J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2018-0028

Journal für die reine und angewandte Mathematik
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On a duality theorem of Schneider–Stuhler

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Abstract. We extend a duality theorem of Schneider and Stuhler about $\mathrm{Ext}^i_G[\pi_1, \pi_2]$ proved for smooth representations of a p-adic group G with central characters to all smooth representations assuming their result for only irreducible representations by generalities in homological algebra.

1. Introduction

Let $\underline{G}$ be a connected reductive algebraic group over F, a non-archimedean local field, and $G = \underline{G}(F)$ the locally compact group of F rational points of $\underline{G}$. In [11, p. 133], Schneider and Stuhler prove a *Duality Theorem* relating $\mathrm{Ext}^i_G[\pi_1, \pi_2]$ with $\mathrm{Tor}^G_{n-i}[D(\pi_1), \pi_2]$, where π_1 is a smooth representation of G of finite length, π_2 is a general smooth representation of G, and $D(\pi_1)$ denotes the Aubert–Zelevinsky involution of π_1.

The theorem of Schneider and Stuhler however assumed that π_1, π_2 had a central character, and $\mathrm{Ext}^i_G[\pi_1, \pi_2]$ is calculated in the category of smooth representations of G with that central character. In the presence of non-compact center, the category of smooth representations of G cannot be decomposed using central characters, and therefore to prove analogous results about $\mathrm{Ext}^i_G[\pi_1, \pi_2]$, where π_1, π_2 are general smooth representations of G, and one of the representations is irreducible, does not seem a consequence of the theorem of Schneider and Stuhler. For some of the applications the second author had in mind in [9] dealing with $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$, where π_1, π_2 are smooth representations of $\mathrm{GL}_n(F)$ with π_1 the restriction to $\mathrm{GL}_n(F)$ of an irreducible smooth representation of $\mathrm{GL}_{n+1}(F)$, it was important not to restrict oneself to smooth representations with a given central character.

In this paper we give a proof of the Schneider–Stuhler duality theorem without assuming any conditions on central characters. In fact, our proof uses the Schneider–Stuhler duality theorem in the simplest possible case, dealing with $\mathrm{Ext}^i_G[\pi_1, \pi_2]$, where π_1, π_2 are smooth and irreducible representations of G.

The idea behind the paper can be summarized as follows. Let A be a finitely generated commutative algebra over $\mathbb{C}$, and B an associative but not necessarily commutative algebra

The work of the second author was supported by a grant of the Government of the Russian Federation for the state support of scientific research, agreement 14.W03.31.0030 dated 15.02.2018.

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containing A in its center such that B is finitely generated as an A-module. Suppose that

$$M \to \mathcal{F}(M)$$

is a functor from the category of B-modules to the category of A-modules. Then one can understand the functor $M \to \mathcal{F}(M)$ via the intermediaries of the functors $M \to \mathcal{F}(M/\mathfrak{m}^k M)$ and $M \to \mathcal{F}(\widehat{M})$, where $\mathfrak{m}$ is a maximal ideal in A, and

$$\widehat{M} = \varprojlim (M/\mathfrak{m}^\ell M).$$

For us, $\mathcal{F}(M) = \mathrm{Ext}^i_B(N, M)$ or $\mathrm{Ext}^j_B(M, N)$ for N a simple B-module on which A acts via $A/\mathfrak{m}$. That we are in this algebraic setup for the extension problem in p-adic groups is by the work of Bernstein on "Bernstein center". The paper assumes that we understand the functor $M \to \mathcal{F}(M)$ for M a simple B-module (through the work of Schneider and Stuhler) and proceeds in steps, first extending their result (now for representations without central characters) to M of finite length as a B-module, then for M finitely generated over B, and finally for M an arbitrary B-module!

2. Aubert–Zelevinsky involution

In this section we discuss the Aubert–Zelevinsky involution $\pi \to D(\pi)$, cf. [2], in some detail as it plays a pivotal role in the Schneider–Stuhler theorem.

We continue with $\underline{G}$ a connected reductive algebraic group over F, a non-archimedean local field, and $G = \underline{G}(F)$ the locally compact group of F rational points of $\underline{G}$. Let π be an irreducible smooth representation of G. Associated to π is the Aubert–Zelevinsky involution $D(\pi)$ of π which is an element of the Grothendieck group of smooth representations of G of finite length, and defined by

$$D(\pi) = \mathrm{Ind}^G_{P_\phi}(R_{P_\phi}\pi) - \sum_{P_1} \mathrm{Ind}^G_{P_1}(R_{P_1}\pi) + \sum_{P_2} \mathrm{Ind}^G_{P_2}(R_{P_2}\pi) \mp \dots,$$

where P_ϕ is a fixed minimal parabolic in G, P_1 the next larger parabolics in G containing P_ϕ, P_2 next larger parabolics etc.; $R_{P_i}\pi$ are the normalized Jacquet modules of the representation π with respect to the parabolic P_i, and Ind denotes normalized parabolic induction. More precisely, it has been proved by Deligne and Lusztig for finite groups of Lie type in [6], and Aubert in [2] for p-adic groups, that for π an irreducible representation of G, the cohomology of the natural complex (that we will call Deligne–Lusztig–Aubert complex)

$$\begin{aligned} 0 \to \pi \to \sum_{|I|=|S|-1} \mathrm{Ind}^G_{P_I}(R_{P_I}\pi) &\to \sum_{|I|=|S|-2} \mathrm{Ind}^G_{P_I}(R_{P_I}\pi) \\ &\to \cdots \to \sum_{|I|=|S|-i} \mathrm{Ind}^G_{P_I}(R_{P_I}\pi) \to 0 \end{aligned}$$

is concentrated in top degree, defining $D(\pi)$ up to a sign; here S denotes the set of simple roots of G with respect to a maximal split torus of G contained in a minimal parabolic P_ϕ with $s = |S|$, and for $I \subset S$, P_I denotes the corresponding parabolic subgroup of G containing P_ϕ; i denotes the largest integer for which $R_{P_I}\pi$ is nonzero for some $I \subset S$ with $|I| = s - i$.

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There is another involution on the category of finite length smooth representations of G due to Bernstein in [4]. Denote this other involution as $\pi \to D'(\pi)$, which for an irreducible smooth representation π of G is defined as

$$D'(\pi) = \mathrm{Ext}^d_G[\pi, \mathcal{H}(G)],$$

where $\mathcal{H}(G)$ is the Hecke algebra of G and d is the only integer for which $\mathrm{Ext}^d_G[\pi, \mathcal{H}(G)]$ is nonzero (that there is only one d for an irreducible representation of G is part of [4]); this definition of D' makes sense for any finite length representation belonging to a particular Bernstein component, and can then be extended by linearity to a general finite length representation π by writing π as a finite sum $\pi = \sum \pi_\alpha$ of representations in different Bernstein components. The Hecke algebra of G being both left and right G-module, $\mathrm{Ext}^d_G[\pi, \mathcal{H}(G)]$ calculated using the left G action on $\mathcal{H}(G)$, is a right G-module.

By [11, Proposition IV.5.2], for an irreducible smooth representation π of G, we have $D'(\pi) \cong D(\pi^\vee)$. It is known from [2] and [11] that $D(\pi)$ takes irreducible representations of G to irreducible representations of G (up to a sign).

In this paper we will simply call $|D(\pi)|$ as the Aubert–Zelevinsky involution, and denote it as $D(\pi)$.

Remark 1. We would like to note two consequences of the Aubert–Zelevinsky involution defined through the above Deligne–Lusztig–Aubert complex. First, it manifestly implies that if π is any irreducible representation of G, then $D(\pi)$ is not only a representation in the Grothendieck group of representations of G, but an honest representation of G (this is the key input for Deligne-Lusztig's paper, as well as Aubert's paper, for their proof of irreducibility of $D(\pi)$). Second, the definition of $D(\pi)$ via the complex makes sense if π is any smooth representation of G with all its subquotients having cuspidal supports in the same Levi subgroup, say M (the complex still has cohomology only in the top degree; Aubert's proof in [2] never used irreducibility of π). Since any smooth representation of G is a direct sum of representations with cuspidal supports in different conjugacy classes of Levi subgroups, $\pi \to D(\pi)$ becomes an *exact* covariant functor from the category of *all* smooth representations of G to the category of smooth representations of G. The known isomorphism $D'(\pi) \cong D(\pi^\vee)$ for π an irreducible representation of G holds good for π any finite length representation of G, better still, there is a functorial isomorphism by [5], and therefore, known properties of D' on finite length representations due to Bernstein in [4] can be transported to $D(\pi)$ for finite length representations of G. In particular, by [4, Theorem 31, parts (2) and (4)], we have

$$D'(D'(\pi)) \cong \pi \quad \text{and} \quad D'(\mathrm{Ind}_P^G \lambda) \cong \mathrm{Ind}_{P^-}^G D'(\lambda),$$

which translates into

$$D(D(\pi)) \cong \pi \quad \text{and} \quad D(\mathrm{Ind}_P^G \lambda) \cong \mathrm{Ind}_{P^-}^G D(\lambda)$$

(where P^- is the parabolic which is opposite of P) hold good for all smooth representations π of G, and λ of M, a Levi subgroup of P, of finite lengths (these are usually asserted only up to semi-simplification).

Here is an example to put the ideas in the previous remark for use in understanding the Aubert–Zelevinsky involution in some explicit cases using non-semi-simple representations in an essential way.

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Proposition 2.1. *Let $G = \underline{G}(F)$ be a reductive p-adic group with $P = MN$ a parabolic in G. Let V be a regular supercuspidal representation of M, i.e., $V^w \ncong V$ for w any nontrivial element of $W_M \backslash W_G / W_M$, where W_G (resp. W_M) is the Weyl group of G (resp. of M) for a maximal split torus in G, where $V^w \ncong V$ means that either w does not preserve M, or if it does, it does not preserve the isomorphism class of V. Then the principal series representation $\mathrm{Ind}_P^G V$ has a unique irreducible quotient representation $Q(V)$, and a unique irreducible sub-representation $S(V)$. The Aubert–Zelevinsky involution of $Q(V)$ is $S(V)$, and that of $S(V)$ is $Q(V)$.*

Proof. By the geometric lemma, the Jacquet module $R_N(\mathrm{Ind}_P^G V)$ with respect to P is up to semi-simplification, the representation of M

$$R_N\big(\mathrm{Ind}_P^G V\big) \cong \sum_{w \in W_M \backslash W_G / W_M} V^w,$$

where the sum is taken over those elements of the double coset space $W_M \backslash W_G / W_M$ which preserve M. By hypothesis, this sum consists of distinct supercuspidal representations of M, and therefore, the Jacquet module of $\mathrm{Ind}_P^G V$ with respect to P is semi-simple, and each component appears with multiplicity 1. By a standard application of Frobenius reciprocity, the uniqueness of the irreducible quotient and sub-representation follows.

Recall that we are denoting the Aubert–Zelevinsky involution as $\pi \to D(\pi)$, and the other involution due to Bernstein as $\pi \to D'(\pi) = \mathrm{Ext}_G^d[\pi, \mathcal{H}(G)]$, with $D(\pi) \cong D'(\pi^\vee)$ on all finite length representations, although it will suffice for us to use it only for irreducible representations to conclude the proposition.

Note that D' is an exact contravariant functor with (cf. [4, Theorem 31])

$$D'\big(\mathrm{Ind}_P^G V\big) = \mathrm{Ind}_{P^-}^G(V^\vee),$$

where $P^- = MN^-$ is the opposite parabolic to $P = MN$. Now $S(V)$ is a subrepresentation of $\mathrm{Ind}_P^G V$, therefore $D'(\mathrm{Ind}_P^G V) = \mathrm{Ind}_{P^-}^G(V^\vee)$ has the irreducible representation $D'(S(V))$ as a quotient. But the irreducible quotient of $\mathrm{Ind}_{P^-}^G(V^\vee)$ being unique, if we can prove that $Q(V)^\vee$ is a quotient of $\mathrm{Ind}_{P^-}^G(V^\vee)$, then it will follow that $D'(S(V)) \cong Q(V)^\vee$, therefore $D(S(V)) \cong Q(V)$ as desired.

Finally, to prove that $Q(V)^\vee$ is a quotient of $\mathrm{Ind}_{P^-}^G(V^\vee)$, dualizing, we need to prove that $Q(V)$ is a sub-representation of $\mathrm{Ind}_{P^-}^G(V)$ given that $Q(V)^\vee$ is a sub-representation of $\mathrm{Ind}_P^G(V^\vee)$. This amounts by the Frobenius reciprocity to the well-known assertion (applied here to $Q(V)$) that $(\pi^\vee)_{N^-} \cong (\pi_N)^\vee$, which is part of the second adjointness theorem of Bernstein, cf. [4, Theorem 21]. □

3. The theorem of Schneider and Stuhler

Let $G = \underline{G}(F)$ be the locally compact group of F rational points of a reductive algebraic group $\underline{G}$, and Z its center. Let $\mathcal{H}(G)$ be the Hecke algebra of G, and for a character $\chi : Z \to \mathbb{C}^\times$, let $\mathcal{H}_\chi = \mathcal{H}_\chi(G)$ be the Hecke algebra of χ-invariant functions on G (locally constant with compact support modulo Z). Integration along Z, i.e.,

$$f(g) \to \int_Z f(gz)\chi(z)\, dz,$$

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defines a surjective algebra homomorphism from $\mathcal{H}(G)$ to $\mathcal{H}_\chi(G)$. The algebras $\mathcal{H}(G)$ and $\mathcal{H}_\chi(G)$ are algebras without units but with a rich supply of idempotents which allows one to define *non-degenerate* representations of these algebras which have the property $\mathcal{H}(G)V = V$ (resp. $\mathcal{H}_\chi(G)V = V$). There is the well-known equivalence of the category of smooth representations of G and non-degenerate representations of $\mathcal{H}(G)$, and similarly the category of smooth representations of G with central character χ and non-degenerate representations of $\mathcal{H}_\chi(G)$.

Let $\mathfrak{R}(G)$ be the abelian category of smooth representations of G, and for a character $\chi : Z \to \mathbb{C}^\times$, let $\mathfrak{R}(G;\chi)$ be the abelian sub-category of smooth representations of G on which Z operates by χ. We use $\mathrm{Ext}^i_G[V, V']$ to denote Ext groups in $\mathfrak{R}(G)$, and $\mathrm{Ext}^i_{G,\chi}[V, V']$ to denote Ext groups in $\mathfrak{R}(G;\chi)$. Similarly, $\mathrm{Tor}^{\mathcal{H}}_i[V, V']$, or $\mathrm{Tor}^G_i[V, V']$, will denote Tor groups for $\mathcal{H}(G)$-modules, and $\mathrm{Tor}^{\mathcal{H}_\chi}_i[V, V']$, or $\mathrm{Tor}^{G,\chi}_i[V, V']$, will denote Tor groups for $\mathcal{H}_\chi$-modules.

Here is the theorem of Schneider and Stuhler [11, p. 133].

Theorem 1. *Let $V \in \mathfrak{R}(G,\chi)$ be an irreducible, admissible representation of G appearing as a sub-quotient of a principal series representation $\mathrm{Ind}_P^G \rho$ for a cuspidal representation ρ of a Levi subgroup M of a parabolic subgroup P of G with $d_s = d_s(V)$ (the rank of the maximal split torus in $Z(M) \cap [G,G]$). Then for any smooth representation $V' \in \mathfrak{R}(G,\chi)$ of G, there is a natural isomorphism*

$$\mathrm{Ext}^i_{G,\chi}[V, V'] \cong \mathrm{Tor}^{\mathcal{H}_\chi}_{d_s - i}[D(V^\vee), V']$$

given by the cap product with $\mathrm{Tor}^{\mathcal{H}_\chi}_{d_s}[D(V^\vee), V] \cong \mathbb{C}$.

The following well-known lemma converts Tor into an Ext eliminating the need of Tor in the above theorem which may be more useful in some contexts (such as in "branching laws").

Lemma 3.1. *The following statements hold:*

(a) *For any two smooth representations π_1, π_2 of a reductive p-adic group G, there is a canonical isomorphism*

$$\mathrm{Ext}^i_G[\pi_1, \pi_2^\vee] \cong \mathrm{Ext}^i_G[\pi_2, \pi_1^\vee] \cong \mathrm{Hom}_{\mathbb{C}}[\mathrm{Tor}^G_i[\pi_1, \pi_2], \mathbb{C}].$$

(b) *Similarly, for any two smooth representations π_1, π_2 of a reductive p-adic group G with a given central character $\chi : Z \to \mathbb{C}^\times$, there is a canonical isomorphism*

$$\mathrm{Ext}^i_{G,\chi}[\pi_1, \pi_2^\vee] \cong \mathrm{Hom}_{\mathbb{C}}[\mathrm{Tor}^{G,\chi}_i[\pi_1, \pi_2], \mathbb{C}] \cong \mathrm{Hom}_{\mathbb{C}}[\mathrm{Tor}^{G,\chi}_i[\pi_2, \pi_1], \mathbb{C}].$$

Proof. Let

$$\cdots \to P_2 \to P_1 \to P_0 \to \pi_2 \to 0$$

be a projective resolution of π_2 in $\mathfrak{R}(G)$. By taking the contragredient, we have an injective resolution of $\pi_2^\vee$:

$$0 \to \pi_2^\vee \to P_0^\vee \to P_1^\vee \to P_2^\vee \to \cdots .$$

Applying the functor $\mathrm{Hom}_G[\pi_1, -]$ to this exact sequence, and omitting the first term, we get the cochain complex

$$\mathrm{Hom}[\pi_1, P_\star] = \left\{0 \to \mathrm{Hom}_G[\pi_1, P_0^\vee] \to \mathrm{Hom}_G[\pi_1, P_1^\vee] \to \mathrm{Hom}_G[\pi_1, P_2^\vee] \to \cdots\right\}$$

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whose cohomology is by definition $\mathrm{Ext}^i_G[\pi_1, \pi_2^\vee]$. For any smooth representation V of G giving rise to a representation of $\mathcal{H} = \mathcal{H}(G)$ on V, define

$$\pi_1 \otimes_{\mathcal{H}} V = \frac{\pi_1 \otimes V}{\{hv_1 \otimes v - v_1 \otimes h^\vee v \mid h \in \mathcal{H},\ v_1 \in \pi_1,\ v \in V\}},$$

where $h \to h^\vee$ is the anti-automorphism of $\mathcal{H}$ given by $h^\vee(g) = h(g^{-1})$. Clearly, we have

$$\mathrm{Hom}_{\mathbb{C}}[\pi_1 \otimes_{\mathcal{H}} V, \mathbb{C}] = \mathrm{Hom}_{\mathcal{H}}[\pi_1, V^\star] = \mathrm{Hom}_{\mathcal{H}}[\pi_1, V^\vee] = \mathrm{Hom}_G[\pi_1, V^\vee],$$

where $V^\star$ is the vector space dual of V, and $V^\vee$ is the smooth dual of V (as a G-module). This allows one to re-write the cochain complex $\mathrm{Hom}_G[\pi_1, P_\star]$ as

$$0 \to \mathrm{Hom}_{\mathbb{C}}[\pi_1 \otimes_{\mathcal{H}} P_0, \mathbb{C}] \to \mathrm{Hom}_{\mathbb{C}}[\pi_1 \otimes_{\mathcal{H}} P_1, \mathbb{C}] \to \mathrm{Hom}_{\mathbb{C}}[\pi_1 \otimes_{\mathcal{H}} P_2, \mathbb{C}] \to \cdots .$$

This complex is just the dual of the complex

$$\cdots \to \pi_1 \otimes_{\mathcal{H}} P_2 \to \pi_1 \otimes_{\mathcal{H}} P_1 \to \pi_1 \otimes_{\mathcal{H}} P_0 \to 0,$$

which calculates $\mathrm{Tor}_i^G[\pi_1, \pi_2]$. Since taking cohomology of a complex over $\mathbb{C}$ commutes with taking duals, this completes the proof of part (a) of the lemma.

Part (b) of the lemma is similarly proved by replacing $\mathfrak{R}(G)$ by $\mathfrak{R}(G;\chi)$, and $\mathcal{H}$ by $\mathcal{H}_\chi$. □

We will use Theorem 1 due to Schneider and Stuhler to prove the following theorem which is the main result of this paper.

Theorem 2. *Let G be a reductive p-adic group, and π an irreducible, admissible representation of G. Let $d(\pi)$ be the largest integer $i \geq 0$ such that there is an irreducible, admissible representation π' of G with $\mathrm{Ext}^i_G[\pi, \pi']$ nonzero.*

(1) *There is a unique irreducible representation π' of G with $\mathrm{Ext}_G^{d(\pi)}[\pi, \pi'] \neq 0$.*

(2) *The representation π' in (1) is nothing but $D(\pi)$, where $D(\pi)$ is the Aubert–Zelevinsky involution of π, and $d(\pi)$ is the split rank of the Levi subgroup M of G which carries the cuspidal support of π.*

(3) $\mathrm{Ext}_G^{d(\pi)}[\pi, D(\pi)] \cong \mathbb{C}$.

(4) *For any smooth representation π' of G, the bilinear pairing*

$$(*) \qquad \mathrm{Ext}^i_G[\pi, \pi'] \times \mathrm{Ext}^j_G[\pi', D(\pi)] \to \mathrm{Ext}_G^{i+j=d(\pi)}[\pi, D(\pi)] \cong \mathbb{C}$$

is non-degenerate in the sense that if $\pi' = \varinjlim \pi'_n$ of finitely generated G-sub-modules π'_n, then

$$\mathrm{Ext}^i_G[\pi, \pi'] = \varinjlim \mathrm{Ext}^i_G[\pi, \pi'_n],$$

a direct limit of finite-dimensional vector spaces over $\mathbb{C}$, and

$$\mathrm{Ext}^j_G[\pi', D(\pi)] = \varprojlim \mathrm{Ext}^j_G[\pi'_n, D(\pi)],$$

an inverse limit of finite-dimensional vector spaces over $\mathbb{C}$, and the pairing in $(*)$ is the direct limit of perfect pairings on these finite-dimensional spaces

$$\mathrm{Ext}^i_G[\pi, \pi'_n] \times \mathrm{Ext}^j_G[\pi'_n, D(\pi)] \to \mathrm{Ext}_G^{i+j=d(\pi)}[\pi, D(\pi)] \cong \mathbb{C}.$$

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(Observe that a compatible family of perfect pairings on finite-dimensional vector spaces $B_n : V_n \times W_n \to \mathbb{C}$ with V_n part of an inductive system, and W_n part of a projective system, gives rise to a natural pairing $B : \varinjlim V_n \times \varprojlim W_n \to \mathbb{C}$ such that the associated homomorphism from $(\varinjlim V_n)^\star$ to $\varprojlim W_n$ is an isomorphism.)

The proof of this theorem will be achieved in two steps. We will take the first step in this section proving the theorem assuming π' to be an irreducible representation of G by a minor modification of the result of Schneider and Stuhler. Having proved the theorem for π' irreducible, in particular, parts (1), (2) and (3) of the theorem are proved in this section, the rest of the paper will prove part (4) of the theorem for a general smooth representation π' of G.

For π' an irreducible representation of G, we note that if this theorem is true for reductive groups $G = G_1$ and $G = G_2$, then it is true for $G = G_1 \times G_2$ by the Künneth theorem, see [10] for a proof of the Künneth theorem.

Also, this theorem is clearly true for tori T (where the Aubert–Zelevinsky involution is trivial) on noting that $\mathrm{Ext}^i_T[\chi_1, \chi_2] = 0$ for all i for any two characters $\chi_1, \chi_2 : T \to \mathbb{C}^\times$, if $\chi_1 \neq \chi_2$, and that

$$\mathrm{Ext}^i_T[\chi, \chi] \cong \mathrm{Ext}^i_T[\mathbb{C}, \mathbb{C}] = \Lambda^i(\mathbb{C}^d),$$

where $T = T^c \times \mathbb{Z}^d$ with T^c the maximal compact subgroup of T.

Let $DG = [G, G]$, the derived subgroup of G, and let Z be the center of G. By the above remarks, the theorem is true for the group $G' = DG \times Z$, and π' an irreducible representation of G'. Since G' is a normal subgroup of G of finite index, it may appear that the generality

$$\mathrm{Ext}^i_G[\pi_1, \pi_2] = \mathrm{Ext}^i_{G'}[\pi_1, \pi_2]^G$$

will prove the theorem for irreducible representation π of G knowing it for irreducible representations π' of G'. This does not seem to be the case since an irreducible representation of G may decompose when restricted to G', and therefore even to conclude

$$\mathrm{Ext}^{d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C}$$

seems not obvious. (A case in point would be when an irreducible representation π of G decomposes as $\pi = 2\pi'$ when restricted to G'. From the information that $\mathrm{Ext}^d_{G'}[\pi', D\pi'] = \mathbb{C}$, and therefore $\mathrm{Ext}^d_{G'}[\pi, D\pi] = \mathbb{C}^4$, we need to deduce that $\mathrm{Ext}^d_G[\pi, D\pi] = \mathbb{C}$, which seems not clear.)

If the above mentioned obstacle to deducing Theorem 2 was not there, we would be using the theorem of Schneider and Stuhler only for semi-simple groups (and only for irreducible representations of them), that of course would have been preferable, but not having succeeded in that direction, we will use the theorem of Schneider and Stuhler for reductive groups (but only for irreducible representations of them) in a slightly different approach.

Proposition 3.2. *Fix a surjective map $\phi : G \to \mathbb{Z}^d$ whose kernel G_ϕ is an open subgroup of G such that G_ϕ contains $G^c = (DG)(F) \cdot Z^c(F)$, where $Z^c(F)$ is the maximal compact subgroup of Z. Then the following statements hold:*

(a) *If π is a finitely generated smooth representation of G with central character $\chi : Z \to \mathbb{C}^\times$, and is a projective module in $\mathfrak{R}(G, \chi)$, $\pi|_{G_\phi}$ is a projective module in $\mathfrak{R}(G_\phi)$.*

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(b) *For the representation $Q_0 = \mathcal{S}(\mathbb{Z}^d) = \mathrm{ind}_{\langle e\rangle}^{\mathbb{Z}^d}(\mathbb{C})$ of $\mathbb{Z}^d$, treated as a representation of G via the map $\phi : G \to \mathbb{Z}^d$, and π a smooth representation of G which is a projective module in $\mathfrak{R}(G,\chi)$, $\pi \otimes Q_0$ is a smooth representation of G which is a projective module in $\mathfrak{R}(G)$.*

Proof. (a) Since π is a finitely generated G-module with central character $\chi : Z \to \mathbb{C}^\times$, and $Z \cdot G_\phi$ is of finite index in G (this is true even if F has +ve characteristic), π is finitely generated as a G_ϕ-module. Since $Z \cap G_\phi$ is a compact abelian group contained in the center of G_ϕ, it decomposes any smooth representation of G_ϕ into a direct sum of eigenspaces for $Z \cap G_\phi$:

$$V = \sum_\alpha V_\alpha,$$

where V_α is the subspace of V on which $Z \cap G_\phi$ acts by the character $\alpha : Z \cap G_\phi \to \mathbb{C}^\times$. Since π is finitely generated as a G_ϕ-module, we have

$$\mathrm{Hom}_{G_\phi}[\pi, V] = \sum_\alpha \mathrm{Hom}_{G_\phi}[\pi, V_\alpha].$$

Therefore to prove the projectivity of π as a G_ϕ-module, it suffices to consider only those surjective homomorphisms

$$\lambda : V_1 \to V_2 \to 0$$

of G_ϕ-modules on which $Z \cap G_\phi$ acts by the restriction of χ to $Z \cap G_\phi$. Since $Z \cap G_\phi$ acts by a character which is $\chi|_{Z\cap G_\phi}$ on both V_1 and V_2, we can let Z operate on V_1 and V_2 by χ, giving a structure of $Z \cdot G_\phi$-module to both V_1 and V_2, making $\lambda : V_1 \to V_2$, $Z \cdot G_\phi$-equivariant. By inducing these representations to G, we get

$$\mathrm{ind}(\lambda) : \mathrm{ind}_{Z\cdot G_\phi}^G(V_1) \to \mathrm{ind}_{Z\cdot G_\phi}^G(V_2) \to 0.$$

Observe that by Frobenius reciprocity,

$$\mathrm{Hom}_G[\pi, \mathrm{ind}_{Z\cdot G_\phi}^G(V_i)] = \mathrm{Hom}_{Z\cdot G_\phi}[\pi, V_i]$$

for both $i = 1, 2$. But Z operates on V_i as well as π by χ, hence,

$$\mathrm{Hom}_G[\pi, \mathrm{ind}_{Z\cdot G_\phi}^G(V_i)] = \mathrm{Hom}_{Z\cdot G_\phi}[\pi, V_i] = \mathrm{Hom}_{G_\phi}[\pi, V_i].$$

It follows that an element ϕ_2 from $\mathrm{Hom}_{G_\phi}[\pi, V_2]$ can be interpreted as an element φ_2 from $\mathrm{Hom}_G[\pi, \mathrm{ind}_{Z\cdot G_\phi}^G(V_2)]$. Since π is a projective module in $\mathfrak{R}(G,\chi)$, and both representations $\mathrm{ind}_{Z\cdot G_\phi}^G(V_1)$ and $\mathrm{ind}_{Z\cdot G_\phi}^G(V_2)$ are in $\mathfrak{R}(G,\chi)$ with a surjection

$$\mathrm{ind}(\lambda) : \mathrm{ind}_{Z\cdot G_\phi}^G(V_1) \to \mathrm{ind}_{Z\cdot G_\phi}^G(V_2) \to 0,$$

φ_2 can be lifted to φ_1 in $\mathrm{Hom}_G[\pi, \mathrm{ind}_{Z\cdot G_\phi}^G(V_1)]$, and hence $\phi_2 \in \mathrm{Hom}_{G_\phi}[\pi, V_2]$ can be lifted to ϕ_1 in $\mathrm{Hom}_{G_\phi}[\pi, V_1]$, proving the projectivity of π in $\mathfrak{R}(G_\phi)$.

(b) Note that

$$\pi \otimes Q_0 = \pi \otimes \mathrm{ind}_{G_\phi}^G(\mathbb{C}) = \mathrm{ind}_{G_\phi}^G(\pi|_{G_\phi}).$$

By part (a) of the proposition, $\pi|_{G_\phi}$ is a projective G_ϕ-module, hence the following most primitive form of the Frobenius reciprocity in the next lemma completes the proof of projectivity of $\pi \otimes Q_0 = \mathrm{ind}_{G_\phi}^G(\pi|_{G_\phi})$ as a G-module. □

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Lemma 3.3. *Let G_ϕ be an open subgroup of a p-adic group G. Let E be a smooth representation of G_ϕ, and F a smooth representation of G. Then*

$$\mathrm{Hom}_G[\mathrm{ind}_{G_\phi}^G E, F] \cong \mathrm{Hom}_{G_\phi}[E, F|_{G_\phi}].$$

Proof. Recall the definition of the induced representation:

$$\mathrm{ind}_{G_\phi}^G E = \{f : G \to E \mid f(hg) = hf(g),\ h \in G_\phi,\ g \in G,\ f \text{ compactly supported}\}.$$

Since G_ϕ is an open subgroup of G, the space of functions $f : G \to E$ with support in G_ϕ (and with $f(hg) = hf(g)$ for all $h \in G_\phi$ and all $g \in G$) forms a G_ϕ-invariant subspace of $\mathrm{ind}_{G_\phi}^G E$ whose G_ϕ/G translates (G operates on the right!) form a direct sum decomposition of $\mathrm{ind}_{G_\phi}^G E$. The proof of the lemma is now clear. □

Proposition 3.4. *Let π_1 and π_2 be two smooth irreducible representations of G with the same central character $\chi : Z \to \mathbb{C}^\times$. Let $\phi : G \to \mathbb{Z}^d$ be a surjective homomorphism with kernel G_ϕ as before with $G_\phi \cap Z$, the maximal compact subgroup of Z. Then*

$$\mathrm{Ext}_G^k[\pi_1, \pi_2] \cong \sum_{k=i+j} \mathrm{Ext}_{G,\chi}^i[\pi_1, \pi_2] \otimes \mathrm{Ext}_{\mathbb{Z}^d}^j[\mathbb{C}, \mathbb{C}],$$

where in $\mathrm{Ext}_{\mathbb{Z}^d}^j[\mathbb{C}, \mathbb{C}]$, $\mathbb{C}$ denotes the trivial module for $\mathbb{Z}^d$.

Proof. Let

$$P_* = \cdots \to P_1 \to P_0 \to \pi_1 \to 0$$

be a projective resolution for π_1 in $\mathfrak{R}(G, \chi)$, and let

$$Q_* = \cdots \to Q_1 \to Q_0 \to \mathbb{C} \to 0$$

be a finite free resolution (such as the Koszul complex) for the trivial module $\mathbb{C}$ for the group $\mathbb{Z}^d$, or for the corresponding A-module for

$$A = \mathbb{C}[z_1, \ldots, z_d, z_1^{-1}, \ldots, z_d^{-1}] = \mathbb{C}[\mathbb{Z}^d] = \mathcal{H}(\mathbb{Z}^d) \cong \mathrm{ind}_{\langle e\rangle}^{\mathbb{Z}^d}(\mathbb{C}).$$

By the previous proposition, it follows that the tensor product $P_* \otimes Q_*$

$$\cdots \to P_1 \otimes Q_0 + P_0 \otimes Q_1 \to P_0 \otimes Q_0 \to \pi_1 \to 0$$

is a projective resolution of π_1 as a smooth G-module, and therefore also a projective resolution of π_1 as a smooth G^0-module, where $G^0 = Z \cdot G_\phi$. Therefore, $\mathrm{Ext}_{G^0}^k[\pi_1, \pi_2]$ is the cohomology of the cochain complex $\mathrm{Hom}_{G^0}[\bigoplus_{i+j=k} P_i \otimes Q_j, \pi_2]$. Recall that each Q_j is a direct sum of $\mathbb{Z}^d$-modules $\mathcal{H}(\mathbb{Z}^d)$, which considered as a G-module via the map $\phi : G \to \mathbb{Z}^d$ with kernel G_ϕ is nothing but $\mathrm{ind}_{G_\phi}^G(\mathbb{C})$. As representations of G^0, we have $Q_j = \mathrm{ind}_{G_\phi}^{G^0}(F_j)$ for F_j a finite-dimensional vector space over $\mathbb{C}$ on which G_ϕ operates trivially. Towards the calculation of $\mathrm{Ext}_{G^0}^i[\pi_1, \pi_2]$, let us note that for F_j a finite-dimensional vector space over $\mathbb{C}$ on which G_ϕ operates trivially,

$$\begin{aligned}
\mathrm{Hom}_{G^0}[P_i \otimes \mathrm{ind}_{G_\phi}^{G^0}(F_j), \pi_2] &= \mathrm{Hom}_{G^0}[\mathrm{ind}_{G_\phi}^{G^0}(P_i|_{G_\phi} \otimes F_j), \pi_2] \\
&= \mathrm{Hom}_{G_\phi}[P_i \otimes F_j, \pi_2] \\
&= \mathrm{Hom}_{G_\phi}[P_i, \pi_2] \otimes \mathrm{Hom}_{G_\phi}[F_j, \mathbb{C}] \\
&= \mathrm{Hom}_{G^0}[P_i, \pi_2] \otimes \mathrm{Hom}_{\mathcal{Z}^0}[Q_j, \mathbb{C}],
\end{aligned}$$

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where $\mathscr{L}^0 = \phi(G^0) \subset \mathbb{Z}^d$, a subgroup of finite index, and in the last equality, we have used the fact that $G^0 = Z \cdot G_\phi$, and that P_i and π_2 are G^0-modules with the same central character $\chi : Z \to \mathbb{C}^\times$.

Summarizing the discussion above, the natural mapping of (tensor product of) the chain complexes

$$\Pi : \mathrm{Hom}_{G^0}[P_i, \pi_2] \otimes \mathrm{Hom}_{\mathscr{L}^0}[Q_j, \mathbb{C}] \to \mathrm{Hom}_{G^0}[P_i \otimes Q_j, \pi_2]$$

is an isomorphism of chain complexes which proves that

$$\mathrm{Ext}^k_{G^0}[\pi_1, \pi_2] \cong \sum_{k=i+j} \mathrm{Ext}^i_{G^0,\chi}[\pi_1, \pi_2] \otimes \mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}].$$

(A small subtlety in the proof of the proposition lies in the fact that the corresponding homomorphism Π of chain complexes with $(G^0, \mathscr{L}^0)$ replaced by $(G, \mathbb{Z}^d)$ is not an isomorphism, but still the conclusion about the Ext groups is true.) Observe that all the modules appearing in the isomorphism Π above carry an action of G (and are projective objects in appropriate categories), and that Π is equivariant under the action of the finite group G/G^0. Since taking cohomology of a complex over $\mathbb{C}$, and taking G/G^0-invariants, which being a finite group, is the same as taking G/G^0-invariants of the complex and taking the cohomology, it follows that

$$\mathrm{Ext}^k_{G}[\pi_1, \pi_2] \cong \sum_{k=i+j} \left[\mathrm{Ext}^i_{G^0,\chi}[\pi_1, \pi_2] \otimes \mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}]\right]^{G/G^0}.$$

Now making the crucial observation that the action of G/G^0 on $\mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}]$ via $\mathbb{Z}^d/\mathscr{L}^0$ is trivial, we deduce that

$$\begin{aligned}\mathrm{Ext}^k_{G}[\pi_1, \pi_2] &\cong \sum_{k=i+j} \left[\mathrm{Ext}^i_{G^0,\chi}[\pi_1, \pi_2] \otimes \mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}]\right]^{G/G^0} \\ &= \sum_{k=i+j} \mathrm{Ext}^i_{G^0,\chi}[\pi_1, \pi_2]^{G/G^0} \otimes \mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}] \\ &= \sum_{k=i+j} \mathrm{Ext}^i_{G,\chi}[\pi_1, \pi_2] \otimes \mathrm{Ext}^j_{\mathbb{Z}^d}[\mathbb{C}, \mathbb{C}],\end{aligned}$$

where in the last equality we are using the fact that the restriction map from $\mathrm{Ext}^j_{\mathbb{Z}^d}[\mathbb{C}, \mathbb{C}]$ to $\mathrm{Ext}^j_{\mathscr{L}^0}[\mathbb{C}, \mathbb{C}]$ is an isomorphism, proving the proposition. □

Remark 2. At this point we have proved Theorem 2 for π' an irreducible smooth representation of any reductive p-adic group G as a consequence of Theorem 1 by combining Lemma 3.1 and Proposition 3.4. The rest of the paper will deduce Theorem 2 for *any* smooth representation π' of G from the irreducible case.

4. Matlis duality and Injective resolutions

The results in this paper need injective resolutions with certain properties which should be well known, but not finding a suitable reference, we have given proofs but it must be emphasized that nothing is new in this section.

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We begin by recalling the Matlis duality, cf. [7], first for a commutative algebra A, and then for a non-commutative algebra B, which is what is needed in this paper. In the commutative case, Matlis duality is a natural functor from the category of modules over a (noetherian) local ring $(A, \mathfrak{m})$ to modules over A turning a noetherian A-module to an artinian A-module, and projective A-modules to injective A-modules. It is especially easy to describe for local rings $(A, \mathfrak{m})$ for which $A/\mathfrak{m} = k$ is contained in A as is the case in all our applications.

Definition 1 (Matlis duality). For a module M over a noetherian local ring $(A, \mathfrak{m})$, the Matlis dual of M, to be denoted as $M^\vee$, is the A-module

$$M^\vee = \mathrm{Hom}_A[M, E(k)],$$

where $E(k)$ is the injective hull of the A-module $k = A/\mathfrak{m}$.

Remark 3. If $k = A/\mathfrak{m}$ can be considered as a subalgebra of A, then

$$E(k) = \varinjlim \mathrm{Hom}_k[A/\mathfrak{m}^\ell, k].$$

Therefore for a finitely generated A-module M, we have

$$\mathrm{Hom}_A[M, E(k)] = \varinjlim \mathrm{Hom}_A[M, \mathrm{Hom}_k[A/\mathfrak{m}^\ell, k]] = \varinjlim \mathrm{Hom}_k[M/\mathfrak{m}^\ell M, k].$$

(Finite generation of M is used in the first equality, whereas the second equality follows from the "adjoint associativity" $\mathrm{Hom}_A(M, \mathrm{Hom}_k(N, P)) \cong \mathrm{Hom}_k(M \otimes_A N, P)$.) Therefore for a finitely generated A-module M, we have

$$M^\vee = \mathrm{Hom}_A[M, E(k)] = \varinjlim \mathrm{Hom}_k[M/\mathfrak{m}^\ell M, k].$$

The Matlis duality $M \to M^\vee$ is an exact contravariant functor from the category of A-modules to the category of A-modules with $M \cong (M^\vee)^\vee$ for M a finite length A-module, taking projective A-modules to injective A-modules, and noetherian A-modules to artinian A-modules. For any finitely generated A-module M, by Remark 3,

$$M^\vee = \varinjlim \mathrm{Hom}_k[M/\mathfrak{m}^\ell M, k]$$

has the property that

$$M^\vee = \bigcup_{k \geq 1} \mathrm{Ann}(\mathfrak{m}^k; M^\vee),$$

where for any A-module M, we let

$$\mathrm{Ann}(\mathfrak{m}^k; M) = \{m \in M \mid \mathfrak{m}^k m = 0\}.$$

If N is a finite artinian A-module, then $N^\vee$ too is one, with $(N^\vee)^\vee \cong N$. Taking a finitely generated projective resolution of the finitely generated A-module $N^\vee$

$$\to P_n \to \cdots \to P_1 \to P_0 \to N \to 0,$$

and applying the Matlis duality to this complex, we have an injective resolution of N by $P_i^\vee$. We summarize this conclusion in the following proposition.

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Proposition 4.1. *Let $(A, \mathfrak{m})$ be a local $k = A/\mathfrak{m}$ algebra which is finitely generated over k. Then any artinian A-module N killed by a power of $\mathfrak{m}$ has an injective resolution*

$$0 \to N \to I_0 \to I_1 \to I_2 \to \cdots,$$

by artinian and injective A-modules I_j with $I_j = \bigcup_{k \geq 1} \mathrm{Ann}(\mathfrak{m}^k; I_j)$ for all j.

The Matlis duality discussed above also makes sense in the non-commutative setting of our paper: thus we have a local ring $(A, \mathfrak{m})$, an (associative) algebra B containing A in its center such that B is finitely generated as an A-module. We will denote by B^0, the *opposite* algebra. If M is a module over B^0, the Matlis dual of M, to be denoted as $M^\vee$, is the B-module

$$M^\vee = \mathrm{Hom}_A[M, E(k)],$$

where $E(k)$ is the injective hull of the A-module $k = A/\mathfrak{m}$. As in the commutative case, it can be seen that if M is finitely generated as a B-module, then

$$M^\vee \cong \varinjlim \mathrm{Hom}_k[M/\mathfrak{m}^\ell M, k].$$

Also, as in the commutative case, $M \to M^\vee$ is an exact contravariant functor from the category of B^0-modules to category of B-modules, taking finitely generated projective B^0-modules to injective B-modules, noetherian B^0-modules to artinian B-modules, and for any B-module M,

$$M^\vee = \bigcup_{k \geq 1} \mathrm{Ann}(\mathfrak{m}^k; M^\vee),$$

where for any B^0-module M, we let

$$\mathrm{Ann}(\mathfrak{m}^k; M) = \{m \in M \mid \mathfrak{m}^k m = 0\}.$$

Further, $N \to N^\vee$ is a bijective correspondence between finite artinian B^0-modules N and finite artinian B-modules $N^\vee$. Taking a projective resolution of a finite B^0-module, and applying the Matlis duality, we have the following proposition:

Proposition 4.2. *Let $(A, \mathfrak{m})$ be a local $k = A/\mathfrak{m}$ algebra which is finitely generated over k. Let B be an algebra containing A in its center such that B is finitely generated as an A-module. Then any artinian B-module N killed by a power of $\mathfrak{m}$ has an injective resolution*

$$0 \to N \to I_0 \to I_1 \to I_2 \to \cdots$$

by artinian and injective B-modules I_j with $I_j = \bigcup_{k \geq 1} \mathrm{Ann}(\mathfrak{m}^k; I_j)$ for all j.

5. Some generalities on Ext groups

In this section, B will be an associative $\mathbb{C}$-algebra with unity containing in its center a finitely generated commutative algebra A with the same unit such that B is a finitely generated A-module. The commutative algebra A over $\mathbb{C}$ comes with a maximal ideal $\mathfrak{m}$. Let

$$\widehat{A} = \varprojlim (A/\mathfrak{m}^n)$$

be the completion of A at $\mathfrak{m}$, and for any module M over A, let

$$\widehat{M} = \varprojlim (M/\mathfrak{m}^n M);$$

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in particular,

$$\widehat{B} = \varprojlim (B/\mathfrak{m}^n B),$$

and for any module M over B too,

$$\widehat{M} = \varprojlim (M/\mathfrak{m}^n M).$$

We begin with a proof of Schur's lemma, well known in representation theory under a countability assumption (which is satisfied here).

Lemma 5.1. *A simple B-module M is finite-dimensional over $\mathbb{C}$ on which A operates by a central character $\omega : A \to \mathbb{C}$ whose kernel is a maximal ideal $\mathfrak{m}$ in A.*

Proof. Being simple, M is finitely generated over B, hence over A. Therefore we can apply Nakayama's lemma to conclude that there is a maximal ideal $\mathfrak{m}$ in A such that $M/\mathfrak{m}M \neq 0$. But A being central in B, $\mathfrak{m}M$ is a B-submodule of M, hence by the simplicity of M as a B-module, $\mathfrak{m}M = 0$. □

Proposition 5.2. *For any finitely generated modules M, N over B,*

$$\mathrm{Ext}^i_B[N, M] \otimes_A \widehat{A} \cong \mathrm{Ext}^i_B[N, \widehat{M}] \cong \varprojlim \mathrm{Ext}^i_B[N, M/\mathfrak{m}^n M].$$

Further, if $\mathfrak{m}$ acts as 0 on N, it also acts by 0 on $\mathrm{Ext}^i_B[N, M]$, and hence,

$$\mathrm{Ext}^i_B[N, M] \cong \mathrm{Ext}^i_B[N, \widehat{M}] \cong \varprojlim \mathrm{Ext}^i_B[N, M/\mathfrak{m}^n M].$$

Proof. Let

$$\to P_n \to \cdots \to P_1 \to P_0 \to N \to 0$$

be a finitely generated projective resolution of N as an B-module. By definition, for any B-module M', $\mathrm{Ext}^i_B[N, M']$ is the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[P_0, M'] \to \mathrm{Hom}_B[P_1, M'] \to \cdots .$$

In particular, $\mathrm{Ext}^i_B[N, M/\mathfrak{m}^k M]$ is the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[P_0, M/\mathfrak{m}^k M] \to \mathrm{Hom}_B[P_1, M/\mathfrak{m}^k M] \to \cdots .$$

Observe that:

(1) Since M and P_i are finitely generated B-modules, and B is noetherian, the modules $\mathrm{Hom}_B[P_i, M/\mathfrak{m}^k M]$ appearing in the above complex are artinian B-modules, hence the above projective system of B-modules satisfies the Mittag-Leffler condition. Cohomology of such a complex commutes with inverse limits.

(2) By the definition of inverse limits, we have the equality

$$\mathrm{Hom}_B[N', \widehat{M}] = \varprojlim \mathrm{Hom}_B[N', M/\mathfrak{m}^n M]$$

for any B-module N'.

These two observations complete the proof of the assertion

$$\mathrm{Ext}^i_B[N, \widehat{M}] \cong \varprojlim \mathrm{Ext}^i_B[N, M/\mathfrak{m}^n M].$$

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Since $\mathrm{Hom}_B[P, M] \otimes_A \widehat{A} \cong \mathrm{Hom}_B[P, \widehat{M}]$ (clearly true for finitely generated projective modules P over B), and since $M \to \widehat{M}$ is an exact functor on the category of B-modules, one similarly proves that

$$\mathrm{Ext}^i_B[N, M] \otimes_A \widehat{A} \cong \mathrm{Ext}^i_B[N, \widehat{M}].$$

If $\mathfrak{m}$ acts by 0 on N, it also acts by 0 on $\mathrm{Ext}^i_B[N, M']$ for any B-module M' (proved for example by observing that it is true for $\mathrm{Hom}_B[N, M']$ and then by the "usual" dimension shifting argument).

The proof of the proposition is now complete. □

Proposition 5.3. *Let V be a simple B-module with central character $\omega : A \to \mathbb{C}$ (assured by Lemma 5.1) with $\mathfrak{m} = \ker(\omega)$ the corresponding maximal ideal in A. Then for any finitely generated B-module M,*

$$\mathrm{Ext}^i_B[M, V] \cong \varinjlim \mathrm{Ext}^i_B[M/\mathfrak{m}^n M, V].$$

Proof. Fix an injective resolution,

$$0 \to V \to I_0 \to I_1 \to \cdots$$

such that for each $\ell \geq 0$,

$$I_\ell = \bigcup_{k \geq 1} \mathrm{Ann}(\mathfrak{m}^k; I_\ell);$$

this is assured by Proposition 4.2. With this property for I_ℓ, and since M is finitely generated as a B-module, we have

$$\mathrm{Hom}_B[M, I_\ell] = \bigcup \mathrm{Hom}_B[M/\mathfrak{m}^k M, I_\ell] = \varinjlim \mathrm{Hom}_B[M/\mathfrak{m}^k M, I_\ell].$$

For any B-module N, $\mathrm{Ext}^i_B[N, V]$ is, by definition, the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[N, I_0] \to \mathrm{Hom}_B[N, I_1] \to \cdots.$$

In particular, $\mathrm{Ext}^i_B[M/\mathfrak{m}^k M, V]$ is the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[M/\mathfrak{m}^k M, I_0] \to \mathrm{Hom}_B[M/\mathfrak{m}^k M, I_1] \to \cdots.$$

As observed before,

$$\mathrm{Hom}_B[M, I_\ell] = \varinjlim \mathrm{Hom}_B[M/\mathfrak{m}^k M, I_\ell].$$

Since cohomology of a cochain complex commutes with arbitrary direct limits, the proof of the proposition is complete. □

6. Direct limit of Ext groups

In this section we prove two lemmas which allow one to understand $\mathrm{Ext}^i_B[V, M]$ and $\mathrm{Ext}^j_B[M, V]$ for an arbitrary B-module M, from their knowledge for finitely generated B-submodules of M.

Lemma 6.1. *Let V be a finitely generated B-module, and M a general B-module. Write $M = \varinjlim M_n$ of finitely generated B-submodules. Then we have*

$$\mathrm{Ext}^i_B[V, M] = \varinjlim \mathrm{Ext}^i_B[V, M_n].$$

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Proof. Let

$$\to P_n \to \cdots \to P_1 \to P_0 \to V \to 0$$

be a projective resolution of V as a B-module consisting of finitely generated B-modules. By definition, $\mathrm{Ext}^i_B[V, M]$ is the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[P_0, M] \to \mathrm{Hom}_B[P_1, M] \to \cdots .$$

Since P_i are finitely generated as B-modules,

$$\mathrm{Hom}_B[P_i, M] = \varinjlim \mathrm{Hom}_B[P_i, M_n].$$

The lemma now follows on noting that cohomology commutes with direct limits. □

Analogously, we have:

Lemma 6.2. *Let V be an irreducible B-module, and M a general B-module. Write $M = \varinjlim M_n$ of finitely generated B-submodules. Then we have*

$$\mathrm{Ext}^i_B[M, V] = \varprojlim \mathrm{Ext}^i_B[M_n, V].$$

Proof. Let

$$0 \to V \to I_0 \to I_1 \to I_2 \to \cdots$$

be an injective resolution of V as a B-module. Since V is an irreducible B-module hence finite-dimensional by Lemma 5.1, one can assume that I_j are artinian as B-modules as provided by Proposition 4.2. By definition, $\mathrm{Ext}^i_B[M, V]$ is the cohomology of the cochain complex

$$0 \to \mathrm{Hom}_B[M, I_0] \to \mathrm{Hom}_B[M, I_1] \to \cdots .$$

Since I_j are artinian, for any finitely generated submodule $M_n \subset M$, $\mathrm{Hom}_B[M_n, I_j]$ are artinian A-modules, therefore the Mittag-Leffler condition holds good for the projective system of cochain complexes:

$$0 \to \mathrm{Hom}[M_n, I_0] \to \mathrm{Hom}[M_n, I_1] \to \mathrm{Hom}[M, I_2] \to \cdots .$$

Since, by definition, $\mathrm{Hom}_B[M, I_i] = \varprojlim \mathrm{Hom}_B[M_n, I_i]$, the proof of the lemma is completed by the generality that cohomology commutes with inverse limits when Mittag-Leffler condition is satisfied. □

7. An algebraic duality theorem

We continue to assume that B is an associative $\mathbb{C}$-algebra with unity containing in its center a finitely generated commutative sub-algebra A with the same unit such that B is a finitely generated A-module.

Theorem 7.1. *Let V and DV be two simple B-modules with the same central character $\omega : A \to \mathbb{C}$ with $\mathfrak{m}$ the kernel of ω. Assume that there is an integer $n = n(V)$ such that for any simple B-module N, the natural (cup product) pairing between finite-dimensional vector spaces over $\mathbb{C}$,*

$$\mathrm{Ext}^i_B[V, N] \times \mathrm{Ext}^{n-i}_B[N, DV] \to \mathrm{Ext}^n_B[V, DV] \cong A/\mathfrak{m} = \mathbb{C},$$

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is perfect. Then for any finitely generated B-module M also, $\mathrm{Ext}^i_B[V, M]$ *and* $\mathrm{Ext}^i_B[M, DV]$ *are finite-dimensional vector spaces over* $\mathbb{C}$*, and the natural (cup product) pairing between the finite-dimensional vector spaces over* $\mathbb{C}$ *is perfect:*

$$\mathrm{Ext}^i_B[V, M] \times \mathrm{Ext}^{n-i}_B[M, DV] \to \mathrm{Ext}^n_B[V, DV] \cong \mathbb{C}. \tag{7.1.1}$$

If M is any B-module, the pairing in (7.1.1) *is non-degenerate in the sense that if* $M = \varinjlim M_n$ *of finitely generated B-submodules M_n, then*

$$\mathrm{Ext}^i_B[V, M] = \varinjlim \mathrm{Ext}^i_B[V, M_n],$$

a direct limit of finite-dimensional vector spaces over $\mathbb{C}$*, and similarly,*

$$\mathrm{Ext}^j_B[M, DV] = \varprojlim \mathrm{Ext}^j_B[M_n, DV],$$

an inverse limit of finite-dimensional vector spaces over $\mathbb{C}$*, and the pairing in* (7.1.1) *is the direct limit of perfect pairings on these finite-dimensional spaces*

$$\mathrm{Ext}^i_B[V, M_n] \times \mathrm{Ext}^j_B[M_n, DV] \to \mathrm{Ext}^{i+j=d(\pi)}_B[V, DV] \cong \mathbb{C}.$$

Proof. By Lemmas 6.1 and 6.2, it suffices to prove the proposition only for M a finitely generated B-module which we assume is the case in the rest of the proof. By generalities, $\mathrm{Ext}^i_B[V, M]$ and $\mathrm{Ext}^j_B[M, DV]$ are finitely generated A-modules on which $\mathfrak{m}$ acts trivially (proved for example by observing that it is true for $\mathrm{Hom}_B[V, M]$ and $\mathrm{Hom}_B[M, DV]$, and then by the "usual" dimension shifting argument). Hence, $\mathrm{Ext}^i_B[V, M]$ and $\mathrm{Ext}^j_B[M, DV]$ are finite-dimensional $\mathbb{C}$-vector spaces. Now we proceed in two steps.

Step 1: We will prove that if the natural pairing

$$\mathrm{Ext}^i_B[V, M] \times \mathrm{Ext}^{n-i}_B[M, DV] \to \mathrm{Ext}^n_B[V, DV] \cong \mathbb{C}$$

is perfect for B-modules M_1 and M_2, it is also the case for any module M which is an extension of M_2 by M_1:

$$0 \to M_1 \to M \to M_2 \to 0.$$

The proof of this follows from the 5-lemma once we observe that the cup product appearing in the statement of the proposition has a naturality property under the boundary maps, call them $\delta^i_1 : \mathrm{Ext}^{i-1}_B[V, M_2] \to \mathrm{Ext}^i_B[V, M_1]$ and $\delta^{n-i}_2 : \mathrm{Ext}^{n-i}_B[M_1, DV] \to \mathrm{Ext}^{n+1-i}_B[M_2, DV]$. Then for $\alpha \in \mathrm{Ext}^{i-1}_B[V, M_2]$ and $\beta \in \mathrm{Ext}^{n-i}_B[M_1, DV]$,

$$\delta^i_1(\alpha) \cup \beta = \alpha \cup \delta^{n-i}(\beta) \in \mathrm{Ext}^n_B[V, DV] \cong \mathbb{C}.$$

Step 2: Given the conclusion in Step 1, for each integer $k \geq 1$, we have a perfect pairing

$$\mathrm{Ext}^i_B[V, M/\mathfrak{m}^k M] \times \mathrm{Ext}^{n-i}_B[M/\mathfrak{m}^k M, DV] \to \mathrm{Ext}^n_B[V, DV] \cong \mathbb{C}.$$

By Proposition 5.2,

$$\mathrm{Ext}^i_B[V, M] = \varprojlim \mathrm{Ext}^i_B[V, M/\mathfrak{m}^k M],$$

and by Proposition 5.3,

$$\mathrm{Ext}^{n-i}_B[M, DV] = \varinjlim \mathrm{Ext}^{n-i}_B[M/\mathfrak{m}^k M, DV].$$

The conclusion of the proposition follows. □

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8. Completing the proof of Theorem 2

In this section we complete the proof of Theorem 2 of this paper. We have already proved parts (1), (2) and (3) of this theorem in Section 3, so we need only prove part (4) of this theorem. The proof of this part of Theorem 2 is a direct consequence of the general algebraic theorem, Theorem 7.1 of the previous section together of course with the theorem of Schneider and Stuhler which allows the conclusion of the theorem for M irreducible.

Proof of part (4) *of Theorem* 2. Since V is a smooth representation of G, $V^{G_n} \neq 0$, where G_n is a sufficiently small congruence subgroup of G (obtained say by using an embedding of $G = \underline{G}(F)$ inside $\mathrm{GL}_m(F)$, and intersecting with a congruence subgroup of $\mathrm{GL}_m(F)$). We now appeal to [3, Corollaire 3.9], according to which the category of smooth representations of G which are generated by their G_n fixed vectors is a direct summand of the category of all smooth representations of G, and this subcategory of smooth representations of G is isomorphic to the category of B-modules for $B = \mathcal{H}(G_n \backslash G/G_n)$, the algebra of locally constant, compactly supported functions on G which are bi-invariant under G_n, under convolution product for the Haar measure on G giving G_n volume 1. Further, B is a finite module over its center A which is a finitely generated $\mathbb{C}$-algebra, cf. [3, Corollaire 3.4], as well as the notes of Bernstein [4]. Moreover, it is easy to see that if M is finitely generated as a G-module, then so is the B-module M^{G_n}. This allows us to use Theorem 7.1 of the previous section by identifying

$$\mathrm{Ext}^i_G[V, M] = \mathrm{Ext}^i_B[V^{G_n}, M^{G_n}],$$

where M^{G_n} (resp. V^{G_n}) is the submodule of M (resp. V) consisting of G_n-fixed vectors which is a module for $B = \mathcal{H}(G_n \backslash G/G_n)$, and completing the proof of Theorem 2. □

Remark 4. It is a theorem of Aizenbud and Sayag in [1] that for a p-adic group G, containing a subgroup H, if one knows that $\dim \mathrm{Hom}_H[\pi_1, \pi_2] < \infty$ for π_1 any irreducible admissible representation of G, and π_2 of H, and if $[G \times H]/\Delta H$ is a spherical variety, then for any compact open subgroup K of H, and any irreducible representation π_1 of G, π_1^K is a finitely generated module over $\mathcal{H}(K \backslash H/K)$. Thus taking G to be any of the groups $\mathrm{GL}(n+1), \mathrm{U}(n+1), \mathrm{SO}(n+1)$ and H to be the corresponding subgroup $\mathrm{GL}(n), \mathrm{U}(n), \mathrm{SO}(n)$, we get a rich supply of finitely generated representations of Hecke algebras that Aizenbud and Sayag call *locally finitely generated*.

9. An application of the duality theorem

In this section we give a sample application of the Schneider–Stuhler duality theorem to branching laws.

The paper [9] suggests that branching problems (say from $\mathrm{SO}_{n+1}(F)$ to $\mathrm{SO}_n(F)$) which have such a simple eventual answer for $\dim \mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, where π_1 is an irreducible admissible representation of $\mathrm{SO}_{n+1}(F)$ and π_2 is an irreducible admissible representation of $\mathrm{SO}_n(F)$ (and assume for instance that they are both tempered, or more generally belong to generic Vogan packets) is because higher Ext's are zero:

$$\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2] = 0 \quad \text{for } i > 0.$$

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The Schneider–Stuhler duality theorem then allows one to answer a natural question: what are the irreducible $\mathrm{SO}_n(F)$-submodules π_2 of an irreducible admissible module π_1 of $\mathrm{SO}_{n+1}(F)$ when restricted to $\mathrm{SO}_n(F)$? The answer is that typically none (except of course supercuspidals), but that there are submodules not in the sense that $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_2,\pi_1] \neq 0$ but in the sense that

$$\mathrm{Ext}^{d(\pi_2)}_{\mathrm{SO}_n(F)}[\pi_2,\pi_1] \neq 0.$$

As a simple application of the Schneider–Stuhler duality theorem, we prove the following proposition giving a complete classification of irreducible submodules π of the tensor product $\pi_1 \otimes \pi_2$ of two representations π_1, π_2 of $\mathrm{GL}_2(F)$ with the product of their central characters trivial. As this proposition shows, it is rare for a non-supercuspidal representation of a subgroup H to appear as a subrepresentation of a representation of a group G when restricted to H (in this case from $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ to the diagonal $\mathrm{GL}_2(F)$); this is to be contrasted with their abundant appearance as a quotient studied in [8]. In fact, it should be considered somewhat of a surprising conclusion that there can be a non-supercuspidal submodule at all!

Proposition 9.1. *Let π_1, π_2 be two irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(F)$ with product of their central characters trivial. Then the following is a complete list of irreducible sub-representations π of $\pi_1 \otimes \pi_2$ as $\mathrm{PGL}_2(F)$-modules:*

(1) *π is a supercuspidal representation of $\mathrm{PGL}_2(F)$, and appears as a quotient of $\pi_1 \otimes \pi_2$.*

(2) *π is a twist of the Steinberg representation, which we assume by absorbing the twist in π_1 or π_2 to be the Steinberg representation* St *of $\mathrm{PGL}_2(F)$. Then* St *is a submodule of $\pi_1 \otimes \pi_2$ if and only if π_1 and π_2 are both irreducible principal series representations, and $\pi_1 \cong \pi_2^\vee$.*

Proof. Since a supercuspidal representation of $\mathrm{PGL}_2(F)$ is a projective module, it appears as a submodule of $\pi_1 \otimes \pi_2$ if and only if it appears as a quotient module, thus the assertion of the proposition for π supercuspidal is clear.

If the representation π of $\mathrm{PGL}_2(F)$ is not supercuspidal, then in the notation of Theorem 1,

$$d(\pi) = 1,$$

and there is a non-degenerate pairing

$$\mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi, \pi_1 \otimes \pi_2] \times \mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, D(\pi)] \to \mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi, D(\pi)] \cong \mathbb{C}$$

allowing one to calculate $\mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi, \pi_1 \otimes \pi_2]$ in terms of $\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, D(\pi)]$ which in turn is calculated in terms of the Euler–Poincaré function (replacing $D\pi$ by π' for notational ease)

$$\mathrm{EP}[\pi_1 \otimes \pi_2, \pi'] = \dim \mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi'] - \dim \mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi'],$$

which is relatively straightforward to calculate – as we briefly indicate in the next paragraph – and then using knowledge about $\dim \mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi']$ from the paper [8] to calculate $\dim \mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi']$.

For example, if π' is an irreducible principal series representation of $\mathrm{PGL}_2(F)$, then

$$D(\pi') = \pi',$$

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and in this case it can be seen that

$$\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \pi'] = \mathrm{Hom}_{\mathrm{PGL}_2(F)}[\pi', \pi_1 \otimes \pi_2] = 0.$$

The assertion on vanishing of Ext^1, one of the main conjectures in [9] in some generality, is easy to prove if one of π_i is supercuspidal; if neither of π_1, π_2 is supercuspidal, then $\pi_1 \otimes \pi_2$ is – by Mackey orbit theory – equal to $\mathrm{ind}_T^{\mathrm{PGL}_2(F)}(\chi)$ (up to an admissible module), where $\chi : T \to \mathbb{C}^\times$ is a character on a maximal split torus T, and the statement on Euler–Poincaré and hence Ext^1 on $\mathrm{PGL}_2(F)$ reduces by Frobenius reciprocity, cf. [9, Proposition 2.5], to an assertion on the split torus.

We next turn to $\pi' = \mathrm{Sp}$ in which case we have

$$D(\mathrm{Sp}) = \mathbb{C}.$$

By the Schneider–Stuhler theorem, calculation of $\mathrm{Hom}_{\mathrm{PGL}_2(F)}[\mathrm{Sp}, \pi_1 \otimes \pi_2]$ is reduced to that of $\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \mathbb{C}]$, where π_1 and π_2 are two irreducible admissible representations of $\mathrm{GL}_2(F)$ with central characters χ and χ^{-1}. For this, note the generality

$$\mathrm{Ext}^1_{\mathrm{PGL}_2(F)}[\pi_1 \otimes \pi_2, \mathbb{C}] \cong \mathrm{Ext}^1_{\mathrm{GL}_2(F),\chi}[\pi_1, \pi_2^\vee].$$

But by another application of the Schneider–Stuhler theorem,

$$\dim \mathrm{Ext}^1_{\mathrm{GL}_2(F),\chi}[\pi_1, \pi_2^\vee] = \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}[\pi_2^\vee, \pi_1]$$

if π_1 or π_2 is an irreducible principal series representation of $\mathrm{GL}_2(F)$. Thus,

$$\mathrm{Hom}_{\mathrm{PGL}_2(F)}[\mathrm{Sp}, \pi_1 \otimes \pi_2] = \mathbb{C}$$

if $\pi_1 \cong \pi_2^\vee$ are irreducible principal series representations of $\mathrm{GL}_2(F)$. The cases of the proposition we missed out by this analysis is when one of π_i is the twist of the Steinberg representation, or when π' is the trivial representation; we leave these to the reader. □

An explicit embedding of Sp into $\mathrm{ind}_T^{\mathrm{PGL}_2(F)}\mathbb{C}$ was constructed in [8, Lemma 5.4], which allows one to get an embedding of Sp inside $\pi_1 \otimes \pi_1^\vee$ for π_1 any principal series representation of $\mathrm{GL}_2(F)$.

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Eingegangen 23. Oktober 2018, in revidierter Fassung 24. Oktober 2018

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