restricted to $GL(n-1)$, it is easy to see that this conjecture is true for finite-dimensional representations of $GL(n)$ over an archimedean field.

**3. General quotient representation.** Here is a general theorem concerning generic representations. This theorem seems to be well known; however, this author has been unable to find a reference to it.

THEOREM 3. *Given irreducible admissible representations $V$ of $GL(n)$ and $W$ of $GL(n-1)$, both of which are generic, there is a nonzero $GL(n-1)$-equivariant map of $V$ onto $W$.*

*Proof.* Let $v \mapsto W_v$ be the Whittaker model of the representation $V$ of $GL(n)$; i.e., $W_v$ is a function on $GL(n)$ such that $W_v(gu) = \psi_n(u) W_v(g)$ for all $u \in U(n)$, and $g \in GL(n)$, and such that $v \mapsto W_v$ is a $GL(n)$-equivariant map. Similarly let $w \mapsto W_w$ be the Whittaker model of the representation of $\tilde{W}$ of $GL(n-1)$ for the additive character $\bar{\psi}_{n-1}$ of $U(n-1)$. Define a bilinear map $B_s$ on $V \otimes \tilde{W}$ depending on a complex parameter $s$ by

$$B_s(v, w) = \int_{GL(n-1)/U(n-1)} W_v(h) W_w(h) v^s \, dh.$$

It is a theorem of Jacquet, Piatetski-Shapiro, and Shalika [JPS2] that the integral defining $B_s(v, w)$ converges for $\mathrm{Re}(s)$ large and is a rational function of $q^{-s}$ (for all $v \in V$, *and* $w \in \tilde{W}$). In fact, there is a polynomial $P(X)$ with $P(0) = 1$ such that $P(q^{-s})B_s(v, w) \in \mathbf{C}[q^s, q^{-s}]$. It follows that $B_s(v, w)$ defined by the above integral for $\mathrm{Re}(s)$ large, has analytic continuation to the entire complex plane. Moreover, [JPS2] shows that there are vectors $v_0 \in V$ and $w_0 \in W$ such that $B_s(v_0, w_0)$ is the constant function 1. Therefore, taking the bilinear form $B_0(v, w)$, or if there is a pole at $s = 0$, taking appropriate term in the Taylor expansion, we have constructed a $GL(n-1)$-invariant bilinear form on $V \otimes \tilde{W}$, or equivalently a $GL(n-1)$-equivariant map from $V$ onto $W$.

The construction of a $GL(2)$-equivariant map from $V$ onto $W$ when the representation $V$ of $GL(3)$ or $W$ of $GL(2)$ is not generic will again depend on the exact sequence (2.1) of $GL(2)$-modules of the previous section, and the Proposition 1 concerning $P_0$ structure of any smooth $P_0$-module. We will also need some lemmas on when certain extensions of $GL(2)$-modules are trivial. For this purpose, we begin with the following.

LEMMA 2. *Let $G$ be any $\ell$-group, and $P$ a projective object in the category of smooth $G$-modules. Then the smooth dual $\tilde{P}$ of $P$ is an injective object in the category of smooth $G$-modules.*

*Proof.* We need to prove that, given an injective map $0 \to V_1 \to V_2$ of smooth $G$-modules, the induced map $\mathrm{Hom}_G[V_2, \tilde{P}] \to \mathrm{Hom}_G[V_1, \tilde{P}]$ is a surjection. For this, let $\mathscr{S}(G)$ denote the $G$-space of locally constant compactly supported functions on $G$ under left multiplication. Clearly, $P$, as any other $G$-module, is the quotient of the $G$-module $\mathscr{S}(G) \otimes W$ where $W$ is a vector space over $\mathbf{C}$ with the trivial action of $G$. From the definition of a projective module, we can find a $G$-submodule $Q$ of $\mathscr{S}(G) \otimes W$ such that $P \oplus Q = \mathscr{S}(G) \otimes W = \operatorname{ind}_{\{e\}}^{G} W$. Therefore taking smooth duals, $\tilde{P} \oplus \tilde{Q} = \operatorname{Ind}_{\{e\}}^{G} W^*$. By Frobenius reciprocity, $\operatorname{Hom}_G[V, \operatorname{Ind}_{\{e\}}^{G} W^*] = \operatorname{Hom}[V, W^*]$, and therefore the map $\operatorname{Hom}_G[V_2, \operatorname{Ind}_{\{e\}}^{G} W^*] \to \operatorname{Hom}_G[V_1, \operatorname{Ind}_{\{e\}}^{G} W^*]$ is a surjection. But since $\tilde{P} \oplus \tilde{Q} = \operatorname{Ind}_{\{e\}}^{G} W^*$, $\operatorname{Hom}_G[V_2, \tilde{P}] \to \operatorname{Hom}_G[V_1, \tilde{P}]$ is also a surjection, as desired.

*Remark* 3. The dual of an injective object need not be projective. For example, if $G = \mathbf{Z}$, a representation of $G$ can be identified to a $\mathbf{C}[X, X^{-1}]$ module. The quotient field $\mathbf{C}(X)$ of $\mathbf{C}[X, X^{-1}]$ is an injective $\mathbf{C}[X, X^{-1}]$ module. The dual of $\mathbf{C}(X)$ is a divisible $\mathbf{C}[X, X^{-1}]$ module and therefore cannot be a free $\mathbf{C}[X, X^{-1}]$ module. However, $\mathbf{C}[X, X^{-1}]$ being a PID, any projective module is free. Therefore, the dual of the injective module $\mathbf{C}(X)$ is not a projective module.

LEMMA 3. *Let $G$ be a reductive p-adic group with $P = MN$ a parabolic subgroup of $G$. Then the Jacquet functor $V \to r_N(V)$ takes a projective object in the category of smooth $G$-modules to a projective object in the category of smooth $M$-modules.*

*Proof.* By Frobenius reciprocity, the Jacquet functor has an adjoint which is also an exact functor. Using this adjoint, the lemma follows by a standard argument.

LEMMA 4. *For any smooth representation $V$ of $GL(2)$ and a representation $\chi$ of $B_0$ which is trivial on $N$, we have*

$$\mathit{Ext}^i_{GL(2)}[\operatorname{ind}_{B_0}^{GL(2)} \chi, \tilde{V}] = \operatorname{Ext}^i_{T_0}[\chi \otimes r_N(V), \mathbf{C}] \qquad \textit{for all } i.$$

*Proof.* For $i = 0$, this is simply Frobenius reciprocity. For general $i$, let us fix a projective resolution of $V$

$$\cdots \to P_1 \to P_0 \to V \to 0.$$

Since taking smooth duals takes exact sequences to exact sequences, it follows from Lemma 2 that $0 \to \tilde{V} \to \tilde{P}_0 \to \tilde{P}_1 \to \cdots$ is an injective resolution of $\tilde{V}$. Therefore by the definition of the Ext groups, $\operatorname{Ext}^i_{GL(2)}[\operatorname{ind}_{B_0}^{GL(2)} \chi, \tilde{V}]$ is the $i$th cohomology of the complex

$$0 \to \operatorname{Hom}_{GL(2)}[\operatorname{ind}_{B_0}^{GL(2)} \chi, \tilde{P}_0] \to \operatorname{Hom}_{GL(2)}[\operatorname{ind}_{B_0}^{GL(2)} \chi, \tilde{P}_1] \to \cdots,$$

which by Frobenius reciprocity is

$$0 \to \operatorname{Hom}_{T_0}[\chi \otimes r_N(P_0), \mathbf{C}] \to \operatorname{Hom}_{T_0}[\chi \otimes r_N(P_1), \mathbf{C}] \to \cdots.$$

By Lemma 3, $r_N(P_i)$ are projective $T$-modules, and as a projective $T$-module is clearly a projective module for $T_0$ also, the lemma follows from the definition of Ext groups.

LEMMA 5. *For any finite-dimensional $k^*$-modules $M_1$ and $M_2$,*

$$\dim \mathrm{Hom}_{k^*}[M_1, M_2] = \dim \mathrm{Ext}^1_{k^*}[M_1, M_2].$$

*Proof.* Since $k^*$ is the product of a compact group by $\mathbf{Z}$, it suffices to prove the lemma after replacing $k^*$ by $\mathbf{Z}$. Now a module for $\mathbf{Z}$ is the same as a module for the laurent polynomial ring $\mathbf{C}[X, X^{-1}]$, with Hom and Ext over $\mathbf{C}[X, X^{-1}]$. To calculate $\mathrm{Ext}^1_{\mathbf{C}[X,X^{-1}]}[M_1, M_2]$, take a projective resolution of $M_1$

$$0 \to P_1 \to P_0 \to M_1 \to 0.$$

Since $M_1$ is a finite-dimensional complex vector space, $P_0$ and $P_1$ have the same dimension over $\mathbf{C}[X, X^{-1}]$, and therefore dimension of $\mathrm{Ext}^1_{\mathbf{C}[X,X^{-1}]}[M_1, M_2]$ which is the complex dimension of the cokernel of the map (between finite-dimensional complex vector spaces) $\mathrm{Hom}_{\mathbf{C}[X,X^{-1}]}[P_0, M_2] \to \mathrm{Hom}_{\mathbf{C}[X,X^{-1}]}[P_1, M_2]$, is also the dimension of the kernel which is $\mathrm{Hom}_{\mathbf{C}[X,X^{-1}]}[M_1, M_2]$.

Lemmas 4 and 5 combine to yield the following corollary.

COROLLARY 1. *For any admissible representation $V$ of $GL(2)$ and a finite-dimensional representation $\chi$ of $B_0$ which factors through $T_0$, we have* $\mathrm{Ext}^1_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\chi, V] \neq 0$ *if and only if* $\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\chi, V] \neq 0$.

We can also prove the following lemma along similar lines.

LEMMA 6. *For any (not necessarily irreducible) principal series representation $P$ of $GL(2)$ and any admissible representation $Q$ of $GL(2)$,* $\mathrm{Ext}^1_{GL(2)}[Q, P] = 0$ *if and only if* $\mathrm{Hom}_{GL(2)}[Q, P] = 0$ *and* $\mathrm{Ext}^1_{GL(2)}[P, Q] = 0$ *if and only if* $\mathrm{Hom}_{GL(2)}[P, Q] = 0$.

To complete the picture about extension problem for $GL(2)$-modules, we augment, this lemma with the following lemma by Steinberg about the extension of the same name which, however, will not be needed in the sequel.

LEMMA 7. *For any character $\mu$ of $k^*$,* $\mathrm{Ext}^1_{GL(2)}[St \cdot \mu, St] = 0$.

*Proof.* If $\mu \neq 1$, this easily follows from the previous lemma. We now prove that $\mathrm{Ext}^1_{GL(2)}[St_2, St_2] = 0$ using a theorem of Borel [Bo] according to which representations of $PGL(2)$ which are generated by Iwahori fixed vectors are classified by the representations of Hecke algebra with respect to the Iwahori subgroup. The structure of this Hecke algebra is well known and especially simple for the case of $PGL(2)$. We recall that this Hecke algebra (for $PGL(2)$) is an associative algebra on two generators $\{X, Y\}$ with the relations $X^2 = (q-1)X + q$, and $Y^2 = 1$ (where $q$ is the cardinality of the residue field of $k$), and that the representation of this Hecke algebra on the Iwahori fixed vectors of $St_2$ is the one-dimensional representation on which both $X$ and $Y$ act by $-1$. From this it is easy to see that the two-dimensional representation of this Hecke algebra on the Iwahori fixed vectors of any extension $0 \to St_2 \to E \to St_2 \to 0$ is the direct sum of two one-dimensional representations, and therefore the lemma follows from the above theorem of Borel.

THEOREM 4. *The only nongeneric representations $V$ of $GL(3)$ which have a generic representation $W$ of $GL(2)$ as a quotient, are as follows.*

(1) *the representations $V = \mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ with $\rho$ one-dimensional, and $W$ any principal series representation $W_{\chi_1,\chi_2}$ of $GL(2)$ with $\chi_1$ or $\chi_2$ equal to $\rho$.*

(2) *the representations $V = L \otimes \alpha$ with $W = St_2 \otimes \alpha$ or the principal series $W = W_{\nu^{-1/2}\alpha,\mu\alpha}$ where $\alpha$ and $\mu$ are arbitrary characters of $k^*$, and the corresponding result for the representation dual to $L$.*

*Proof.* We begin the proof by investigating which irreducible principal series representations $W_{\chi_1,\chi_2}$ of $GL(2)$ induced from the character $(\chi_1, \chi_2)$ of the Borel subgroup of $GL(2)$ appear as a quotient of $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ for $\rho$ a one-dimensional representation of $GL(2)$ (identified to a character $\rho$ of $k^*$: $\rho(x) = \rho(\det x)$ for $x \in GL(2)$).

It follows from the exact sequence (2.1) together with Lemma 6 that $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ has a $GL(2)$-equivariant map onto $W_{\chi_1,\chi_2}$ if and only if either $\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}, W_{\chi_1,\chi_2}] \neq 0$, or $\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi, W_{\chi_1,\chi_2}] \neq 0$. By Frobenius reciprocity and the fact that the Jacquet functor with respect to $B$ of $W_{\chi_1,\chi_2}$ consists of the two characters $\chi_1(a)\chi_2(d)$, $\chi_2(a)\chi_1(d)$ of the torus $T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, it follows that $\mathrm{Hom}_{GL(2)}[\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi], W_{\chi_1,\chi_2}]$ is nonzero if and only if either $\chi_1 = \rho$ or $\chi_2 = \rho$ (an $\chi$ arbitrary).

We now investigate which of the representations $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ with $\rho$ a one-dimensional representation of $GL(2)$ have $St_2$ as a quotient. From the exact sequence (2.1), it follows that $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ has a $GL(2)$-equivariant map onto $St_2$ only if either $\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}, St_2] \neq 0$, or $\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi, St_2] \neq 0$. By Frobenius reciprocity and the fact that the Jacquet functor with respect to $B$ (resp. $\bar{B}$) of $St_2$ is the character $\delta^{1/2}$ (resp. $\delta^{-1/2}$), it follows that

$$\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{B_0}^{GL(2)}\rho|_{B_0}, St_2] = \mathrm{Hom}_{k^*}[\rho \cdot \nu^{1/2}, \mathbf{C}],$$

$$\mathrm{Hom}_{GL(2)}[\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi, St_2] = \mathrm{Hom}_T[\rho(d) \cdot \chi(a)|a|^{-1/2}, |a|^{1/2}|d|^{-1/2}].$$

Therefore $\mathrm{Hom}_{GL(2)}[\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi], St_2]$ is nonzero only if either $\rho = \nu^{-1/2}$ and $\chi$ arbitrary, or $\rho = \nu^{-1/2}$, and $\chi = \nu$. If $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ is irreducible, then $\mathrm{Hom}_{GL(2)}[\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi], St_2]$ is nonzero. This clearly means that no irreducible $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ can have $St_2$ as a quotient. Since $\mathrm{ind}_{\bar{B}}^{GL(2)}\rho_\chi$ is a quotient representation of $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$, we find that $\mathrm{Ind}_P^{GL(3)}[\nu^{-1/2} \otimes \nu]$ has a $GL(2)$-equivariant map onto $St_2$. We now have the exact sequence,

$$0 \to \mathbf{C} \to \mathrm{Ind}_P^{GL(3)}[\nu^{-1/2} \otimes \nu] \to L \to 0,$$

and therefore $L$ has a $GL(2)$-equivariant map onto $St_2$.

We already know that the only irreducible principal series representation $W_{\chi_1,\chi_2}$ of $GL(2)$ which are quotients of $\mathrm{Ind}_P^{GL(3)}[\nu^{-1/2} \otimes \nu]$ have $\chi_1$ or $\chi_2$ equal to $\nu^{1/2}$. Clearly all these, and precisely these, are quotients of $L$.

Finally, it is clear from the exact sequence (2.1) that no supercuspidal representation of $GL(2)$ can be a quotient of $\mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ with $\rho$ one-dimensional, completing proof of the theorem.

*Remark* 4. We can reinterpret Theorem 4 in terms of the representation of the Weil-Deligne group as follows. The representation $V = \mathrm{Ind}_P^{GL(3)}[\rho \otimes \chi]$ with $\rho$ one-dimensional has a representation $W$ of $GL(2)$ as a quotient if and only if for the three-dimensional representation $\sigma_V$ of $W_k'$ associated to $V$, and the two-dimensional representation $\sigma_W$ of $W_k'$ associated to $W$, $\sigma_V \otimes \sigma_W^*$ contains the two-dimensional representation of $W_k'$ associated to the trivial representation of $GL(2)$. The representation $V = L \otimes \alpha$ has $W$ as a quotient if and only if $\sigma_V \otimes \sigma_W^*$ contains the two-dimensional representation of $W_k'$ associated to either of the representations $St_2 \cdot \nu^{\pm 1}$ of $GL(2)$.

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