NOTES ON CENTRAL EXTENSIONS

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ABSTRACT. These are the notes for some lectures given by this author at Harish-Chandra Research Institute, Allahabad in March 2014 for a workshop on Schur multipliers. The lectures aimed at giving an overview of the subject with emphasis on groups of Lie type over finite, real and p-adic fields. The author thanks Prof. Pooja Singla for the first draft of these notes and Shiv Prakash Patel for the second draft of these notes.

1. INTRODUCTION

Definition 1. *Let G be a group and A an abelian group. A group E is called a central extension of G by A if there is a short exact sequence of groups,*

$$1 \to A \xrightarrow{i} E \xrightarrow{\mathsf{p}} G \to 1 \tag{1}$$

such that image of i is contained in the center of E.

Isomorphism classes of central extensions of G by A are parametrized by $H^2(G, A)$, where A is considered to be a trivial G-module. Indeed, if $s : G \to E$ is a section of p then $\beta : G \times G \to A$ given by $\beta(g_1, g_2) := s(g_1)s(g_2)s(g_1g_2)^{-1}$ defines a 2-cocycle on G with values in A. On the other hand, if $\beta : G \times G \to A$ is a 2-cocycle on G with values in A, then the binary operation on $E = G \times A$ defined by $(g_1, a_1)(g_2, a_2) := (g_1g_2, a_1a_2\beta(g_1, g_2))$ makes E into a group, which is a central extension of G by A.

A closely related group which comes up in the study of central extensions is the *Schur multiplier* of a group G defined to be $H_2(G, \mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group $\operatorname{Ext}^1(H_1(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = 0$, hence by the *universal coefficient theorem* recalled below, $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, i.e. Pontryagin dual of $H_2(G, \mathbb{Z})$ classifies central extensions of G by $\mathbb{Q}/\mathbb{Z}$.

Theorem 2 (Universal Coefficient theorem)**.** *Let G be a group and A an abelian group, considered as a trivial G-module. Then we have a short exact sequence of*

abelian groups as follows:

$$0 \longrightarrow \mathrm{Ext}^1(H_1(G,\mathbb{Z}), A) \longrightarrow H^2(G, A) \longrightarrow \mathrm{Hom}(H_2(G,\mathbb{Z}), A) \longrightarrow 0.$$

This sequence is split (though not in a natural way).

Let us consider the following two extreme cases of G.

(1) $G^{ab} := G/[G,G] = H_1(G,\mathbb{Z}) = 1.$
(2) $G = G^{ab}.$

Let us consider the first case, i.e. $G^{ab} = 1$, in which case $H_1(G,\mathbb{Z}) = 1$. In this case, the universal coefficient theorem reduces to,

$$H^2(G, A) \cong \mathrm{Hom}(H_2(G,\mathbb{Z}), A). \tag{2}$$

In particular, there exists a central extension $\tilde{G}$ of G by $A = H_2(G,\mathbb{Z})$ corresponding to the identity, $\mathrm{Id} \in \mathrm{Hom}(H_2(G,\mathbb{Z}), H_2(G,\mathbb{Z}))$:

$$1 \to H_2(G,\mathbb{Z}) \to \tilde{G} \to G \to 1. \tag{3}$$

This central extension $\tilde{G}$ of G by $H_2(G,\mathbb{Z})$ is universal in the sense that any central extension E of G by A is given by a push-out diagram for a group homomorphism $\varphi : H_2(G,\mathbb{Z}) \to A$ as follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(G,\mathbb{Z}) & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 0 \\
& & \varphi \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0,
\end{array}
$$

where $E = [\tilde{G} \times A]/\Delta H_2(G,\mathbb{Z}).$

In the second extreme case, we have $G = G^{ab}$, i.e. G is abelian. For an abelian group G, we have

$$H_2(G,\mathbb{Z}) \cong \Lambda^2 G := \frac{G \otimes G}{\{g \otimes g : g \in G\}}. \tag{4}$$

The universal coefficient theorem for an abelian group G gives a split exact sequence as follows:

$$0 \longrightarrow \mathrm{Ext}^1(G, A) \longrightarrow H^2(G, A) \longrightarrow \mathrm{Hom}(\Lambda^2 G, A) \longrightarrow 0. \tag{*}$$

This exact sequence can be nicely interpreted in terms of extensions. Recall that $H^2(G, A)$ classifies central extensions E of G by A. Among these central extensions E, those which are abelian correspond to the subgroup $\mathrm{Ext}^1(G, A)$ of $H^2(G, A)$. The

map from $H^2(G, A)$ to $\mathrm{Hom}(\Lambda^2 G, A)$ is given by taking arbitrary lifts g_1, g_2 elements of G to E, and taking their commutator in E which is an element of A. This clearly gives a homomorphism from $\Lambda^2 G$ to A. In particular, if A is a divisible abelian group such as $A = \mathbb{Q}/\mathbb{Z}$, then since $\mathrm{Ext}^1(G, A) = 0$ (for G abelian),

$$H^2(G, A) \cong \mathrm{Hom}(\Lambda^2 G, A).$$

The exact sequence (*) is known to be split. We show by an example that the exact sequence (*) is not canonically split. For this, take $G = V = \mathbb{Z}/2 + \mathbb{Z}/2$, and $A = \mathbb{Z}/2$. It is known that $H^*(\mathbb{Z}/2, \mathbb{Z}/2)$ is the polynomial algebra on $H^1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$, and therefore by the Kunneth theorem, $H^*(\mathbb{Z}/2 + \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Sym}^*[V^\vee] \cong \mathbb{Z}/2[X, Y]$. In particular, $H^2(\mathbb{Z}/2 + \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Sym}^2[V^\vee]$. Further, $\mathrm{Ext}^1(V, \mathbb{Z}/2) \cong V^\vee$. The exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1(G, A) \longrightarrow H^2(G, A) \longrightarrow \mathrm{Hom}(\Lambda^2 G, A) \longrightarrow 0,$$

becomes,

$$0 \longrightarrow V^\vee \longrightarrow \mathrm{Sym}^2[V^\vee] \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

where the map from $V^\vee$ to $\mathrm{Sym}^2[V^\vee]$ is given by $v \to v \otimes v$. It is known that for $\mathrm{Aut}(G) = \mathrm{Aut}(V) = \mathrm{GL}_2(\mathbb{Z}/2)$, the above is a nonsplit exact sequence of $\mathrm{Aut}(V)$-modules.

2. The Dual point of view

Recall that a central extension of a group G by an abelian group A is another group E such that A is contained in the center of E and $E/A \cong G$, i.e. we have an exact sequence of groups

$$1 \to A \to E \to G \to 1.$$

A common theme in categorical mathematics is that many notions in an abstract category remain meaningful by reversing arrows! Reversing arrows in the above exact sequence, we get:

$$1 \to G \to E \to A \to 1,$$

i.e., now G sits as a normal subgroup inside a group E with quotient A, which we don't necessarily assume to be abelian, and then change notation from A to Q.

We ask the following question from a dual point of view to the central extension.

Question 3. *For a group G and another group Q, what are the way in which G sits in a group E as a normal subgroup with quotient $E/G \cong Q$? In other words, what are the isomorphism classes of extension of groups E which give rise to a short exact sequence of groups as follows:*

$$1 \to G \to E \to Q \to 1.$$

Note that given a normal subgroup G of a group E with quotient Q, there exists a natural homomorphism $\phi : Q \to \mathrm{Out}(G)$, where $\mathrm{Out}(G)$ is the group of outer automorphism of G which is defined as $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Aut}(G)$ is the group of all automorphisms of G and $\mathrm{Inn}(G)$ is the normal subgroup of the group of inner automorphisms of G. The mapping $\phi : Q \to \mathrm{Out}(G)$ is defined by choosing an arbitrary lift $\tilde{a}$ of $a \in Q$ in E, and using the automorphism of G given by conjugation by $\tilde{a}$, the automorphism of G considered as an element of $\mathrm{Out}(G)$ being independent of the choice of the lift $\tilde{a}$ of a.

The group $\mathrm{Out}(G)$ is an important invariant of a group G, which has been much studied in all branches of mathematics where groups play a role.

Let C be the center of G. An element $\xi \in H^2(Q, C)$ corresponds to a central extension

$$0 \to C \to E \to Q \to 0.$$

This gives rise to a push-out diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \tilde{E} & \longrightarrow & Q & \longrightarrow & 0,
\end{array}
$$

where $\tilde{E} := [E \times G]/\Delta(C)$. Thus we have a natural map

$$H^2(Q, C) \to \mathrm{Ext}^1(Q, G). \tag{5}$$

As described above, $0 \to G \to \tilde{E} \xrightarrow{\mathrm{p}} Q \to 0$ gives rise to a natural map,

$$\mathrm{Ext}^1(Q, G) \to \mathrm{Hom}(Q, \mathrm{Out}(G)). \tag{6}$$

Given a homomorphism $\varphi : Q \to \mathrm{Out}(G)$, we have a pull-back diagram of the natural exact sequence

$$1 \to G/C \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1,$$

given by

$$1 \longrightarrow G/C \longrightarrow E = \operatorname{Aut}(G) \times_{\operatorname{Out}(G)} Q \longrightarrow Q \longrightarrow 1$$

$$1 \longrightarrow G/C \longrightarrow \operatorname{Aut}(G) \longrightarrow \operatorname{Out}(G) \longrightarrow 1.$$

(with vertical map φ from E to $\operatorname{Aut}(G)$)

This gives us a morphism of sets $\operatorname{Hom}(Q, \operatorname{Out}(G)) \to \operatorname{Ext}^1(Q, G/C)$.

Associated to the short exact sequence

$$1 \to C \to G \to G/C \to 1,$$

we have an exact sequence of pointed sets

$$0 \longrightarrow \operatorname{Ext}^1(Q, C) \longrightarrow \operatorname{Ext}^1(Q, G) \longrightarrow \operatorname{Ext}^1(Q, G/C) \longrightarrow \cdots$$

$$H^2(Q, C) \qquad\qquad \operatorname{Hom}(Q, \operatorname{Out}(G))$$

giving rise to

$$0 \to H^2(Q, C) \to \operatorname{Ext}^1(Q, G) \to \operatorname{Hom}(Q, \operatorname{Out}(G)).$$

The following theorem summarizes the above discussion.

Theorem 4. *Let C be the center of a group G. Then for any group Q, we have an exact sequence of pointed sets:*

$$0 \to H^2(Q, C) \to \operatorname{Ext}^1(Q, G) \to \operatorname{Hom}(Q, \operatorname{Out}(G)).$$

making $\operatorname{Ext}^1(Q, G)$ into a principal homogeneous space with fibres abelian groups $H^2(Q, C)$ and base which is the subset of $\operatorname{Hom}(Q, \operatorname{Out}(G))$ consisting of those homomorphisms from Q to $\operatorname{Out}(G)$ which is realized by an extension of Q by G.

If the center of G is trivial, then

$$\operatorname{Ext}^1(Q, G) \cong \operatorname{Hom}(Q, \operatorname{Out}(G)).$$

Proof. We only make some remarks on the last assertion in the theorem. If $C = \{e\}$, then the morphism of sets $\operatorname{Hom}(Q, \operatorname{Out}(G)) \to \operatorname{Ext}^1(Q, G/C)$, gives rise to a morphism of sets

$$\operatorname{Hom}(Q, \operatorname{Out}(G)) \to \operatorname{Ext}^1(Q, G).$$

This can be easily seen to be inverse of the morphism of sets in (6) and therefore if $C = \{e\}$, we have an isomorphism of sets

$$\operatorname{Ext}^1(Q, G) \cong \operatorname{Hom}(Q, \operatorname{Out}(G)).$$

$\square$

Example. As an example, we consider possible extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{GL}_n(\mathbb{C})$

$$1 \to \mathrm{GL}_n(\mathbb{C}) \to E \to \{\pm 1\} \to 1,$$

such that -1 acts on $\mathrm{GL}_n(\mathbb{C})$ via the outer automorphism given by $g \mapsto^t g^{-1}$. We note that the outer automorphism group of $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to $\mathbb{Z}/2$ generated by $g \mapsto^t g^{-1}$. By the above theorem, the number of extensions corresponding to this outer automorphism of $\mathrm{GL}_n(\mathbb{C})$ is exactly the number of elements in $H^2(\{\pm 1\}, Z(\mathrm{GL}_n(\mathbb{C}))) = H^2(\{\pm 1\}, \mathbb{C}^\times)$, where $\{\pm 1\}$ acts on $\mathbb{C}^\times$ by $x \mapsto x^{-1}$. Since the cohomology of a finite cyclic group is periodic with period 2, the 2nd cohomology reduces to the 0th (Tate) cohomology, which is easily seen to be $\mathbb{Z}/2$ in this case. This means that there are exactly two extensions of $\mathbb{Z}/2$ by $\mathrm{GL}_n(\mathbb{C})$, one of which is the trivial one, i.e., a semi-direct product, and the other one given by generators and relations as $\{\mathrm{GL}_n(\mathbb{C}); j\}$ with $j^2 = -1, jgj^{-1} = j_0^t g^{-1} j_0^{-1}$ with j_0 the anti-diagonal matrix with entries $(1, -1, 1, -1, \cdots)$.

3. Examples

Examples (Examples of central extensions of certain abelian groups:)**.**

(1) Central extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$: There are two extensions in this case. One of them is the trivial extension, i.e., the direct product, and the other one is the non-trivial extension,

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

(2) Let $\mathbb{F}_q$ be the finite field with $q = p^n$-elements. There is a non-trivial central extension $H(\mathbb{F}_q^{2n})$ of $\mathbb{F}_q^{2n}$ by $\mathbb{F}_q$ for all positive integer n, called the Heisenberg group, i.e.

$$0 \to \mathbb{F}_q \to H(\mathbb{F}_q^{2n}) \to \mathbb{F}_q^{2n} \to 0.$$

The simplest realization of $H(\mathbb{F}_q^{2n})$ is the group of $(n + 1) \times (n + 1)$ upper triangular unipotent matrices with only non-diagonal nonzero entries (from $\mathbb{F}_q$) in the first row and last column.

Example (Central extension of $\{\pm 1\}^{n-1}$)**.** Let $\mathrm{O}_n(\mathbb{R})$ be defined as isometry group of the quadratic form given by $q(x) = x_1^2 + \cdots + x_n^2$, and let $\mathrm{SO}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})$. Consider the $\{\pm 1\}^n \hookrightarrow \mathrm{O}_n(\mathbb{R})$ as the group of diagonal matrices with entries $+1$ or -1. Identify $\{\pm 1\}^{n-1}$ to be the subgroup of $\{\pm 1\}^n$ consisting of those elements of $\{\pm 1\}^n$ with even number of -1's. Define F_{n-1} to be the 2-fold cover of $\{\pm 1\}^{n-1}$

which is obtained from the pull back of $\{\pm 1\}^{n-1} \hookrightarrow SO_n(\mathbb{R})$ through the spin cover $\mathrm{Spin}_n(\mathbb{R}) \to SO_n(\mathbb{R})$, i.e.

$$
\begin{array}{ccc}
F_{n-1} & \longrightarrow & \mathrm{Spin}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
\{\pm 1\}^{n-1} & \longrightarrow & SO_n(\mathbb{R}).
\end{array}
$$

Then F_{n-1} is a non-trivial 2-fold cover of $\{\pm 1\}^{n-1}$. Let us describe the group F_{n-1} more explicitly (via the usual Clifford algebra construction which we hide).

We will describe F_{n-1} as a subgroup of another group E_n defined below. As a set E_n is

$$
E_n = \{\epsilon e_A \mid \epsilon \in \{\pm 1\}, A \subset \{1, \cdots, n\}\},
$$

where $\{\pm 1\}$ lies in the center of E_n and

$$
F_{n-1} = \{\epsilon e_A \mid \#(A) = \text{even }\}.
$$

For $A = \{e_{i_1}, \cdots, e_{i_j}\}$ with $i_1 < \cdots < i_j$, we write $e_A := e_{i_1} \cdots e_{i_j}$ with convention that $e_\emptyset = 1$. The elements e_i's satisfy the relation $e_i e_j = -e_j e_i$ for $i \neq j$ and $e_i^2 = 1$. For $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and $A, B \subset \{1, \cdots, n\}$, define the multiplication as

$$
(\epsilon_1 e_A)(\epsilon_2 e_B) = \epsilon_1 \epsilon_2 \epsilon(A, B) e_{(A \cup B) - (A \cap B)},
$$

where $\epsilon(A, B)$ is determined by the relations relations among e_i's given above.

Theorem 5.

(a) E_n and F_{n-1} define a non-trivial central extension of $\{\pm 1\}^n$ and $\{\pm 1\}^{n-1}$ by $\{\pm 1\}$ respectively. F_{n-1} is isomorphic to the one obtained from 2-fold cover $\mathrm{Spin}(n) \to SO(n)$.

(b) $[E_n, E_n] = \{\pm 1\}$.

(c) The center of E_n is $\{\pm 1\}$ if n is even, and $\{\pm 1, \pm e_1 \cdots e_n\}$ if n is odd, whereas the center of F_{n-1} is $\{\pm 1\}$ if n is odd, and $\{\pm 1, \pm e_1 \cdots e_n\}$ if n is even.

Remark 6. *The group E_n for n even, and F_{n-1} for n odd is what's called an extra special 2 group. They have a unique irreducible representation of dimension > 1, which is equal to $2^{[n/2]}$ where $[n/2]$ refers to the integral part of $n/2$.*

Example (Central extensions of alternating groups). Let S_n be the symmetric group on a finite set of n elements. Let $A_n \subset S_n$ be the subgroup of even permutations. It is known that $H_2(A_n, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ if $n = 5$ or $n > 7$; $H_2(A_6, \mathbb{Z}) \cong H_2(A_7, Z) \cong \mathbb{Z}/6\mathbb{Z}$. In particular, A_n for $n \geq 5$ have a unique 2-fold cover $\tilde{A}_n$. We construct $\tilde{A}_n$ using the spin cover of $SO(n)$ below.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of an n-dimensional quadratic space over $\mathbb{R}$. We identify S_n with the group of matrices, which permute the basis vectors $\{e_1, \cdots, e_n\}$. This gives rise to an embedding $S_n \hookrightarrow O_n(\mathbb{R})$ and hence $A_n \hookrightarrow SO_n(\mathbb{R})$. Define $\tilde{A}_n$ to be the 2-fold cover of A_n which is obtained from the pull-back of the $A_n \to SO_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{R}) \to SO_n(\mathbb{R})$, i.e.

$$
\begin{array}{ccc}
\tilde{A}_n & \longrightarrow & \mathrm{Spin}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
A_n & \longrightarrow & SO_n(\mathbb{R}).
\end{array}
$$

Then $\tilde{A}_n$ is a non-trivial 2-fold cover of A_n.

Let us directly describe the 2-fold cover of A_n, which arises in the above fashion. We will define a two fold cover of S_n such that the 2-fold cover of A_n is obtained from that of restriction to A_n. The group S_n has a presentation on $n-1$ generators, say $t_1, \cdots, t_{n-1}$ with the following relations:

(a) $t_i^2 = 1$ for $1 \le i \le n-1$.
(b) $t_{i+1}t_i t_{i+1} = t_i t_{i+1} t_i$ for $1 \le i \le n-2$.
(c) $t_j t_i = t_i t_j$ for $|i - j| > 1$.

We use these relations to describe a two fold cover $\tilde{S}_n$ of S_n. The group $\tilde{S}_n$ has generators $z, t_1, \cdots, t_n$ with the following relation:

(a) $z^2 = 1$.
(b) $t_i t_i = z$ for $1 \le i \le n-1$.
(c) $t_{i+1}t_i t_{i+1} = t_i t_{i+1} t_i$ for $1 \le i \le n-2$.
(d) $t_j t_i = t_i t_j$ for $|i - j| > 1$.

Theorem 7. *Let $\tilde{A}_n$ be the 2-fold cover of A_n which is the restriction of the above defined cover $\tilde{S}_n$ of S_n.*

(1) *The $\tilde{A}_n$ is a non-trivial 2-fold cover of A_n if and only if $n \ge 4$.*
(2) *For $n \ge 4$, up to an isomorphism, $\tilde{A}_n$ is the only non-trivial 2-fold cover of A_n, which is isomorphic to the 2-fold cover of A_n obtained from the pull back of 2-fold cover $\mathrm{Spin}_n(\mathbb{R}) \to SO_n(\mathbb{R})$.*

4. REAL GROUPS

For G a semisimple real group, we randomly pick a few examples.

(1) $\mathrm{SU}(n)$ is simply connected.
(2) Simply connected cover of $\mathrm{SO}(n)$ is $\mathrm{Spin}(n)$, which is a 2-fold cover of $\mathrm{SO}(n)$, i.e. we have

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

(3) $\pi_1(\mathrm{SL}_n(\mathbb{R})) = \mathbb{Z}/2$ if $n > 2$, and equals $\mathbb{Z}$ for $n = 2$.
(4) $\pi_1(\mathrm{Spin}(p,q)) = 1, \mathbb{Z}, \mathbb{Z}/2$ depending on whether $\min(p,q) \leq 1$, $\min(p,q) = 2$, $\min(p,q) > 2$.

Remark 8. *Information on the (topological) fundamental group of a semi-simple simply connected real algebraic group $G(\mathbb{R})$ seems not to be clearly spelled out in the literature. It is known that the only options for $\pi_1(G(\mathbb{R})) = \pi_1(K)$ for K a maximal compact subgroup of $G(\mathbb{R})$, are $1, \mathbb{Z}/2, \mathbb{Z}$, and the only cases when $\pi_1(K) = \mathbb{Z}$ are the Hermitian symmetric cases. The only cases in which the fundamental group is trivial is for $\mathrm{SU}^*(2n), \mathrm{Sp}(p,q), \mathrm{Spin}(n,1)$, the non-split inner form of E_6, and the rank 1 form of F_4. It is known that real forms G' of a group G are in bijective correspondence with conjugacy classes of involutive automorphisms σ on the compact real form K of G, such that the maximal compact subgroup of G' is $K' = K^\sigma$. Since K' has the same fundamental group as G', the question on $\pi_1(G')$ amounts to a question in algebraic groups: for an involution on a simple simply connected algebraic group, when is the fixed points semi-simple, and when it is semi-simple and simply connected.*

5. TWO QUESTIONS

We pose two questions here which we are not sure are already answered in existing literature!

Question 9. *Let G be any finite or compact connected Lie group. Is any two fold cover of G obtained from the pull-back of $\mathrm{Spin}(n)$, the two fold spin cover of $\mathrm{SO}(n)$, through a map $\varphi : G \to \mathrm{SO}(n)$?*

Question 10. *Is $H^*(G, \mathbb{Z}/2\mathbb{Z})$ generated as an algebra by $\omega_i(\rho)$, as $\rho : G \to \mathrm{O}(n)$ varies over all orthogonal representations of G and $\omega_i(\rho)$'s are Stiefel-Whitney classes of orthogonal representations ρ?*

Remark 11. *The question seems analogous to the Hodge conjecture in Algebraic Geometry which is about generators of certain cohomologies by Chern classes of vector bundles.*

6. Relation of central extensions to K_2

Theorem 12 (Steinberg). *If $n \geq 3$, then $H_2(\mathrm{SL}_n(F), \mathbb{Z}) \cong K_2(F)$ provided that we exclude $n = 3$ and $|F| = 2$ or 4, and $n = 4$ and $|F| = 2$.*

Remark 13. For $a, b \in F^\times$ and $n \geq 3$, let

$$
r_a = \begin{pmatrix} a & & & & \\ & a^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tau_b = \begin{pmatrix} b & & & & \\ & 1 & & & \\ & & b^{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in \mathrm{SL}_n(F).
$$

Let $1 \to C \to E \xrightarrow{\mathsf{p}} \mathrm{SL}_n(E) \to 1$ be a central extension of $\mathrm{SL}_n(F)$. Note that $[\tilde{r}_a, \tilde{\tau}_b]$ is a well defined element in E, where $\tilde{r}_a$ and $\tilde{\tau}_b$ are arbitrary element in the inverse images $\mathsf{p}^{-1} r_a$ and $\mathsf{p}^{-1} \tau_b$ respectively. Since r_a and τ_b commute in $\mathrm{SL}_n(F)$, $[\tilde{r}_a, \tilde{\tau}_b] \in C$. This defines a map $(a, b) \mapsto \{a, b\} := [\tilde{r}_a, \tilde{\tau}_b] \in C$ from $F^\times \times F^\times \to C$. Moreover,

$$
\{a, b_1 b_2\} = [\tilde{r}_a, \tilde{\tau}_{b_1 b_2}] = [\tilde{r}_a, \tilde{\tau}_{b_1} \tilde{\tau}_{b_2}] = [\tilde{r}_a, \tilde{\tau}_{b_1}] \cdot [\tilde{r}_a, \tilde{\tau}_{b_2}] \cdot [\tilde{\tau}_{b_1}, [\tilde{\tau}_{b_2}, \tilde{r}_a]]^{-1} = \{a, b_1\}\{a, b_2\}.
$$

Similarly $\{a_1 a_2, b\} = \{a_1, b\}\{a_2, b\}$. Therefore this map is bi-multiplicative in both the coordinates. An important point to note is that instead of coroots r_a, τ_b we could have taken any other two distinct coroots with nonzero inner product for defining the element $\{a, b\}$. This follows because the Weyl group operates transitively on the set of distinct coroots with nonzero inner product. (If the coroots are orthogonal to each other, then they belong to distinct commuting $\mathrm{SL}_2(F)$, and hence their commutator in any central extension is trivial.) Furthermore, it can be verified that $\tilde{r}_a \tilde{r}_b = \{a, b\}\tilde{r}_{ab}$ which allows one to prove the last two of the following identities (the first being trivial); for all this —which although is no more than matrix manipulation, is quite tricky, and due to Steinberg— see the book of Milnor [3].

(1) $\{a, b\}\{b, a\} = 1$.
(2) $\{a, -a\} = 1$.
(3) $\{a, 1 - a\} = 1$.

Thus the map $(a, b) \mapsto \{a, b\}$ factors through

$$
K_2(F) := \frac{F^\times \otimes F^\times}{\{a \otimes (1 - a) \mid a(1 - a) \neq 0\}}.
$$

Since $\mathrm{SL}_n(F)$, $n \geq 3$ is easily seen to be a perfect group, $\mathrm{SL}_n(F)$ has a universal central extension with $H_2(\mathrm{SL}_n(F), \mathbb{Z})$, as the center of the universal central extension. The

above analysis with the universal central extension of $\mathrm{SL}_n(F)$ by $H_2(\mathrm{SL}_n(F), \mathbb{Z})$ gives a map $K_2(F) \to H_2(\mathrm{SL}_n(F), \mathbb{Z})$ which by a theorem of Steinberg is an isomorphism.

The following more general theorem is due to Matsumoto.

Theorem 14. *Let G be a simple, simply connected split algebraic group over an infinite field F. Then*
$$H_2(G(F), \mathbb{Z}) \cong K_2(F)$$
except for groups of type $C_n, n \geq 1$, when $H_2(G(F), \mathbb{Z})$ has $K_2(F)$ as a quotient, and in fact $K_2(F)$ is the maximal quotient of $H_2(G(F), \mathbb{Z})$ on which $\mathrm{Aut}(G)(F)$ acts trivially.

Remark 15. *It is easy to see that $K_2(\mathbb{F}_q) = \{1\}$ for a finite field $\mathbb{F}_q$. Also, it is also known that simple simply connected algebraic groups G, $G(\mathbb{F}_q)$ have no non-trivial central extensions except in a small number of cases that we enumerate later. But the author of these notes has not seen any uniform theorem proving this such as the very precise theorem 12 above due to Steinberg (the problem is for small fields such as $\mathbb{F}_2, \mathbb{F}_3,$ and $\mathbb{F}_4$).*

Remark 16. *There seems no such precise theorem for quasi-split groups over general fields. Deodhar has defined in [2] what he calls a Moore group which depends only on the field of which $H_2(G(F), \mathbb{Z})$ is a quotient of.*

Theorem 17. *If G is simply connected simple algebraic group defined over $\mathbb{F}_q$, then*

(1) *$G^{ab} = \{e\}$, i.e. G is perfect, except the following:*
$$\mathrm{SL}_2(\mathbb{F}_2) \cong S_3, \ \mathrm{SL}_2(\mathbb{F}_3) \cong \tilde{A}_4, \ \mathrm{Sp}_4(\mathbb{F}_2), \ G_2(\mathbb{F}_2), \ \mathrm{SU}_3(\mathbb{F}_2).$$

(2) *If $G = G^{ab}$, then G is its own universal central extension, except if G is one of the following:*

$$\mathrm{SL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_5), \ \mathrm{SL}_2(\mathbb{F}_9), \ \mathrm{SL}_3(\mathbb{F}_2) \cong \mathrm{PSL}_2(\mathbb{F}_7), \ \mathrm{SL}_3(\mathbb{F}_4),$$

$$\mathrm{SL}_4(\mathbb{F}_2) \cong A_8, \ \mathrm{Spin}_7(\mathbb{F}_2) \cong \mathrm{Sp}_6(\mathbb{F}_2), \ \mathrm{Sp}_4(\mathbb{F}_2) \cong S_6, \ \mathrm{Spin}_7(\mathbb{F}_3), \ \mathrm{Spin}_8(\mathbb{F}_2),$$

$$F_4(\mathbb{F}_2), \ G_2(\mathbb{F}_3), \ G_2(\mathbb{F}_4), \ \mathrm{SU}_4(\mathbb{F}_2), \ \mathrm{SU}_4(\mathbb{F}_3), \ \mathrm{SU}_6(\mathbb{F}_2), \ {}^2E_6(\mathbb{F}_2).$$

Remark 18. Let Z be the center of a connected algebraic group G defined over $\mathbb{F}_q$ which we assume is a finite (algebraic) group. It is curious to observe that the central extension
$$1 \to Z(\mathbb{F}_q) \to G(\mathbb{F}_q) \to G(\mathbb{F}_q)/Z(\mathbb{F}_q) \to 1$$

has a "dual extension", given by the following part of the long exact sequence associated to the exact sequence $1 \to Z \to G \to G/Z \to 1$ of algebraic groups:

$$1 \to Z(\mathbb{F}_q) \to G(\mathbb{F}_q) \to (G/Z)(\mathbb{F}_q) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), Z) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), G) \to \cdots ,$$

but $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), G) = \{1\}$ by Lang's theorem. Thus we have

$$1 \to Z(\mathbb{F}_q) \to G(\mathbb{F}_q) \to (G/Z)(\mathbb{F}_q) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), Z) \to 1.$$

It is well known that for $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}}$, and for a module A of $\hat{\mathbb{Z}}$ with $A^\vee = Hom(A, \mathbb{Q}/\mathbb{Z})$ with natural $\hat{\mathbb{Z}}$ structure, there is a perfect pairing:

$$H^1(\hat{\mathbb{Z}}, A) \times H^0(\hat{\mathbb{Z}}, A^\vee) \to H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z},$$

and hence

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), Z) \cong Z(\mathbb{F}_q).$$

Therefore the above part of the long exact can be written as the following short exact sequence:

$$1 \to G(\mathbb{F}_q)/Z(\mathbb{F}_q) \to (G/Z)(\mathbb{F}_q) \to Z(\mathbb{F}_q) \to 1.$$

7. Central extension of Algebraic groups

The following basic theorem is due to C. Moore, Matsumoto, Deodhar, G. Prasad, Raghunathan, and Rapinchuk.

Theorem 19. *Let G be an absolutely simple, simply connected algebraic group which is isotropic over k, a non-Archimedean local field, with $\mu(k)$ the cyclic group of roots of unity in k, or $k = \mathbb{R}$, and G split but not of type $C_n, n \geq 1$. Then there exists a natural isomorphism $H^2(G(k), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mu(k), \mathbb{Q}/\mathbb{Z})$.*

Given this theorem, two important questions immediately come to mind (which is also the way the theorem is proved):

(1) What is the functorial nature of the group $H^2(G(k), \mathbb{Q}/\mathbb{Z})$ as the field k varies?
(2) What is the functorial nature of the group $H^2(G(k), \mathbb{Q}/\mathbb{Z})$ as the group G varies?

The following theorem due to C. Moore, Deodhar, G. Prasad, Raghunathan, and Rapinchuk is useful to answer such questions.

Theorem 20. *Let G be an absolutely simple, simply connected algebraic group which is isotropic over k, a local field. Let S be a maximal split torus in G, and α a root of G with respect to S. Let G_α be the (simply connected semi-simple) group generated by the root subgroups U_α and $U_{-\alpha}$. Then if α is a long root in the relative root system of G with respect to S (which if the root system is not reduced means that $\alpha/2$ is a root), the restriction map from $H^2(G(k), \mathbb{Q}/\mathbb{Z})$ to $H^2(G_\alpha(k), \mathbb{Q}/\mathbb{Z})$ is injective.*

Regarding question 1 above, let E/F be a finite separable extension of non-Archimedean local fields. Let m be an integer such that the m-th roots of unity are contained in $F^\times$. Let $(-, -)_m^F$ denote the Hilbert symbol on F with values in the m-th roots of unity in F, and similarly, let $(-, -)_m^E$ denote the Hilbert symbol on E with values in the m-th roots of unity in E. Then it is known that,

$$(a, b)_m^E = (a, b)_m^{F\,d},$$

where $a, b \in F^\times$, and d is the degree of the field extension E/F. By the way central extensions of $G(E)$ and $G(F)$ are constructed using Hilbert symbols, it follows that the restriction from $H^2(G(E), \mathbb{Q}/\mathbb{Z})$ to $H^2(G(F), \mathbb{Q}/\mathbb{Z})$ lands inside $d \cdot H^2(G(F), \mathbb{Q}/\mathbb{Z})$, in particular any central extension of $G(E)$ by $\mu_d \subset F^\times$ becomes trivial when restricted to $G(F)$ whenever the degree of E over F is a multiple of d.

Regarding question 2 above, there is a general recipe due to Deligne. For this assume that G and H are simple, simply connected, split groups with maximal tori T and S, and a morphism $\phi : H \to G$ defined over F taking S to T. Assume that $\alpha^\vee : \mathbb{G}_m \to S \subset H$ is a coroot in H corresponding to a long root α for H which under ϕ goes to the coroot $\phi(\alpha^\vee)$ for G. Fix a Weyl group invariant positive definite integral bilinear form on the cocharacter group of T such that the corresponding quadratic form Q_G takes the value 1 on any coroot corresponding to a long root of T in G. Let $d = Q_G(\phi(\alpha^\vee))$. Then the restriction from $H^2(G(F), \mathbb{Q}/\mathbb{Z})$ to $H^2(H(F), \mathbb{Q}/\mathbb{Z})$ has its image equal to $d \cdot H^2(H(F), \mathbb{Q}/\mathbb{Z})$.

Remark 21. *Although it is not meaningful to talk about central extensions of $G(\bar{k})$ for the algebraic closure $\bar{k}$ of a non-Archimedean local field k, since there are none which are non-trivial, for each finite Galois extension K of k, there is a rich supply of central extensions, say $E(K)$ of $G(K)$ which carry the Galois action too; the only (slight) issue is that the $\mathrm{Gal}(K/k)$-invariants in $E(K)$ is an extension of $G(k)$ is not necessarily a non-trivial extension—still a perfectly fine context to think about basechange issues.*

8. Some results about Galois groups of number fields

The notion of Schur multiplier, and the dual notion of constructing extensions of a group are also studied for Galois groups of number fields and local fields, and have important implications in the subject. We simply state two most important results on these.

Theorem 22 (Tate). *For any number field F, $H^2(Gal(\bar{F}/F), \mathbb{C}^\times) = \{1\}$.*

Theorem 23 (Neukirch). *The group of outer automorphism $\mathrm{Out}(Gal(\bar{\mathbb{Q}}/\mathbb{Q})) = \{1\}$.*

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