

Euler-Poincaré Characteristic for the Oscillator Representation

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Dedicated to Roger Howe on his 70th birthday

Abstract Suppose (G, G') is a dual pair of subgroups of a metaplectic group. The dual pair correspondence is a bijection between (subsets of the) irreducible representations of G and G' , defined by the non-vanishing of $\text{Hom}(\omega, \pi \times \pi')$, where ω is the oscillator representation. Alternatively one considers $\text{Hom}_G(\omega, \pi)$ as a G' -module. It is fruitful to replace Hom with Ext^i , and general considerations suggest that the Euler-Poincaré characteristic $\text{EP}(\omega, \pi)$, the alternating sum of $\text{Ext}^i(\omega, \pi)$, will be a more elementary object. We restrict to the case of p -adic groups, and prove that $\text{EP}(\omega, \pi)$ is a well defined element of the Grothendieck group of finite length representations of G' , and show that it is indeed more elementary than $\text{Hom}(\omega, \pi)$. We expect that computation of EP, together with vanishing results for higher Ext groups, will be a useful tool in computing the dual pair correspondence, and will help to elucidate the structure of $\text{Hom}(\omega, \pi)$.

Keywords Oscillator representation • Euler-Poincaré characteristic • Howe correspondence • Dual pairs

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1 Introduction

Consider a reductive dual pair of subgroups (G, G') of the metaplectic group $\widetilde{\mathrm{Sp}}(2n)$, a two-fold cover of $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, F)$ for F a local field. Let ω be the oscillator representation of $\widetilde{\mathrm{Sp}}(2n)$. The dual pair correspondence, due to Roger Howe, is a bijection between subsets of the duals of G, G' ; we say $\pi \leftrightarrow \pi'$ if $\mathrm{Hom}_{(G, G')}(\omega, \pi \boxtimes \pi') \neq 0$. The fact that this is a bijection is a deep theorem, first proved by Howe over \mathbb{R} [6], and for p -adic fields by Waldspurger [12], Gan-Takeda [5] and Gan-Sun [4].

This correspondence plays an important role in the theory of automorphic forms. It is typically subtle and difficult to compute explicitly, and there is a wealth of literature on the subject. For example, see [10].

It is natural to try to simplify the problem by generalizing: instead of $\mathrm{Hom}_{(G, G')}(\omega, \pi \boxtimes \pi')$, one considers $\mathrm{Ext}_{(G, G')}^i(\omega, \pi \boxtimes \pi')$. This suggests the possibility of studying the Euler-Poincaré characteristic

$$\mathrm{EP}(\omega, \pi \boxtimes \pi') = \sum_i (-1)^i \mathrm{Ext}_{(G, G')}^i(\omega, \pi \boxtimes \pi').$$

There are a number of technical issues to overcome in order to carry this out, the first of which is to show that $\mathrm{Ext}_{(G, G')}^i(\omega, \pi \boxtimes \pi')$ is finite dimensional, and 0 for sufficiently large i . Some of the general theory required can be found in [11].

In the theory of dual pairs it is fruitful to consider the question from a less symmetric point of view. Fix an irreducible representation π of G , and consider the maximal π -isotypic quotient of ω . As a $G \times G'$ -module, this quotient is isomorphic to $\pi \boxtimes \Theta(\pi)$ for a smooth G' -module $\Theta(\pi)$, whose (algebraic) dual is $\mathrm{Hom}_G(\omega, \pi)$. The main step in the proof of the duality correspondence is to show that $\Theta(\pi)$ is a finite length G' -module, with unique irreducible quotient $\theta(\pi)$. Then $\pi \leftrightarrow \theta(\pi)$ is the dual pair correspondence. As the discussion above suggests, the fine structure of $\Theta(\pi)$ is also of some interest.

So in our setting we consider the spaces $\mathrm{Ext}_G^i(\omega, \pi)$ as G' -modules. We specialize now to the p -adic case. In order to stay in the category of smooth representations, it is better to take the smooth vectors $\mathrm{Ext}_G^i(\omega, \pi)^\infty$. We would like to know that $\mathrm{EP}(\omega, \pi)^\infty := \sum_i (-1)^i \mathrm{Ext}_G^i(\omega, \pi)^\infty$ is a well-defined element of the Grothendieck group of finite length representations of G' .

For simplicity of exposition, we will only consider the case of type II dual pairs $(G, G') = (\mathrm{GL}(m), \mathrm{GL}(n))$ in this Introduction, directing the reader to the main body of the paper for type I dual pairs. Fix an irreducible representation π of $\mathrm{GL}(m)$. A preliminary result is that $\mathrm{EP}_G(\omega, \pi)^\infty$ is well defined.

Proposition 1.1 *$\mathrm{Ext}_G^i(\omega, \pi)^\infty$ is a finite length module for G' for all i , and equal to 0 if $i > \mathrm{rank}(G)$. Hence $\mathrm{EP}(\omega, \pi)^\infty$ is a well-defined element of the Grothendieck group.*

See Proposition 5.17. When $i = 0$ this is the well-known fact that $\Theta(\pi)$ has finite length. See Proposition 4.1.

Now suppose $m \leq n$ and that P is a parabolic subgroup of $\mathrm{GL}(n)$ with Levi factor $\mathrm{GL}(m) \times \mathrm{GL}(n - m)$. It is well known that for π an irreducible representation of $\mathrm{GL}(m)$,

$$\mathrm{Hom}_{\mathrm{GL}(m) \times \mathrm{GL}(n)}(\omega, \pi \boxtimes i_P^G(\pi \boxtimes 1)) \neq 0 \quad (1)$$

where i_P^G denotes normalized smooth induction from P to G . See [10].

Hence a naive guess for the explicit dual correspondence for type II dual pairs would be that for $n \geq m$ (which we can assume without loss of generality since $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$ play a symmetrical role), the map in (1) is surjective and realizes the maximal π -isotypic quotient of ω , and that the induced representation has a unique irreducible quotient. If this is the case, then $\Theta(\pi) = i_P^{\mathrm{GL}(n)}(\pi \boxtimes 1)$, and $\theta(\pi)$ is the unique irreducible quotient of this induced representation. Generically, of course, the induced representation is irreducible and this is true. However, in general, the induced representation may be reducible, and computing the explicit dual pair correspondence amounts to understanding the image of the map in (1), and the structure of the induced representation. See [10, conjecture on bottom of page 64] and [9].

Our first main result is that the corresponding naive guess does in fact hold if one replaces $\mathrm{Hom}_G(\omega, \pi)$ with $\mathrm{EP}_G(\omega, \pi)^\infty$.

Theorem 1.3 *Consider the oscillator representation ω for the dual pair $(\mathrm{GL}(m), \mathrm{GL}(n))$. Let π be an irreducible representation of $\mathrm{GL}(m)$. Then*

$$\mathrm{EP}_{\mathrm{GL}(m)}(\omega, \pi)^\infty = \begin{cases} 0 & n < m \\ i_P^{\mathrm{GL}(n)}(\pi \boxtimes 1) & n \geq m \end{cases}$$

where P is a parabolic subgroup of $\mathrm{GL}(n)$ with Levi subgroup $\mathrm{GL}(m) \times \mathrm{GL}(n - m)$.

We refer to results of this type as *the theta correspondence for dummies*: replacing Hom with EP makes this more elementary statement true. See [1]. Hopefully such easy results for EP , together with vanishing results for higher Ext groups, will give results about $\Theta(\pi)$ and $\theta(\pi)$. See Question 6.3 and Example 6.5. Moreover non-trivial higher Ext groups may help to clarify the structure of $\Theta(\pi)$.

Similar results hold for type I dual pairs. In order to keep the notation simple we consider only the case of orthogonal-symplectic dual pairs, see Sect. 7. It is clear that the proofs go through for general type I dual pairs, using [10].

It would be interesting to consider the case of real groups, say in the context of (\mathfrak{g}, K) -modules, where we expect results similar to what we obtain here for p -adic groups.

2 Some Background

In this section we introduce notation and prove some basic results needed later on. For background on representations of p -adic groups, see [3] and [7].

Suppose G is a p -adic group. Let $\mathcal{C} = \mathcal{C}_G$ be the category of smooth representations of G , and let $Ob(\mathcal{C})$ denote the objects of this category. For X any G -module, let $X^\infty \in Ob(\mathcal{C})$ be the submodule of smooth vectors. It is a union of X^K , the space of K -fixed vectors, as K runs over all open compact subgroups of G . In some cases X is a module for two different groups G and H , in which case we will be specific about denoting X^∞ as $X^{G-\infty}$ or $X^{H-\infty}$. We work entirely in the setting of smooth representations.

For $Y \in Ob(\mathcal{C})$ let $Y^* = \text{Hom}_{\mathbb{C}}(Y, \mathbb{C})$ be the algebraic dual, and let $Y^\vee = (Y^*)^\infty \in Ob(\mathcal{C})$ be the smooth dual. For H a closed subgroup of G , let Res_H^G be the restriction functor from \mathcal{C}_G to \mathcal{C}_H , let Ind_H^G be (smooth) induction from \mathcal{C}_H to \mathcal{C}_G , and let ind_H^G be compact induction.

Lemma 2.1 *Let K be an open compact subgroup of G . Suppose $X \in Ob(\mathcal{C}_K)$ and $Y \in Ob(\mathcal{C}_G)$. Then*

- (1) $\text{Hom}_G(\text{ind}_K^G(X), Y) \simeq \text{Hom}_K(X, \text{Res}_K^G(Y))$.
- (2) $\text{ind}_K^G(X)$ is a projective G -module.

Proof The first statement is a standard version of Frobenius reciprocity; it implies that $\text{Hom}_G(\text{ind}_K^G(X), -)$ is an exact functor, so $\text{ind}_K^G(X)$ is projective. \square

Lemma 2.2 *Let X be a smooth module for $G \times H$. If X^K is a projective H -module for every open compact subgroup K of G , then X is a projective H -module.*

Proof Fix an open compact subgroup K of G . For τ , a smooth irreducible representation of K , let X_τ be the τ -isotypic subspace of X . Let K_τ be the kernel of τ . Then X_τ is a direct summand of X^{K_τ} and is, therefore, projective. Furthermore $X = \bigoplus_\tau X_\tau$ is a direct sum of projective modules, hence projective. \square

Lemma 2.3 *Let Q be a closed subgroup of G such that $Q \backslash G$ is compact. Let X be a smooth representation of $Q \times H$, projects as an H -module. Then $\text{Ind}_Q^G X$, with the natural action of H , is a projective H -module.*

Proof By Lemma 2.2, it suffices to prove that $(\text{Ind}_Q^G X)^K$ is H -projective for every open compact subgroup K of G . Write $G = \cup_i Qg_iK$ for a finite set of elements g_i in G . Let $K_i = g_iKg_i^{-1} \cap Q$. Then

$$(\text{Ind}_Q^G X)^K = \bigoplus_i X^{K_i}$$

where the isomorphism is given by evaluating $f \in \text{Ind}_Q^G X$ at the points g_i . The Lemma follows since X^{K_i} are summands of X and hence H -projective. \square

Let $\mathcal{S}(G)$ be the Schwartz space of the locally constant compactly supported functions on G . This is a module for $G \times G$ by the left and right translation actions.

Lemma 2.4 (1) $\mathcal{S}(G)$ is a projective module for the right action of G .
 (2) For any smooth (left) G -module X , $\text{Hom}_G(\mathcal{S}(G), X)^\infty \simeq X$ as (left) G -modules, where $\text{Hom}_G(\mathcal{S}(G), X)$ is defined to be the space of homomorphisms $\lambda : \mathcal{S}(G) \rightarrow X$ with $\lambda(R_g f) = g\lambda(f)$ for all $f \in \mathcal{S}(G)$.

Proof Projectivity of $\mathcal{S}(G)$ is usually attributed to Blanc [2]. We give an independent and rather simple proof. Let K be an open compact subgroup of G , acting from the left. Then $\mathcal{S}(G)^K = \text{ind}_K^G(\mathbb{C})$ and this is projective by Lemma 2.1. Hence $\mathcal{S}(G)$ is a projective G -module by Lemma 2.2.

We now prove (2). For any compact open subgroup K of G we have

$$\begin{aligned} \text{Hom}_G(\mathcal{S}(G), X)^K &\simeq \text{Hom}_G(\mathcal{S}(K \backslash G), X) \\ &\simeq \text{Hom}_G(\text{ind}_K^G(\mathbb{C}), X) \\ &\simeq \text{Hom}_K(\mathbb{C}, X) \\ &\simeq X^K. \end{aligned}$$

The isomorphism $\text{Hom}_G(\mathcal{S}(G), X)^K \simeq X^K$ is given by $\varphi \mapsto \frac{1}{\text{vol}(K)}\varphi(1_K)$, where 1_K is the characteristic function of K .

These isomorphisms as K varies are compatible, and therefore give an isomorphism $T : \text{Hom}_G(\mathcal{S}(G), X)^\infty \simeq X$, defined by $T(\varphi) = \frac{1}{\text{vol}(K)}\varphi(1_K)$ for $\varphi \in \text{Hom}_G(\mathcal{S}(G), X)^\infty$, where K is any compact open subgroup of G which leaves φ invariant. \square

Now assume that G is reductive, $P = MN$ is a parabolic subgroup of G , and δ_P is the modulus character of P : $\delta_P(mn) = |\det(\text{Ad}_n(m))|$ [3, 3.1]. For $X \in \text{Ob}(\mathcal{C}_M)$ we write $i_P^G(X)$ for normalized induction (X is pulled back to P): $i_P^G(X) = \text{Ind}_P^G(\delta_P^{\frac{1}{2}}X)$. Then i_P^G preserves unitarity, and

$$i_P^G(X)^\vee = i_P^G(X^\vee). \quad (2)$$

For $X \in \text{Ob}(\mathcal{C}_G)$ write $r_P^G(X) \in \text{Ob}(\mathcal{C}_M)$ for the normalized Jacquet module of X : $r_P^G(X) = H_0(N, X)\delta_P^{-\frac{1}{2}}$.

Lemma 2.6 *Let $P = MN$ be a parabolic subgroup of G . Let V be an $M \times H$ -module. Then, for every smooth H -module U , with trivial action of G , we have the following natural isomorphism of smooth G -modules*

$$\text{Hom}_H(i_P^G(V), U)^\infty \cong i_P^G(\text{Hom}_H(V, U)^{M-\infty}).$$

Proof It suffices to show that we have an isomorphism of $G \times H$ -modules

$$\text{Hom}_{\mathbb{C}}(i_P^G(V), U)^\infty \cong i_P^G(\text{Hom}_{\mathbb{C}}(V, U)^{M-\infty})$$

and then the proposition follows by taking H -fixed vectors on both sides. Note that, if $U = \mathbb{C}$, this is the well-known statement $i_P^G(V)^\vee \cong i_P^G(V^\vee)$. The proof is the same. More precisely, $f \in i_P^G(\text{Hom}_{\mathbb{C}}(V, U)^{M-\infty})$ defines $\ell_f \in \text{Hom}_{\mathbb{C}}(i_P^G(V), U)^\infty$ by

$$\ell_f(f') = \int_{P \backslash G} f(g)(f'(g)) dg$$

for every $f' \in i_P^G(V)$. One checks that $f \mapsto \ell_f$ is an isomorphism by doing so at the level of K -fixed vectors. The map is determined by fixing a density on G . \square

3 Euler-Poincaré Characteristic

For background on statements in this section, see [11]. Assume that G is a reductive group.

Lemma 3.1 *The category \mathcal{C} of smooth representations of G has enough projectives and enough injectives. Therefore, for $X, Y \in \text{Ob}(\mathcal{C})$, we can define the complex vector spaces $\text{Ext}_G^i(X, Y)$ for all $i \geq 0$ with the following properties:*

1. $\text{Ext}_G^0(X, Y) \simeq \text{Hom}_G(X, Y)$;
2. $\text{Ext}_G^i(X, Y) = 0$ if $i > 0$ and Y is injective. In general, $\text{Ext}_G^i(X, Y)$ can be computed using an injective resolution of Y ;
3. $\text{Ext}_G^i(X, Y) = 0$ if $i > 0$ and X is projective. In general, $\text{Ext}_G^i(X, Y)$ can be computed using a projective resolution of X .

Now suppose $P = MN$ is a parabolic subgroup of G . Let $\bar{P} = M\bar{N}$ be the opposite parabolic. We have two versions of Frobenius reciprocity.

Lemma 3.2 *For $X \in \text{Ob}(\mathcal{C}_G)$ and $Y \in \text{Ob}(\mathcal{C}_M)$*

1. $\text{Ext}_G^i(X, i_P^G(Y)) \simeq \text{Ext}_M^i(r_P^G(X), Y)$
2. $\text{Ext}_G^i(i_{\bar{P}}^G(X), Y) \simeq \text{Ext}_M^i(X, r_{\bar{P}}^G(Y))$

We also need a version of the Kunnetth formula [11].

Lemma 3.3 *Suppose G_1, G_2 are reductive p -adic groups, and X_i, Y_i are smooth representations of G_i . Furthermore assume that X_1 is admissible. Then*

$$\text{Ext}_{G_1 \times G_2}^i(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \simeq \bigoplus_{j+k=i} \text{Ext}_{G_1}^j(X_1, Y_1) \otimes \text{Ext}_{G_2}^k(X_2, Y_2)$$

See [11]. For $X, Y \in \text{Ob}(\mathcal{C}_G)$ assume that $\text{Ext}_G^i(X, Y)$ are finite dimensional for all $i \geq 0$, and 0 for i large enough. Then the Euler-Poincaré characteristic is defined to be

$$\text{EP}_G(X, Y) = \sum_i (-1)^i \text{Ext}_G^i(X, Y). \quad (3)$$

This is a well-defined element of the Grothendieck group of finite dimensional vector spaces. If

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow 0$$

is an exact sequence of smooth G -modules, and $Y \in \text{Ob}(\mathcal{C}_G)$, then

$$\sum_j (-1)^j \text{EP}_G(X_j, Y) = \sum_j (-1)^j \text{EP}_G(Y, X_j) = 0.$$

If

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

is a filtration by smooth G -submodules, with successive quotients $W_i = X_i/X_{i-1}$, then

$$\text{EP}(X, Y) = \sum_{i=1}^n \text{EP}(W_i, Y), \quad \text{EP}(Y, X) = \sum_{i=1}^n \text{EP}(Y, W_i). \quad (4)$$

Proposition 3.6 *If G is a reductive group and X, Y are smooth G -modules of finite length, then $\text{Ext}_G^i(X, Y)$ are finite dimensional for all i and 0 for i greater than the split rank of G . Moreover, if G has non-compact center, then $\text{EP}_G(X, Y) = 0$.*

Proof The finite dimensionality of $\text{Ext}^i(X, Y)$ and vanishing beyond the split rank are well-known general facts. We only prove the vanishing of Euler-Poincaré characteristic for groups having non-compact center. It suffices to prove this statement for a normal subgroup G_0 of finite index in G which we assume has the form $G_0 = G_1 \times G_2$ where $G_1 \supseteq G^{\text{der}}$ and $G_2 \subseteq Z(G)$ with $G_2 \cong \mathbb{Z}$. (These groups are not necessarily algebraic.) Decomposing X and Y as direct sums of irreducible representations for G_0 , it suffices to assume that X and Y themselves are irreducible as G_0 -modules. Write $X = U_1 \otimes U_2$, and $Y = V_1 \otimes V_2$ where U_1, V_1 are irreducible modules for G_1 which are just the restrictions of the smooth modules X, Y of G_0 to G_1 ; U_2, V_2 are the one dimensional representations on which G_2 operates by the central characters for the action of G_0 on X and Y , respectively, restricted to $G_2 \simeq \mathbb{Z} \subset Z(G_0)$. It suffices to prove that

$$\text{EP}_{G_1 \times G_2}(U_1 \boxtimes U_2, V_1 \boxtimes V_2) = 0.$$

By the Künneth formula,

$$\text{EP}_{G_1 \times G_2}(U_1 \boxtimes U_2, V_1 \boxtimes V_2) = \text{EP}_{G_1}(U_1, V_1) \otimes \text{EP}_{G_2}(U_2, V_2)$$

Since U_2, V_2 are one dimensional representations of $G_2 = \mathbb{Z}$, $\text{EP}_{G_2}(U_2, V_2) = \text{EP}_{\mathbb{C}[\mathbb{Z}]}(U_2, V_2) = 0$, and the proposition follows. \square

Suppose that H is another p -adic reductive group, $X \in \text{Ob}(\mathcal{C}_{G \times H})$ and $Y \in \text{Ob}(\mathcal{C}_H)$. Then G acts on $\text{Ext}_H^i(X, Y)$ via its action on X . This module is not necessarily smooth, so we take its smooth vectors for the action of G :

$$\text{Ext}_H^i(X, Y)^\infty \in \text{Ob}(\mathcal{C}_G).$$

We would like to use $\text{Ext}_H^i(X, Y)^\infty$, to construct the Euler-Poincaré characteristic as an element of the Grothendieck group of finite length representations. For this we need to know that $\text{Ext}_H^i(X, Y)^\infty$ is a finite length smooth G -module.

Definition 3.7 Fix $X \in \text{Ob}(\mathcal{C}_{G \times H})$ and $Y \in \text{Ob}(\mathcal{C}_H)$. Assume

$$\text{Ext}_H^i(X, Y)^\infty \text{ is a finite length smooth } G\text{-module for all } i. \quad (5)$$

Define $\text{EP}_H(X, Y)^\infty = \sum_i (-1)^i \text{Ext}_H^i(X, Y)^\infty$. This is a well-defined element in the Grothendieck group of finite length representations.

In practice we will always assume Y (but not X) has finite length as an H -module.

4 The Theta Correspondence

Consider a dual pair of subgroups (G, H) of the metaplectic group $\widetilde{\text{Sp}}(2n)$. Fix an additive character of our p -adic field F and let ω be the corresponding oscillator representation of $\widetilde{\text{Sp}}(2n)$.

Consider the theta correspondence for the dual pair (G, H) . See [10]. Suppose π is an irreducible representation of G . Let

$$\omega(\pi) = \bigcap_{f \in \text{Hom}_G(\omega, \pi)} \text{Ker}(f).$$

This is a $G \times H$ -submodule of ω . Set

$$\omega[\pi] = \omega / \omega(\pi).$$

This $G \times H$ -module is the maximal π -isotypic quotient of ω . By [10], there is a smooth H -module $\Theta(\pi)$, unique up to equivalence, such that $\omega[\pi] \simeq \pi \boxtimes \Theta(\pi)$. By the Howe conjecture, now proved in generality by Gan-Takeda [5] and Gan-Sun [4], $\Theta(\pi)$ has a unique irreducible quotient, which is denoted by $\theta(\pi)$.

Proposition 4.1

$$\Theta(\pi)^* \simeq \text{Hom}_G(\omega, \pi) \text{ and } \Theta(\pi) \simeq \text{Hom}_G(\omega, \pi)^\vee.$$

Proof By the definition of the maximal isotypic quotient,

$$\begin{aligned} \mathrm{Hom}_G(\omega, \pi) &\simeq \mathrm{Hom}_G(\pi \boxtimes \Theta(\pi), \pi) \\ &\simeq \mathrm{Hom}_G(\pi, \pi) \boxtimes \mathrm{Hom}_\mathbb{C}(\Theta(\pi), \mathbb{C}) \\ &\simeq \Theta(\pi)^* \end{aligned}$$

This proves the first assertion. Taking the smooth vectors on both sides gives $\mathrm{Hom}_G(\omega, \pi)^\infty \simeq \Theta(\pi)^\vee$. By [10, III.5] $\Theta(\pi)$ is a finite length H -module, so admissible, so by [7, Proposition 7] $\Theta(\pi)^{\vee\vee} \simeq \Theta(\pi)$. Take the smooth dual of both sides to conclude $\Theta(\pi) \simeq \mathrm{Hom}_G(\omega, \pi)^\vee$. \square

5 Type II Dual Pairs

Consider the oscillator representation ω of the dual pair $(\mathrm{GL}(m), \mathrm{GL}(n))$. Let ω_0 be the geometric representation of this dual pair on $\mathcal{S}(M_{m \times n})$:

$$\omega_0(g, h)(f)(x) = f(g^{-1}xh) \quad ((g, h) \in \mathrm{GL}(m) \times \mathrm{GL}(n)). \quad (6)$$

We use the standard normalization of the oscillator representation: set

$$\xi(g, h) = |\det(g)|^{-n/2} |\det(h)|^{m/2} \quad (7)$$

and define the oscillator representation to be:

$$\omega = \omega_0 \otimes \xi. \quad (8)$$

This is a unitary representation of $\mathrm{GL}(m) \times \mathrm{GL}(n)$ on which $\mathrm{GL}(1)$ embedded as scalar matrices $(\lambda I_m, \lambda I_n) \in \mathrm{GL}(m) \times \mathrm{GL}(n)$ acts trivially.

Let $t = \min(m, n)$. Consider the filtration

$$0 = \omega_{t+1} \subset \omega_t \subset \cdots \subset \omega_0 = \omega \quad (9)$$

where ω_k is the Schwartz space of functions supported on matrices in $M_{m \times n}(F)$ of rank $\geq k$. Let Ω_k be the set of matrices in $M_{m \times n}(F)$ of rank k . Then $\omega_k / \omega_{k+1} \simeq \mathcal{S}(\Omega_k)$ ($0 \leq k \leq t$).

For π an irreducible representation of $\mathrm{GL}(m)$, we will show that $\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)^\infty$ is well defined, and at the same time compute it; then we will calculate $\mathrm{EP}_{\mathrm{GL}(m)}(\omega, \pi)^\infty$ as the direct sum of these.

5.1 $\mathbf{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)^\infty$

First of all by [10]:

$$\mathcal{S}(\Omega_k) \simeq \xi \mathrm{Ind}_{P_k \times Q_k}^{\mathrm{GL}(m) \times \mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1).$$

Note that the induction is unnormalized, and we've included the character ξ . Here the parabolic subgroups and Levi factors are:

$$M_k = \mathrm{GL}(k) \times \mathrm{GL}(m-k) \subset P_k = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}(m)$$

and

$$L_k = \mathrm{GL}(k) \times \mathrm{GL}(n-k) \subset Q_k = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \subset \mathrm{GL}(n).$$

We rearrange terms:

$$M_k \times L_k \simeq [\mathrm{GL}(k) \times \mathrm{GL}(k)] \times [\mathrm{GL}(m-k) \times \mathrm{GL}(n-k)],$$

and $\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1$ is a representation of $M_k \times L_k$ with respect to this decomposition. So

$$\mathrm{Ext}_{\mathrm{GL}(m)}^i(\mathcal{S}(\Omega_k), \pi) = \mathrm{Ext}_{\mathrm{GL}(m)}^i(\xi \mathrm{Ind}_{P_k \times Q_k}^{\mathrm{GL}(m) \times \mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1), \pi).$$

To compute $\mathrm{Ext}_{\mathrm{GL}(m)}^i$, we only need the action of $\mathrm{GL}(m)$. So write

$$\mathrm{Ind}_{P_k \times Q_k}^{\mathrm{GL}(m) \times \mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1) = \mathrm{Ind}_{P_k}^{\mathrm{GL}(m)} \{ \mathrm{Ind}_{Q_k}^{\mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1) \boxtimes 1 \}.$$

Now apply Frobenius reciprocity, Lemma 3.2(2). Write $\nu_k(j)$ for the character $|\det|^j$ of $\mathrm{GL}(k)$.

$$\begin{aligned} & \mathrm{Ext}_{\mathrm{GL}(m)}^i(\xi \mathrm{Ind}_{P_k}^{\mathrm{GL}(m)} \{ \mathrm{Ind}_{Q_k}^{\mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1) \boxtimes 1 \}, \pi) \\ &= \mathrm{Ext}_{\mathrm{GL}(m)}^i(\nu_m(-\frac{n}{2}) i_{P_k}^{\mathrm{GL}(m)} \{ \delta_{P_k}^{-\frac{1}{2}} \mathrm{Ind}_{Q_k}^{\mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1) \boxtimes 1 \}, \pi) \otimes \nu_n(-\frac{m}{2}) \\ &= \mathrm{Ext}_{\mathrm{GL}(k) \times \mathrm{GL}(m-k)}^i(\mathrm{Ind}_{Q_k}^{\mathrm{GL}(n)} (\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1) \boxtimes 1, \nu_m(\frac{n}{2}) \delta_{P_k}^{-\frac{1}{2}} r_{P_k}^{\mathrm{GL}(m)}(\pi)) \otimes \nu_n(-\frac{m}{2}) \end{aligned}$$

Here $\mathrm{GL}(k)$ is acting on $\mathrm{Ind}_{Q_k}^{\mathrm{GL}(n)}(*)$ by its action on $\mathcal{S}(\mathrm{GL}(k))$, and $\mathrm{GL}(m-k)$ is acting trivially (the second \boxtimes).

Write

$$v_m\left(\frac{n}{2}\right)\delta_{P_k}^{-\frac{1}{2}}r_{\overline{P}_k}^{\text{GL}(m)}(\pi) = \sum_{j=1}^{\ell} \sigma_j \boxtimes \tau_j \quad (10)$$

with $\sigma_j \boxtimes \tau_j$ an irreducible representation of $\text{GL}(k) \times \text{GL}(m-k)$.

Now we compute

$$\text{Ext}_{\text{GL}(k) \times \text{GL}(m-k)}^i(\text{Ind}_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k)) \boxtimes 1) \boxtimes 1, \sigma_j \boxtimes \tau_j),$$

where the first 1 denotes the trivial representation of $\text{GL}(n-k)$, and the second 1 denotes the trivial representation of $\text{GL}(m-k)$. By the Kunnetth formula (Lemma 3.3), this is equal to

$$\sum_{p=0}^i \text{Ext}_{\text{GL}(k)}^p(\text{Ind}_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k)) \boxtimes 1), \sigma_j) \otimes \text{Ext}_{\text{GL}(m-k)}^{i-p}(1, \tau_j) \quad (11)$$

By Lemma 2.3 the induced representation $\text{Ind}_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k)) \boxtimes 1)$ is projective as a representation of $\text{GL}(k)$. Therefore all terms in (11) with $p > 0$ are 0, and (summing over j again) we get

$$\begin{aligned} & \text{Ext}_{\text{GL}(m)}^i(\mathcal{S}(\Omega_k), \pi) \simeq \\ & \sum_{j=1}^{\ell} \text{Hom}_{\text{GL}(k)}(\text{Ind}_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k)) \boxtimes 1), \sigma_j) \otimes \text{Ext}_{\text{GL}(m-k)}^i(1, \tau_j) \otimes v_n\left(-\frac{m}{2}\right) \end{aligned} \quad (12)$$

Now take the $\text{GL}(n)$ -smooth vectors on both sides. We want to apply Lemma 2.6 to the first factor on the right-hand side, so first we replace $\text{Ind}_{Q_k}^{\text{GL}(n)}$ with normalized induction.

$$\text{Ind}_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k)) \boxtimes 1) \simeq i_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k))v_k\left(\frac{n-k}{2}\right) \boxtimes v_{n-k}\left(-\frac{k}{2}\right)) \quad (13)$$

and the first term on the right-hand side of (12) is

$$\begin{aligned} & \text{Hom}_{\text{GL}(k)}(i_{Q_k}^{\text{GL}(n)}(\mathcal{S}(\text{GL}(k))v_k\left(\frac{n-k}{2}\right) \boxtimes v_{n-k}\left(-\frac{k}{2}\right)), \sigma_j)^\infty \\ & \simeq i_{Q_k}^{\text{GL}(n)}(\text{Hom}_{\text{GL}(k)}(\mathcal{S}(\text{GL}(k))v_k\left(\frac{n-k}{2}\right), \sigma_j)^\infty \boxtimes v_{n-k}\left(\frac{k}{2}\right)) \\ & \simeq i_{Q_k}^{\text{GL}(n)}(\sigma_j v_k\left(\frac{-n+k}{2}\right) \boxtimes v_{n-k}\left(\frac{k}{2}\right)) \end{aligned}$$

where the final isomorphism is by Lemma 2.4(2).

This proves the following intermediate result. Recall that $\sigma_j \boxtimes \tau_j$, an irreducible representation of $\text{GL}(k) \times \text{GL}(m-k)$, is given in (10).

Proposition 5.9 *For an irreducible smooth representation π of $\mathrm{GL}(m)$, we have an isomorphism of $\mathrm{GL}(n)$ -modules:*

$$\mathrm{Ext}_{\mathrm{GL}(m)}^i(\mathcal{S}(\Omega_k), \pi)^\infty \simeq \sum_{j=1}^{\ell} i_{Q_k}^{\mathrm{GL}(n)}(\sigma_j \nu_k(\frac{-n+k-m}{2}) \boxtimes \nu_{n-k}(\frac{k-m}{2})) \otimes \mathrm{Ext}_{\mathrm{GL}(m-k)}^i(1, \tau_j),$$

with $\mathrm{GL}(n)$ acting trivially on the last factor. In particular, $\mathrm{Ext}_{\mathrm{GL}(m)}^i(\mathcal{S}(\Omega_k), \pi)^\infty$ is a finite length $\mathrm{GL}(n)$ -module.

Note that the right-hand side is 0 if $i > m - k$.

An important special case is $m = k, i = 0$. The Levi factor of Q_m is $\mathrm{GL}(m) \times \mathrm{GL}(n - m)$. Also $\ell = 1$, and (10) is simply $\nu_m(\frac{n}{2})\pi = \sigma_1$. Plugging this in gives

$$\begin{aligned} \mathrm{Hom}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_m), \pi)^\infty &\simeq i_{Q_m}^{\mathrm{GL}(n)}(\pi \boxtimes 1) \\ \mathrm{Ext}_{\mathrm{GL}(m)}^i(\mathcal{S}(\Omega_m), \pi)^\infty &= 0 \quad (i > 0). \end{aligned} \tag{14}$$

Now we can conclude that the Euler-Poincaré characteristic is well defined, and (5.9) yields:

$$\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)^\infty \simeq \sum_{j=1}^{\ell} i_{Q_k}^{\mathrm{GL}(n)}(\sigma_j \nu_k(\frac{-n+k-m}{2}) \boxtimes \nu_{n-k}(\frac{k-m}{2})) \otimes \mathrm{EP}_{\mathrm{GL}(m-k)}(1, \tau_j).$$

By Proposition 3.6, $\mathrm{EP}_{\mathrm{GL}(m-k)}(1, \tau_j) = 0$ unless $k = m$, and if $k = m$ (14) gives

$$\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_m), \pi)^\infty = i_{Q_m}^{\mathrm{GL}(n)}(\pi \boxtimes 1)$$

This proves:

Proposition 5.11 *Let Ω_k be the set of $m \times n$ matrices (over F) of rank $k \leq \min(m, n)$. Consider the action of $\mathrm{GL}(m) \times \mathrm{GL}(n)$ on $\mathcal{S}(\Omega_k)$ given by (6). Then for an irreducible smooth representation π of $\mathrm{GL}(m)$,*

$$\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)^\infty = \begin{cases} 0 & k < m \\ i_{Q_m}^{\mathrm{GL}(n)}(\pi \boxtimes 1) & k = m \end{cases}$$

Recall that ω has a filtration with successive quotients $\mathcal{S}(\Omega_k)$ ($0 \leq k \leq \min(m, n)$).

Proposition 5.12 *For an irreducible smooth representation π of $\mathrm{GL}(m)$,*

$$\mathrm{EP}_{\mathrm{GL}(m)}(\omega, \pi)^\infty = \sum_{k=0}^{\min(m, n)} \mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)^\infty$$

This would be a trivial consequence of Proposition 5.11, except for a technical issue related to taking smooth vectors, so we defer the proof to the next sub-section. Together with Proposition 5.11, this implies the main result of this section.

Theorem 5.13 *Let π be an irreducible representation of $\mathrm{GL}(m)$. Then*

$$\mathrm{EP}_{\mathrm{GL}(m)}(\omega, \pi)^\infty = \begin{cases} 0 & n < m \\ i_{Q_m}^{\mathrm{GL}(n)}(\pi \boxtimes 1) & n \geq m \end{cases}$$

where Q_m is a parabolic subgroup of $\mathrm{GL}(n)$ with Levi subgroup $\mathrm{GL}(m) \times \mathrm{GL}(n-m)$.

5.2 $\mathrm{EP}_{\mathrm{GL}(m)}(\omega, \pi)^\infty$

We prove Proposition 5.12.

Lemma 5.14 *Suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of smooth $G \times H$ -modules, and X is a smooth H -module. Then the sequence of smooth G -modules*

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_H(C, X)^\infty &\rightarrow \mathrm{Hom}_H(B, X)^\infty \rightarrow \mathrm{Hom}_H(A, X)^\infty \rightarrow \dots \\ \dots \rightarrow \mathrm{Ext}_H^i(C, X)^\infty &\rightarrow \mathrm{Ext}_H^i(B, X)^\infty \rightarrow \mathrm{Ext}_H^i(A, X)^\infty \rightarrow \dots \end{aligned}$$

is exact.

Proof The given sequence is exact before taking the smooth vectors. We claim this remains true after taking the smooth vectors.

This is not immediate since the functor $\pi \rightarrow \pi^\infty$ from the category of all G -modules to the category of smooth G -modules is only left exact. Indeed, for G a non-discrete topological group, consider the right exact sequence of G -modules

$$\mathbb{C}[G] \rightarrow \mathbb{C} \rightarrow 0$$

where $\mathbb{C}[G]$ is the group algebra of G , i.e. the space of finite sums $\sum_{g \in G} c_g g$, where $c_g \in \mathbb{C}$, and the first arrow is $\sum_{g \in G} c_g g \mapsto \sum_{g \in G} c_g$. Since G is non-discrete, $\mathbb{C}[G]^\infty = 0$, so the sequence is not exact after taking smooth vectors.

The essence of the proof below is that for G -modules of the form $\mathrm{Hom}_H(Y, X)$ where X is a smooth module of H and Y of $G \times H$, taking smooth vectors for G is an exact functor.

Let K be an open compact subgroup of G . Since taking K -fixed vectors is an exact functor on smooth representations of G [3], the sequence

$$\begin{aligned}
0 \rightarrow \mathrm{Hom}_H(C^K, X) \rightarrow \mathrm{Hom}_H(B^K, X) \rightarrow \mathrm{Hom}_H(A^K, X) \rightarrow \dots \\
\dots \rightarrow \mathrm{Ext}_H^i(C^K, X) \rightarrow \mathrm{Ext}_H^i(B^K, X) \rightarrow \mathrm{Ext}_H^i(A^K, X) \rightarrow \dots
\end{aligned} \tag{15}$$

is exact.

Let Y be a smooth $G \times H$ -module. Taking an injective resolution of X :

$$0 \rightarrow X \rightarrow X_0 \rightarrow X_1 \rightarrow \dots,$$

we can calculate $\mathrm{Ext}_H^i(Y, X)$ by using the chain complex $\mathrm{Hom}_H(Y, X_i)$. Write $Y = \bigoplus_\tau Y_\tau$ as a direct sum over irreducible representations of K , where K acts on Y_τ as a multiple of τ . Then $\mathrm{Hom}_H(Y, X_i)$ is a direct product of spaces $\mathrm{Hom}_H(Y_\tau, X_i)$ over all τ , for every i . Let $\mathrm{Ext}_H^i(Y_\tau, X)$ be the cohomology of the complex $\mathrm{Hom}_H(Y_\tau, X_i)$. Then $\mathrm{Ext}_H^i(Y, X)$ is a direct product of spaces $\mathrm{Ext}_H^i(Y_\tau, X)$ over all τ , for every i .

For every τ , K acts on the complex $\mathrm{Hom}_H(Y_\tau, X_i)$ through a finite quotient, hence $\mathrm{Ext}_H^i(Y_\tau, X)^K$ is calculated using the complex $\mathrm{Hom}_H(Y_\tau, X_i)^K = \mathrm{Hom}_H(Y_\tau^K, X_i)$. Note that $Y_\tau^K = 0$ unless τ is the trivial representation of K and then $Y_\tau^K = Y^K$. Since $\mathrm{Ext}_H^i(Y, X)$ is a direct product of spaces $\mathrm{Ext}_H^i(Y_\tau, X)$, it follows that $\mathrm{Ext}_H^i(Y, X)^K \simeq \mathrm{Ext}_H^i(Y^K, X)$ for all i . This proves that the following is an exact sequence for all K :

$$\begin{aligned}
0 \rightarrow \mathrm{Hom}_H(C, X)^K \rightarrow \mathrm{Hom}_H(B, X)^K \rightarrow \mathrm{Hom}_H(A, X)^K \rightarrow \dots \\
\dots \rightarrow \mathrm{Ext}_H^i(C, X)^K \rightarrow \mathrm{Ext}_H^i(B, X)^K \rightarrow \mathrm{Ext}_H^i(A, X)^K \rightarrow \dots
\end{aligned} \tag{16}$$

The lemma now follows since the subspace of G -smooth vectors is a union of subspaces of K -fixed vectors for all K . \square

Proposition 5.17 *Suppose π is an irreducible representation of G . Then for $0 \leq k \leq \min(m, n)$, $\mathrm{Ext}_G^i(\omega_k, \pi)^\infty$ is a G' -module of finite length, and $\mathrm{EP}(\omega_k, \pi)^\infty$ is a well-defined element of the Grothendieck group of finite length representations of G' .*

Proof Return to the filtration (9), and for $0 \leq k \leq t = \min(m, n)$ consider the exact sequence

$$0 \rightarrow \omega_{k+1} \rightarrow \omega_k \rightarrow \mathcal{S}(\Omega_k) \rightarrow 0 \tag{17}$$

Apply Lemma 5.14 to this, with $X = \pi$. Part of the resulting long exact sequence is:

$$\begin{aligned}
\mathrm{Ext}_{\mathrm{GL}(m)}^i(\mathcal{S}(\Omega_k), \pi)^\infty \rightarrow \mathrm{Ext}_{\mathrm{GL}(m)}^i(\omega_k, \pi)^\infty \rightarrow \\
\rightarrow \mathrm{Ext}_{\mathrm{GL}(m)}^i(\omega_{k+1}, \pi)^\infty \rightarrow \mathrm{Ext}_{\mathrm{GL}(m)}^{i+1}(\mathcal{S}(\Omega_k), \pi)^\infty
\end{aligned}$$

By Proposition 5.9 the first and last terms have finite length. Since $\omega_{r+1} = 0$ we conclude $\text{Ext}_{GL(m)}^i(\omega_r, \pi)^\infty$ has finite length. By downward induction on k the same holds for $\text{Ext}_{GL(m)}^i(\omega_k, \pi)^\infty$ for all k . The assertion about EP follows from this and Lemma 3.1(2). \square

Proof of Proposition 5.12 By the preceding Lemma we conclude

$$\text{EP}(\omega_k, \pi)^\infty = \text{EP}(\mathcal{S}(\Omega_k), \pi)^\infty + \text{EP}(\omega_{k+1}, \pi)^\infty.$$

The Proposition follows by repeated applications of this, starting with $k = 0$ and going through $k = \min(m, n)$. \square

6 Example: $(\text{GL}(1), \text{GL}(n))$

Let $(G, H) = (\text{GL}(1), \text{GL}(n))$, acting on $\omega = \mathcal{S}(V)$ where $V = F^n$. The filtration is $0 \subset \mathcal{S}(\Omega_1) \subset \mathcal{S}(V) = \omega$, where $\Omega_1 = V - \{0\}$.

Lemma 6.1 *The representation $\omega = \mathcal{S}(V)$ of $\text{GL}(1)$ is a projective $\text{GL}(1)$ -module.*

Proof Let K and K' be maximal compact subgroups of $\text{GL}(1)$ and $\text{GL}(n)$, respectively. Let

$$\omega = \bigoplus_\tau \omega_\tau$$

be the decomposition of ω into K' -isotypic components. It suffices to prove that each summand is projective. If τ is not the trivial representation, then $\mathcal{S}(\Omega_1)_\tau = \omega_\tau$. Since $\mathcal{S}(\Omega_1)$ is projective, it follows that ω_τ is projective. It remains to deal with the trivial K' -type. The group K' stabilizes a lattice chain in V , and any smooth K' -invariant function is a linear combination of characteristic functions of the lattices in the chain. Since $\text{GL}(1)$ acts transitively on the lattices in the chain with the stabilizer K , it follows that

$$\omega^{K'} \cong \text{ind}_K^{\text{GL}(1)}(1)$$

and so is projective. \square

Recall the definition of the character $\xi = |\det|^{-\frac{n}{2}} \boxtimes |\det|^{\frac{1}{2}}$ of $G \times H$ (7). We have the exact sequence

$$0 \rightarrow \mathcal{S}(\Omega_1) \rightarrow \omega \rightarrow \xi \rightarrow 0 \quad (18)$$

which gives the exact sequence of $\text{GL}(n)$ -modules:

$$\begin{aligned} 0 &\rightarrow \text{Hom}_{\text{GL}(1)}(\xi, \chi) \rightarrow \text{Hom}_{\text{GL}(1)}(\omega, \chi) \\ &\rightarrow \text{Hom}_{\text{GL}(1)}(\mathcal{S}(\Omega_1), \chi) \rightarrow \text{Ext}_{\text{GL}(1)}^1(\xi, \chi) \rightarrow 0. \end{aligned}$$

By Lemma 5.14, the sequence remains exact after taking the smooth vectors; take the smooth dual to give the exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Ext}_{\text{GL}(1)}^1(\xi, \chi)^\vee &\rightarrow \text{Hom}_{\text{GL}(1)}(\mathcal{S}(\Omega_1), \chi)^\vee \\ &\rightarrow \text{Hom}_{\text{GL}(1)}(\omega, \chi)^\vee \rightarrow \text{Hom}_{\text{GL}(1)}(\xi, \chi)^\vee \rightarrow 0. \end{aligned}$$

By (14) and Proposition 4.1,

$$\begin{aligned} \text{Hom}_{\text{GL}(1)}(\mathcal{S}(\Omega_1), \chi)^\vee &\simeq i_{Q_1}^{\text{GL}(n)}(\chi^\vee \boxtimes 1) \\ \text{Hom}_{\text{GL}(1)}(\omega, \chi)^\vee &\simeq \Theta(\chi). \end{aligned}$$

By Proposition 3.6, $\text{EP}_{\text{GL}(1)}(\xi, \chi) = 0$, i.e.

$$\text{Ext}_{\text{GL}(1)}^1(\xi, \chi) = \text{Hom}_{\text{GL}(1)}(\xi, \chi) = \begin{cases} |\det|^{-\frac{1}{2}} & \chi = |\det|^{-\frac{n}{2}} \\ 0 & \text{else.} \end{cases}$$

If $\chi \neq |\det|^{-\frac{n}{2}}$, we conclude

$$\Theta(\chi) \simeq i_{Q_1}^{\text{GL}(n)}(\chi^\vee \boxtimes \mathbb{C}).$$

On the other hand, taking $\chi = |\det|^{-\frac{n}{2}}$ gives the exact sequence

$$0 \rightarrow |\det|^{\frac{1}{2}} \rightarrow i_{Q_1}^{\text{GL}(n)}(|\det|^{\frac{n}{2}} \boxtimes \mathbb{C}) \rightarrow \Theta(|\det|^{-\frac{n}{2}}) \rightarrow |\det|^{\frac{1}{2}} \rightarrow 0,$$

which implies

$$\theta(|\det|^{-\frac{n}{2}}) = |\det|^{\frac{1}{2}}.$$

Note that $\Theta(|\det|^{-\frac{n}{2}})$ and $i_{Q_1}^{\text{GL}(n)}(|\det|^{\frac{n}{2}} \boxtimes \mathbb{C})$ have the same image in the Grothendieck group. However $|\det|^{\frac{1}{2}}$ is a quotient of the former, and a submodule of the latter.

In particular if $n = 1$ this proves

$$\Theta(\chi) = \theta(\chi) = \chi^\vee \quad (\text{for all } \chi).$$

If $n = 2$ taking $\chi = |\det|^{-1}$ shows that $\Theta(|\det|^{-1})$ has Steinberg $|\det|^{\frac{1}{2}}$ as a submodule, and $|\det|^{\frac{1}{2}} \simeq \theta(|\det|^{-1})$ as a quotient (the opposite composition series of the induced representation).

Question 6.3 Consider a dual pair (G, G') in the metaplectic group $\widetilde{\text{Sp}}(2n)$, such that the split rank of G is not greater than the split rank of G' . Let ω be the oscillator representation of $\widetilde{\text{Sp}}(2n)$. Is it true that ω is a projective module in the category of

smooth representations of G ? In fact all we need is an affirmative answer to the (presumably weaker question): for every irreducible tempered representation π of G , is

$$\mathrm{Ext}_G^i(\omega, \pi) = 0 \quad \text{for all } i > 0? \quad (19)$$

More generally, does the same conclusion hold if π is contained in an extended (Vogan) L -packet containing a generic representation? A similar question is posed in [11, Conjecture 2]. For F an Archimedean field, similar questions may be posed in the category of (\mathfrak{g}, K) -modules.

Assuming (19) then $\Theta(\pi)^\vee = \mathrm{EP}_G(\omega, \pi)$; the right-hand side is a more elementary object, and easier to compute. A similar discussion applies when reducing the computation of Hom to EP in branching problems, as discussed in [11, Sect. 2].

Example 6.5 Consider the dual pair $(\mathrm{GL}(m), \mathrm{GL}(n))$ with $m \leq n$, and suppose π is an irreducible representation of $\mathrm{GL}(m)$. Assume $\mathrm{Ext}_{\mathrm{GL}(m)}^i(\omega, \pi) = 0$ for $i > 0$. Then by Theorem 5.13

$$\Theta(\pi) = i_Q^{\mathrm{GL}(n)}(\pi^\vee \boxtimes 1) \quad (20)$$

(equality in the Grothendieck group of finite length $\mathrm{GL}(n)$ -modules). The computation of $\Theta(\pi)$ (as opposed to its irreducible quotient $\theta(\pi)$) seems not to be available in the literature, even in the case of type II dual pairs.

If π is unitary, then $\theta(\pi) = \Theta(\pi) = i_Q^{\mathrm{GL}(n)}(\pi^\vee \boxtimes 1)$ since the induced representation is irreducible. Also if $m = 1$ then (19) holds by Lemma 6.1. In all of these cases (20) holds.

7 Type I Dual Pairs

Let $\mathrm{O}(N)$ be the isometry group of a nondegenerate quadratic space of dimension N . Let ω be the oscillator representation for the dual pair $(\mathrm{Sp}(2m), \mathrm{O}(N))$. We'll ignore the issue of covers, which play a nonessential role, while making the notation more cumbersome. If N is even, the covers can be avoided altogether.

Suppose $P(t) = M(t)N(t)$ is the stabilizer of a t -dimensional isotropic subspace of the symplectic space, and π is an irreducible representation of $M(t)$. We want to compute

$$\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi))^\infty \quad (21a)$$

as well as EP.

By Frobenius reciprocity (Lemma 3.2(1))

$$\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi)) \simeq \mathrm{Ext}_{M(t)}^i(i_{P(t)}^{\mathrm{Sp}(2m)}(\omega), \pi). \quad (21b)$$

Write

$$M(t) = \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t). \quad (21c)$$

For $0 \leq j \leq \min(t, n)$, let

$$P(t, j) = M(t, j)N(t, j) \subset M(t) \quad (21d)$$

be a parabolic subgroup of $M(t)$ where

$$M(t, j) = \mathrm{GL}(t - j) \times \mathrm{GL}(j) \times \mathrm{Sp}(2m - 2t) \quad (21e)$$

and $N(t, j) \subset \mathrm{GL}(t) \subset M(t)$. Let

$$Q(j) = L(j)U(j) \subset \mathrm{O}(N) \quad (21f)$$

be a parabolic subgroup of $\mathrm{O}(N)$ with Levi factor

$$L(j) = \mathrm{GL}(j) \times \mathrm{O}(N - 2j).$$

Let n be the Witt index of V , so V is the direct sum of an anisotropic kernel V_0 and a hyperbolic space of dimension $2n$. We will also consider the family of orthogonal spaces with the same anisotropic kernel V_0 . Such a space is determined by its Witt index, so we write $\omega_{m', n'}$ for the oscillator representation for the dual pair $(\mathrm{Sp}(2m'), \mathrm{O}(N'))$, where the orthogonal has dimension $N' = \dim(V_0) + 2n'$, anisotropic kernel V_0 and Witt index n' . With this convention $\omega = \omega_{m, n}$.

By [10, 3.IV.5], the representation $r_{P(t)}^{\mathrm{Sp}(2m)}(\omega_{m, n})$ of $M(t) \times \mathrm{O}(N) = \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t) \times \mathrm{O}(N)$ has a filtration

$$0 = F_{t+1} \subset F_t \subset F_{t-1} \subset \cdots \subset F_0 = r_{P(t)}^{\mathrm{Sp}(2m)}(\omega_{m, n}) \quad (21g)$$

with subquotients

$$F_j/F_{j+1} \simeq i_{P(t, j) \times Q(j)}^{M(t) \times \mathrm{O}(N)}(\xi(t, j) \otimes \mathcal{S}(\mathrm{GL}(j)) \boxtimes \omega_{m-t, n-j}) \quad (21h)$$

for some character $\xi(t, j)$ of $\mathrm{GL}(t - j)$. The actions are:

$\mathrm{GL}(j) \subset M(t, j)$ acts on $\mathcal{S}(\mathrm{GL}(j))$ on the left,

$\mathrm{GL}(t - j) \subset M(t, j)$ acts by some character $\xi(t, j)$,

$\mathrm{GL}(j) \subset L(j)$ acts on $\mathcal{S}(\mathrm{GL}(j))$ on the right,

$(\mathrm{Sp}(2m - 2t), \mathrm{O}(N - 2j))$ acts by the oscillator representation $\omega_{m-t, n-j}$.

So we need to compute

$$\mathrm{Ext}_{M(t)}^i (i_{P(t,j) \times Q(j)}^{M(t) \times O(N)} (\xi(t,j) \otimes \mathcal{S}(\mathrm{GL}(j)) \boxtimes \omega_{m-t,n-j}), \pi). \quad (21i)$$

By the second version of Frobenius reciprocity (Lemma 3.2(2)), applied to the induction from $P(t,j)$ to $M(t)$, this is isomorphic to

$$i_{Q(j)}^{O(N)} (\mathrm{Ext}_{M(t,j)}^i (\xi(t,j) \otimes \mathcal{S}(\mathrm{GL}(j)) \boxtimes \omega_{m-t,n-j}), r_{\overline{P}(t,j)}^{M(t)}(\pi)). \quad (21j)$$

Recall $M(t,j) = \mathrm{GL}(t-j) \times \mathrm{GL}(j) \times \mathrm{Sp}(2m-2t)$, and the tensor product appearing above is with respect to this decomposition. Write

$$\pi = \pi_1 \boxtimes \pi_2 \quad (21k)$$

with respect to the decomposition (21c), and then

$$r_{\overline{P}(t,j)}^{M(t)}(\pi) = \sum_{k=1}^{\ell} \sigma_{j,k} \boxtimes \tau_{j,k} \boxtimes \pi_2 \quad (21l)$$

as a representation of $\mathrm{GL}(t-j) \times \mathrm{GL}(j) \times \mathrm{Sp}(2m-2t)$. So we need to compute

$$\mathrm{Ext}_{M(t,j)}^i (\xi(t,j) \otimes \mathcal{S}(\mathrm{GL}(j)) \boxtimes \omega_{m-t,n-j}, \sigma_{j,k} \boxtimes \tau_{j,k} \boxtimes \pi_2) \quad (21m)$$

This is a representation of $L(j) = \mathrm{GL}(j) \times \mathrm{O}(N-2j)$. We need to know that the space of $L(j)$ -smooth vectors has finite length.

By the Kunneth formula, this is a sum of (external tensor products of) terms of the following three types. The first is

$$\mathrm{Ext}_{\mathrm{GL}(t-j)}^a (\xi(t,j), \sigma_{j,k}) \quad (21n)$$

where $\xi(t,j)$ is a character and $\sigma_{j,k}$ is irreducible. By Proposition 3.6 this is finite dimensional (and trivial as a representation of $L(j)$). The second type is

$$\mathrm{Ext}_{\mathrm{GL}(j)}^b (\mathcal{S}(\mathrm{GL}(j)), \tau_{j,k}) \quad (21o)$$

where $\mathrm{GL}(j)$ is acting $\mathcal{S}(\mathrm{GL}(j))$ on the right, and $\tau_{j,k}$ irreducible. By Lemma 2.4 this is 0 if $b > 0$, and for $b = 0$ the space of smooth vectors (with $\mathrm{GL}(j)$ acting on the left on $\mathcal{S}(\mathrm{GL}(j))$ and hence on $\mathrm{Hom}_{\mathrm{GL}(j)}(\mathcal{S}(\mathrm{GL}(j)), \tau_{j,k})$) is $\tau_{j,k}$. The third term is

$$\mathrm{Ext}_{\mathrm{Sp}(2m-2t)}^c (\omega_{m-t,n-j}, \pi_2)^\infty, \quad (21p)$$

which if π_2 is supercuspidal is nonzero only if $c = 0$, in which case it is of finite length for the corresponding orthogonal group.

We have proved the following intermediate result in the preceding paragraphs.

Lemma 7.2 *Suppose $\pi = \pi_1 \boxtimes \pi_2$ is an irreducible representation of $M(t) = \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t)$ with π_2 a supercuspidal representation of $\mathrm{Sp}(2m - 2t)$. Then $\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega_{m,n}, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi))$ is a finite length $O(N)$ -module for all i .*

Proposition 7.3 *Consider the oscillator representation ω for the dual pair $(\mathrm{Sp}(2m), O(N))$, and suppose π is an irreducible representation of $\mathrm{Sp}(2m)$. Then $\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega, \pi)^\infty$ is a finite length $O(N)$ -module for all i .*

Proof The proof of the Proposition is by induction on i , by an argument similar to [11, Sect. 5, Lemma 3]. Thus we assume that we have proved the Proposition for all $i \leq d$ and all π irreducible, and therefore $\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega, \pi)^\infty$, $i \leq d$, also has finite length as an $O(N)$ -module for π of finite length as $\mathrm{Sp}(2m)$ -module. The case $i = 0$ is in [8] and [10].

We now need the following lemma which is proved in the same manner as Lemma 5.14.

Lemma 7.4 *Suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of smooth H -modules, and X is a smooth $G \times H$ -module. Then the sequence*

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_H(X, A)^\infty \rightarrow \mathrm{Hom}_H(X, B)^\infty \rightarrow \mathrm{Hom}_H(X, C)^\infty \rightarrow \dots \\ \dots \rightarrow \mathrm{Ext}_H^i(X, A)^\infty \rightarrow \mathrm{Ext}_H^i(X, B)^\infty \rightarrow \mathrm{Ext}_H^i(X, C)^\infty \rightarrow \dots \end{aligned}$$

is exact.

Let π be any irreducible representation of $\mathrm{Sp}(2m)$. Then π sits in an exact sequence

$$0 \rightarrow \pi \rightarrow I \rightarrow J \rightarrow 0$$

where I is fully induced from a supercuspidal representation. Lemma 7.4 now gives an exact sequence

$$\mathrm{Ext}_{\mathrm{Sp}(2m)}^i(\omega, J)^\infty \rightarrow \mathrm{Ext}_{\mathrm{Sp}(2m)}^{i+1}(\omega, \pi)^\infty \rightarrow \mathrm{Ext}_{\mathrm{Sp}(2m)}^{i+1}(\omega, I)^\infty.$$

The first term is of finite length for $i \leq d$ by the inductive hypothesis, and this holds by Lemma 7.2 for the last term since I is fully induced from a supercuspidal representation. This implies that the middle term has finite length for $i + 1 \leq d + 1$. \square

We conclude that the space of $L(j)$ -smooth vectors in (21m) has finite length, and $\mathrm{EP}^{L(j)-\infty}$ is well defined. Therefore we can take the EP version of (21m), which by the Kunneth formula equals the tensor product of:

$$\begin{aligned} \mathrm{EP}_{\mathrm{GL}(t-j)}(\xi(t, j), \sigma_{j,k}), \\ \mathrm{EP}_{\mathrm{GL}(j)}(\mathcal{S}(\mathrm{GL}(j)), \tau_{j,k})^\infty \simeq \tau_{j,k}, \\ \mathrm{EP}_{\mathrm{Sp}(2m-2t)}(\omega_{m-t, n-j}, \pi_2). \end{aligned} \tag{22}$$

The first term is 0 unless $j = t$, in which case $N(t, j)$ is trivial, and (211) is simply π_1 . Here is the conclusion.

Theorem 7.6 *Suppose $O(N)$ is the isometry group of an orthogonal space of dimension N and Witt index n . Consider the oscillator representation $\omega_{m,n}$ for the dual pair $(\mathrm{Sp}(2m), O(N))$. Fix $0 \leq t \leq \min(m, n)$ and let $P(t) = M(t)N(t)$ be the stabilizer in $\mathrm{Sp}(2m)$ of an isotropic subspace of dimension t . Fix an irreducible representation π of $M(t) \simeq \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t)$. Consider the space*

$$\mathrm{EP}_{\mathrm{Sp}(2m)}(\omega_{m,n}, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi))^\infty$$

where the smooth vectors are with respect to $O(N)$. This is an element of the Grothendieck group of finite length $O(N)$ -modules.

If $t > n$, then

$$\mathrm{EP}_{\mathrm{Sp}(2m)}(\omega_{m,n}, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi))^\infty = 0.$$

Suppose $t \leq n$, and let $Q(t) = L(t)U(t)$ be the stabilizer of a t -dimensional isotropic subspace of the orthogonal space, so

$$L(t) \simeq \mathrm{GL}(t) \times O(N - 2t).$$

Write $\pi = \pi_1 \boxtimes \pi_2$ for $M(t) = \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t)$. Then

$$\mathrm{EP}_{\mathrm{Sp}(2m)}(\omega_{m,n}, i_{P(t)}^{\mathrm{Sp}(2m)}(\pi))^\infty \simeq i_{Q(t)}^{O(N)}(\pi_1 \boxtimes \mathrm{EP}_{\mathrm{Sp}(2m-2t)}(\omega_{m-t, n-t}, \pi_2)^\infty).$$

This can be stated more succinctly as follows. Let $\omega_{M(t), M'(t)}$ be the oscillator representation for the dual pair $(M(t), M'(t)) = (\mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t), \mathrm{GL}(t) \times O(N - 2t))$.

Corollary 7.7 *Consider the oscillator representation $\omega_{G, G'}$ of the dual pair $(G, G') = (\mathrm{Sp}(2m), O(N))$. For $t \leq m$, let $P(t) = M(t)N(t) \subset \mathrm{Sp}(2m)$ be the stabilizer of an isotropic space of dimension t . Similarly, if $t \leq n$, let $Q(t) = L(t)U(t) \subset O(N)$ be the stabilizer of an isotropic space of dimension t .*

If $t \leq \min(m, n)$, let $\omega_{M(t), L(t)}$ be the oscillator representation for the dual pair $(M(t), L(t)) = (\mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t), \mathrm{GL}(t) \times O(N - 2t))$.

Fix an irreducible representation π of $M(t)$. Then

$$\mathrm{EP}_G(\omega_{G, G'}, i_P^G(\pi))^\infty \simeq \begin{cases} 0 & t > n \\ i_{Q(t)}^{G'}(\mathrm{EP}_{M(t)}(\omega_{M(t), L(t)}, \pi)^\infty) & t \leq n. \end{cases}$$

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