

GENERIC REPRESENTATIONS FOR SYMMETRIC SPACES

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ABSTRACT. For a connected quasi-split reductive algebraic group G over a field k , which is either a finite field or a non-archimedean local field, θ an involutive automorphism of G over k , let $K = G^\theta$. Let $K^1 = [K^0, K^0]$, the commutator subgroup of K^0 , the connected component of identity of K . In this paper, we provide a simple condition on (G, θ) for there to be an irreducible admissible generic representation π of G with $\text{Hom}_{K^1}[\pi, \mathbb{C}] \neq 0$. The condition is most easily stated in terms of a real reductive group $G_\theta(\mathbb{R})$ associated to the pair (G, θ) being quasi-split.

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1. INTRODUCTION

Let G be a connected reductive algebraic group over a field k , which is either a finite field or a non-archimedean local field. Let θ be an involutive automorphism of G over k and $K = G^\theta$, the subgroup of G on which θ acts trivially. The pair (G, θ) or (G, K) is called a symmetric space over k .

A well-known question of much current interest is to spectrally decompose $L^2(K(k) \backslash G(k))$, or the related algebraic question to understand irreducible admissible representations π of $G(k)$ with $\text{Hom}_{K(k)}[\pi, \mathbb{C}] \neq 0$. Irreducible admissible representations π of $G(k)$ with $\text{Hom}_{K(k)}[\pi, \mathbb{C}] \neq 0$ are often called *distinguished* representations of $G(k)$ (with respect to $K(k)$). We refer to the work of

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Lusztig [16] for a definitive work for k a finite field, elaborated by Henderson in [12] for $G = \mathrm{GL}_n(k)$ and $\mathrm{U}_n(k)$. For k a non-archimedean local field, there are various works, see e.g. Hakim and Murnaghan in [10], Kato and Takano in [14]. For a general spectral decomposition, see Sakellaridis-Venkatesh [20]. For $k = \mathbb{R}$, the spectral decomposition of $L^2(K(\mathbb{R}) \backslash G(\mathbb{R}))$ is well understood due to the works of Flensted-Jensen, Oshima et al., see [19] for a survey.

Complete results about spectral decomposition of $L^2(K(k) \backslash G(k))$, or about distinguished representations of $G(k)$, will naturally require a full classification of irreducible admissible representations of $G(k)$. However, for many purposes, less precise, but general results such as multiplicity one property (i.e., $\dim \mathrm{Hom}_{K(k)}[\pi, \mathbb{C}] \leq 1$) when available, are of great importance. As another general question, one might mention the question of whether there exists a discrete series representation of $G(k)$ distinguished by $K(k)$, or whether there exists a tempered representation of $G(k)$ distinguished by $K(k)$. The paper [3] of Ash-Ginzburg-Rallis defines a pair (G, L) for a subgroup L of G to be a *vanishing pair* if there are no cuspidal representations of G distinguished by L , and provides many examples of such pairs without a general criterion about them.

In this paper, we give a general criterion as to when there is a generic representation of $G(k)$ distinguished by $K(k)$ (assuming of course that G is quasi-split over k , a condition which is always satisfied if k is a finite field). Although generic representations are a very special class of representations where the geometric methods of this paper apply, it appears to us that distinguished generic representations hold key to all distinguished tempered representations in that the following are equivalent.

- (1) Existence of distinguished tempered representations.
- (2) Existence of distinguished generic representations.

We consider a more precise form of the equivalence above as the (symmetric space) analogue of Shahidi's conjecture on the existence of a generic representation in a tempered L -packet.

Question 1. Let (G, θ) be a symmetric space over a local field k with G quasi-split over k . Assume that for $K = G^\theta$, K^0 the connected component of identity of K is split over k . Then if a tempered representation of $G(k)$ is distinguished by $K(k)$, then so is some generic member of its L -packet? In particular, if there are no generic representations of $G(k)$ distinguished by $K(k)$, then there are no tempered representations of $G(k)$ distinguished by $K(k)$?

Remark 1. We show by an example that the question above will have a negative answer without assuming K^0 to be split. For this we note Corollary 6 of [9] according to which for $G = \mathrm{SO}(4, 2)$ (where we are using $\mathrm{SO}(p, q)$ to denote the orthogonal group of *any* quadratic space of dimension $(p + q)$, and rank $\min\{p, q\}$), there are cuspidal representations of G distinguished by $H = \mathrm{SO}(4, 1)$ although our main theorem below will show that there are no generic representations of $\mathrm{SO}(4, 2)$ distinguished by $\mathrm{SO}(4, 1)$; on the other hand, by

[9] there are no cuspidal representations of $G = \mathrm{SO}(4,2)$ distinguished by $H = \mathrm{SO}(3,2)$. Similarly, for E/F a quadratic extension of non-archimedean local fields, there are examples of distinguished cuspidal representations for $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$ in the paper of Lei Zhang [26], although this paper proves that there no distinguished generic representations for $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(E))$. For $(\mathrm{Sp}_{4n}(F), \mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F))$ by [3], there are no distinguished cuspidal representations, and by this paper, there are no distinguished generic representations.

Before we come to the statement of the main theorem of this paper, we want to discuss a bit of the *universality* of reductive groups with involutions over general algebraically closed fields of characteristic not 2. Recall that reductive groups are supposed to have an existence independent of a field, for example Sp_{2n} is a reductive group, and we can talk of $\mathrm{Sp}_{2n}(\mathbb{C})$ as well as $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_p)$ (made precise either by smooth models over open subsets of $\mathrm{Spec}(\mathbb{Z})$ containing the point p , or by saying that a reductive group over *any* algebraically closed field is given by a root datum). In a similar spirit, there is a notion of reductive groups with involution (G, θ) which makes sense independent of the algebraically closed field (of characteristic not 2) over which these are defined.

More precisely, for the purposes of this paper, an involution θ and its conjugates under the group $G(k)$ play similar role (the action of the group $G(k)$ being $(g \cdot \theta)(x) = (g\theta g^{-1})(x) = g\theta(g^{-1}xg)g^{-1}$). If $\mathrm{Aut}(G)(k)[2]$ denotes the set of elements $\theta \in \mathrm{Aut}(G)(k)$ with $\theta^2 = 1$, then the object of interest for this paper is the orbit space,

$$\mathrm{Aut}(G)(k)[2]/G(k).$$

The following (presumably well-known) proposition lies at the basis of this paper and allows one to compare symmetric spaces over different algebraically closed fields. In this proposition we will use the notion of a *quasi-split* symmetric space (G, θ) , which will play an important role in all of this paper. A symmetric space (G, θ) over a field k will be said to be quasi-split if there exists a Borel subgroup B of $G(k)$ such that B and $\theta(B)$ are opposite Borel subgroups of G , i.e., $B \cap \theta(B)$ is a maximal torus of G . Most often in this paper, we will use this concept only over algebraically closed fields (even if (G, θ) is defined over a finite or non-archimedean local field).

Proposition 1. *If \bar{k}_1 and \bar{k}_2 are any two algebraically closed fields of characteristic not 2, then for any connected reductive algebraic group G , there exists a canonical identification of finite sets*

$$\mathrm{Aut}(G)(\bar{k}_1)[2]/G(\bar{k}_1) \longleftrightarrow \mathrm{Aut}(G)(\bar{k}_2)[2]/G(\bar{k}_2).$$

Under this identification of conjugacy classes of involutions, if $\theta_1 \leftrightarrow \theta_2$, then in particular,

- (1) *the connected components of identity $(G^{\theta_1})^0$ and $(G^{\theta_2})^0$ are reductive algebraic groups, and correspond to each other in the sense defined earlier.*
- (2) *the symmetric space $(G(\bar{k}_1), \theta_1)$ is quasi-split if and only if $(G(\bar{k}_2), \theta_2)$ is quasi-split.*

We next note the following basic result (called the Cartan classification) about real groups, cf. [21], Theorem 6 of Chapter III, §4.

Proposition 2. *Let θ be an involutive automorphism of a connected reductive group G over \mathbb{C} with $K = G^\theta(\mathbb{C})$. Then there exists a naturally associated reductive group G_θ defined over \mathbb{R} with $G_\theta(\mathbb{C}) = G(\mathbb{C})$, and such that $G_\theta(\mathbb{R}) \cap K(\mathbb{C})$ is a maximal compact subgroup of $G_\theta(\mathbb{R})$. The isomorphism class of the real reductive group G_θ depends only on $G(\mathbb{C})$ -conjugacy class of the involution $\theta \in \text{Aut}(G)$. All real reductive groups are obtained by this construction (as $G_\theta(\mathbb{R})$).*

Example 1. Let $G(\mathbb{C}) = \text{GL}_{m+n}(\mathbb{C})$, $\theta = \theta_{m,n}$ the involution $g \rightarrow \theta_{m,n} g \theta_{m,n}$ where $\theta_{m,n}$ is the diagonal matrix in $\text{GL}_{m+n}(\mathbb{C})$ with first m entries 1, and last n entries -1 . In this case, the real reductive group G_θ is the group $\text{U}(m, n)$ with maximal compact $\text{U}(m) \times \text{U}(n)$ which is the compact form of $G^\theta = \text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$.

Here is the main theorem of this paper.

Theorem 1. *Let θ be an involutive automorphism of a connected reductive group G over k which is either a finite or a non-archimedean local field of characteristic not 2 with $K = G^\theta$. Then if G is quasi-split over k , and if $G(k)$ has a generic representation distinguished by $K^1(k)$ for $K^1 = [K^0, K^0]$ where K^0 is the connected component of identity of K , one of the following equivalent conditions (for k of odd characteristic) hold good.*

- (1) *There exists a Borel subgroup B of $G(\bar{k})$ such that $B \cap \theta(B)$ is a maximal torus of G , i.e., the symmetric space (G, θ) is quasi-split over \bar{k} .*
- (2) *Using the identification of groups and involutions over different fields given in Proposition 1, suppose (G, θ) over \bar{k} is associated to (G', θ') over \mathbb{C} , then the associated real reductive group $G'_{\theta'}(\mathbb{R})$ is quasi-split over \mathbb{R} .*

Remark 2. The equivalence of the two conditions in Theorem 1 is part of Proposition 1, thus the essence of this theorem is that if $G(k)$ has a generic representation distinguished by $K(k)$, then there exists a Borel subgroup B of $G(\bar{k})$ such that B and $\theta(B)$ are opposite Borel subgroups of G , i.e., $B \cap \theta(B)$ is a maximal torus of G , which is what most of this paper does.

We summarize the flow of arguments in the paper. The paper tries to understand when the space of locally constant compactly supported functions $\mathcal{S}(K(k) \backslash G(k))$ on $K(k) \backslash G(k)$ has a Whittaker model (for k a finite or a non-archimedean local field). This involves understanding orbits of $U(k)$, a maximal unipotent subgroup in the quasi-split group $G(k)$, on $K(k) \backslash G(k)$. The space $\mathcal{S}(K(k) \backslash G(k))$ has a Whittaker model if and only if one of the orbits of $U(k)$ on $K(k) \backslash G(k)$ supports a Whittaker model, i.e., there exists an orbit of $U(k)$ with stabilizer say $U_x(k)$ such that the Whittaker functional $\psi : U(k) \rightarrow \mathbb{C}^\times$ is trivial on $U_x(k)$. Looking at the action of U on $K \backslash G$ over the algebraic closure \bar{k} of k , we can hope to analyze all possible stabilizers for the action of U on $K \backslash G$ which is what the paper does to prove Theorem 1. The non-obvious assertion which goes in the proof of this theorem is that if U has a certain orbit on $K \backslash G$ of the

form $U_x \backslash U$ which allows a Whittaker model, so the Whittaker character is trivial on U_x (notice that such an orbit of U can be even 1-dimensional), we prove that in fact U must have an orbit on $K \backslash G$ with trivial stabilizer. This we prove using detailed structure of reductive groups (over an algebraically closed field) which come equipped with an involution, eventually working with root systems with involutions. We know of no algebraic geometric explanation of why ‘small’ orbits of U on $K \backslash G$ of a certain kind forces U to have a large orbit on $K \backslash G$, in fact one with no stabilizer. This is connected with the well-known fact in representation theory that irreducible representations of $G(\mathbb{F}_q)$ with Whittaker model are ‘large’, i.e., have dimension of the order of $q^{d(U)}$ where $d(U)$ is the dimension of a maximal unipotent subgroup of G assumed to have connected center, cf. Theorem 10.7.7 in [8], although it is not clear from the definition that this is forced. (One can prove that representations of $GL_n(\mathbb{F}_q)$ with Whittaker model are ‘large’ using the structure of the mirabolic subgroup of $GL_n(\mathbb{F}_q)$.)

Proposition 1 identifying involutions on different algebraically closed fields of characteristic not 2 is of independent interest, and is proved in section 6 of the paper. In section 7, we propose a definition of a symmetric space in characteristic 2, and suggest that our main theorem of the paper extends to symmetric spaces in characteristic 2.

Most of the paper is devoted to proving that if there is a generic distinguished representation for (G, θ) , the symmetric space must be quasi-split. Section 9 deals with the converse: if the symmetric space is quasi-split, we prove that there is a generic irreducible representation of $G(k)$ distinguished by $G^\theta(k)$.

We end the introduction with the following question which we answer in this paper (in the affirmative) for symmetric varieties.

Question 2. Let G be a reductive algebraic group over \mathbb{C} operating on a spherical variety X . Let U be a maximal unipotent subgroup of G . Suppose U has an orbit on X of the form U/U_x such that no simple root space of U is contained in $U_x \cdot [U, U]$. Then is there an orbit of U on X with trivial stabilizer? What if we drop the assumption on X to be spherical? By Theorem 10.7.7 of [8] about dimension of generic representations of $G(\mathbb{F}_q)$ recalled earlier, one can deduce that $\dim X \geq \dim U$.

2. GENERALITIES ON GROUPS WITH INVOLUTION

In this section we collect together facts on groups with involutions (over an algebraically closed field \bar{k} of characteristic not 2), all well-known for a long time. We refer to the article [24] of Springer as a general reference to this section.

Given a connected reductive group G with an involution θ on it, let K^0 be the identity component of $K = G^\theta$. One defines a torus T in G to be θ -split if $\theta(t) = t^{-1}$ for all $t \in T$. A parabolic P in G is said to be θ -split if P and $\theta(P)$ are opposite parabolics, i.e., $P \cap \theta(P)$ is a Levi subgroup for both P and $\theta(P)$.

Here are some facts about groups with involutions.

- (1) Unless G is a torus, there are always nontrivial θ -split tori in G .

- (2) Maximal θ -split tori in G are conjugate under K^0 ; their common dimension is called the rank of the symmetric space (G, θ) .
- (3) If $Z_G(A)$ (resp $Z_{K^0}(A)$) is the connected component of identity of the centralizer of a maximal θ -split torus A in G (resp. in K^0), then $Z_G(A) = Z_{K^0}(A) \cdot A$.
- (4) Minimal θ -split parabolics in G are conjugate under K^0 .
- (5) If A is a maximal θ -split torus in G , then its centralizer in G is a Levi subgroup for a minimal θ -split parabolic P in G .

Definition 1. (*Split and quasi-split symmetric spaces*) A symmetric space (G, θ) over a field k is said to be split if a maximal θ -split torus is a maximal torus of G . A symmetric space (G, θ) is said to be quasi-split if one of the two equivalent conditions hold good:

- (1) For a maximal θ -split torus A , $Z_G(A)$ is a maximal torus of G .
- (2) There exists a θ -split Borel subgroup in G .

Remark 3. If (G, θ) is a quasi-split symmetric space, A a maximal θ -split torus in G , B a θ -split Borel subgroup in G with $T = Z_G(A)$, a maximal torus contained in B , then in terms of Lie algebras, we have $\mathfrak{g} = \mathfrak{b} + \theta(\mathfrak{b})$, hence $\mathfrak{g} = \mathfrak{g}^\theta + \mathfrak{b}$. Therefore, for a quasi-split symmetric space (G, θ) ,

$$\dim(K) = \dim(U) + \dim T - \dim(A),$$

in particular, if (G, θ) is a quasi-split symmetric space,

$$\dim B > \dim(K) \geq \dim(U),$$

which can be improved to the equality:

$$\dim(K) = \dim(U),$$

if (G, θ) is a split symmetric space (any quasi-split symmetric space is split if G has no outer automorphism).

It is well-known that all the notions and facts above have analogues in the theory of algebraic groups over \mathbb{R} from which they are derived; in particular, if $\bar{k} = \mathbb{C}$, then a group $G(\mathbb{C})$ with involution θ is split or quasi-split if and only if the group $G_\theta(\mathbb{R})$ is split or quasi-split.

Two symmetric spaces (G, θ) and (G, θ') are said to be conjugate if θ and θ' are conjugate by an element of G , whereas two symmetric spaces (G, θ) and (G, θ') are said to be isomorphic if θ and θ' are conjugate by an element of $\text{Aut}(G)$.

Just like uniqueness of split and quasi-split groups over any field (with a given splitting field etc.), given a reductive group G over \mathbb{C} , the set of conjugacy classes of quasi-split symmetric spaces (G, θ) over \mathbb{C} is in bijective correspondence with the set of involutions in $\text{Out}(G) = \text{Aut}(G)/G$, whereas isomorphism classes of quasi-split symmetric spaces (G, θ) over \mathbb{C} is in bijective correspondence with the conjugacy classes of involutions in $\text{Out}(G) = \text{Aut}(G)/G$, cf. Theorem 6.14 of [1]. As an example, for $G = \text{GL}_n(\mathbb{C})$, there are exactly two quasi-split symmetric spaces (G, θ) since $\text{Out}(\text{GL}_n(\mathbb{C})) = \mathbb{Z}/2$.

The following definition is inspired by the theory of real groups.

Definition 2. Let $T = HA$ be a maximal torus for G left invariant by θ which operates as identity on H and as $t \rightarrow t^{-1}$ on A . Then a root space U_α for T is said to be imaginary if $\alpha : T \rightarrow k^\times$ is trivial on A , real if α is trivial on H , and complex if it is neither real or imaginary. If α is imaginary and $U_\alpha \subset G^\theta$, then α is said to be a compact-imaginary root.

The following well-known lemma is a consequence of Theorem 7.5 of [25] applied to the connected algebraic group $B \cap \theta(B)$ which contains a maximal torus of G . (The last assertion in the lemma is a consequence of conjugacy of maximal tori in any connected algebraic group.)

Lemma 1. For a symmetric space (G, θ) over a field k of characteristic not 2 and B a Borel subgroup of G , there exists a maximal torus T of B which is θ -invariant. Further, any two θ -invariant maximal tori of B are conjugate under $B^\theta(k)$.

The following lemma appears as Lemma 2.4 in [2].

Lemma 2. Let θ be an involution on a reductive group G over an algebraically closed field \bar{k} . If θ operates trivially on a maximal torus $T \subset G$, then θ must be conjugation by an element of T .

Proof. Since θ operates trivially on T , it operates trivially on the character group of T , and hence takes any root of T to itself. Therefore, θ takes all root spaces of T to itself, and therefore preserves any Borel subgroup B of G containing T . Clearly, there is $t \in T$ such that the automorphism $\theta' = \text{Ad}(t) \circ \theta$ acts trivially on all simple root spaces of T in B , and hence on B . Thus we have an automorphism θ' of G acting trivially on B . It is well-known that such an automorphism must be identity since $\theta'(g) \cdot g^{-1} : G \rightarrow G$ descends to give a morphism from G/B to G , but there are no non-constant morphisms from a projective variety to an affine variety, proving that $\theta'(g) = g$ for all $g \in G$, hence θ is inner-conjugation by an element of T as desired. \square

Lemma 3. For a symmetric space (G, θ) over an algebraically closed field \bar{k} , let A be a θ -split torus in G . Then G is θ -quasi-split if and only if $Z_G(A) = \{g \in G \mid gag^{-1} = a, \forall a \in A\}$ which is θ -invariant is θ -quasi-split.

Proof. To say that G is θ -quasi-split is equivalent to say that there exists a maximal θ -split torus in G , say A_0 , whose centralizer in G is a maximal torus in G . We can assume that $A \subset A_0$, therefore $A_0 \subset Z_G(A)$, and is a maximal θ -split torus in $Z_G(A)$. It follows that if the centralizer of A_0 in G is a torus, then the centralizer of A_0 in $Z_G(A)$ is a torus too. Conversely, since the centralizer of A_0 in G is contained in $Z_G(A)$, if the centralizer of A_0 in $Z_G(A)$ is a torus, the centralizer of A_0 in G is the same torus. \square

3. WHITTAKER MODEL

In this section k is a finite or a non-archimedean local field, $G(k)$ is the group of k -rational points of a connected quasi-split reductive group G with $U(k)$ the

k -rational points of a fixed maximal connected unipotent group U of G , and $\psi : U(k) \rightarrow \mathbf{C}^\times$ a non-degenerate character of $U(k)$, and B the normalizer of U in G . By Lemma 1, the Borel subgroup B contains a θ -invariant torus T defined over k . Whenever we use root spaces in U or in B , it is with respect to such a θ -invariant torus T . It may be noted that since tori in B are conjugate under $U(k)$, although the notion of a root space depends on the choice of T , but the notion of a simple root space is independent of the choice of T . We will abuse notation to denote B, T, U also for $B(k), T(k), U(k)$.

For any smooth representation π of $G(k)$, $\pi_{U,\psi}$ denotes the largest quotient of π on which U operates by the character ψ . The representation π is said to be *generic* if $\pi_{U,\psi} \neq 0$. An important property of the functor $\pi \rightarrow \pi_{U,\psi}$ is that it is exact. In what follows, $\mathcal{S}(X)$ denotes the space of compactly supported locally constant functions on a topological space X .

Proposition 3. *If there exists an irreducible admissible representation π of $G(k)$ which is distinguished by $K(k)$ and for which $\pi_{U,\psi}^\vee \neq 0$, then*

$$\mathcal{S}(K(k) \backslash G(k))_{U,\psi} \neq 0.$$

Proof. By Frobenius reciprocity, the representation π of $G(k)$ is distinguished by $K(k)$ if and only if π^\vee appears as a quotient of $\text{ind}_{K(k)}^{G(k)} \mathbf{C}$:

$$\text{Hom}_{G(k)}[\text{ind}_{K(k)}^{G(k)} \mathbf{C}, \pi^\vee] \cong \text{Hom}_{G(k)}[\pi, \text{Ind}_{K(k)}^{G(k)} \mathbf{C}] \cong \text{Hom}_{K(k)}[\pi, \mathbf{C}].$$

Since twisted Jacquet functor is exact, if π^\vee is a quotient of $\mathcal{S}(K(k) \backslash G(k))$, and if $\pi_{U,\psi}^\vee \neq 0$, then clearly

$$\mathcal{S}(K(k) \backslash G(k))_{U,\psi} \neq 0,$$

proving the proposition. □

Proposition 4. *With the notation as before, $\mathcal{S}(K(k) \backslash G(k))_{U,\psi} \neq 0$ if and only if there exists an orbit of $U(k)$ on $K(k) \backslash G(k)$ passing through say $K(k) \cdot x \in K(k) \backslash G(k)$, so of the form $U_x(k) \backslash U(k)$ with $U_x(k) = x^{-1}K(k)x \cap U(k)$ such that ψ is trivial on $U_x(k) \cdot [U, U](k)$.*

Proof. It is a consequence of Theorem 6.9 of [4] that if the group $H(k)$ of k -rational points of any algebraic group H (over k) operates on an algebraic variety $X(k)$ algebraically, and there is a distribution on $X(k)$ supported on a closed subset $X' \subset X(k)$ on which $H(k)$ operates via a character $m : H(k) \rightarrow \mathbf{C}^\times$, then there must be an orbit of $H(k)$ on $X' \subset X(k)$ which carries a distribution on which $H(k)$ operates via the character $m : H(k) \rightarrow \mathbf{C}^\times$. (An essential component of this theorem of Bernstein-Zelevinsky is their Theorem A, proved in the appendix to [4] for all non-archimedean local fields, that the action of $H(k)$ on $X(k)$ is always *constructible*.)

Therefore if $\mathcal{S}(K(k) \backslash G(k))_{U,\psi} \neq 0$, there is an orbit of $U(k)$ on $K(k) \backslash G(k) \subset (K \backslash G)(k)$ carrying a distribution on which $U(k)$ operates via $\psi : U(k) \rightarrow \mathbf{C}^\times$

(thus we are applying Theorem 6.9 of [4] in the notation above to $H = U$, $X = K \backslash G$, $X' = K(k) \backslash G(k) \subset (K \backslash G)(k)$).

If the orbit of $U(k)$ carrying a distribution on which $U(k)$ operates via $\psi : U(k) \rightarrow \mathbb{C}^\times$ is $U_x(k) \backslash U(k)$, by Frobenius reciprocity

$$\mathrm{Hom}_{U(k)}[\mathrm{ind}_{U_x(k)}^{U(k)} \mathbb{C}, \psi] \cong \mathrm{Hom}_{U(k)}[\psi^{-1}, \mathrm{Ind}_{U_x(k)}^{U(k)} \mathbb{C}] \cong \mathrm{Hom}_{U_x(k)}[\psi^{-1}, \mathbb{C}].$$

Therefore, for $\mathrm{ind}_{U_x(k)}^{U(k)} \mathbb{C}$ to have Whittaker model for the character ψ , ψ must be trivial on $U_x(k)$. But ψ being a character on $U(k)$, it is automatically trivial on $[U, U](k)$, proving the proposition. \square

Proposition 5. *Let G be a quasi-split reductive algebraic group over a field k which is either a finite field or a non-archimedean local field. Let θ be an involution on G with $K = G^\theta$. Then if $\mathcal{S}(K(k) \backslash G(k))$ has a Whittaker model there exists a maximal unipotent subgroup U of G such that $U^\theta(k) \cdot [U, U](k)$ contains no simple root space of $U(k)$.*

Proof. This proposition is a direct consequence of the previous proposition. \square

The earlier discussion motivates us to make the following definition.

Definition 3. (Property [G]) *A symmetric space (G, θ) over a field k is said to have property [G] (G for generic) if there exists a Borel subgroup B of G with unipotent radical U , and a θ -invariant maximal torus T inside B such that none of the simple root subgroups of B with respect to T are contained in $U^\theta \cdot [U, U]$.*

Remark 4. A maximal unipotent subgroup $U(k)$ of $G(k)$ determines the Borel subgroup $B(k)$ as the normalizer of $U(k)$. Any two maximal tori in $B(k)$ are conjugate under $U(k)$, and therefore the simple root spaces for $U(k)$ are independent of the choice of a maximal torus in $B(k)$; thus in the definition above of property [G], the choice of T is redundant. A similar remark applies when we talk of root spaces in U in other places in the paper.

Definition 4. (Algebraic Whittaker character) *Let G be a quasi-split group over any field k and U a maximal unipotent subgroup of G defined over k . A homomorphism of algebraic groups $\ell : U \rightarrow \mathbb{G}_a$ defined over k will be said to be an algebraic Whittaker character if it is nontrivial restricted to each simple root subspace of U (with respect to a maximally split torus A over k normalizing U).*

An algebraic Whittaker character over \bar{k} is unique up to conjugacy by $T(\bar{k})$. If k is either a finite field or is a local field, and ℓ is defined over k , then fixing a nontrivial character $\psi_0 : k \rightarrow \mathbb{C}^\times$ allows one to construct a Whittaker character in the usual sense $\psi_0 \circ \ell : U(k) \rightarrow \mathbb{C}^\times$. The map $\ell \rightarrow \psi_0 \circ \ell$ is a bijection between algebraic Whittaker characters and Whittaker characters on $U(k)$. Thus, algebraic Whittaker characters are more basic objects being defined over *any* field k capturing all the attributes of a Whittaker character.

The discussion so far in this section is summarized in the following proposition.

Proposition 6. *Let k be either a finite field, or a non-archimedean local field. Let (G, K) be a symmetric space with G quasi-split over k . If for any point $x \in K(\bar{k}) \setminus G(\bar{k})$, for $U_x(\bar{k}) = x^{-1}K(\bar{k})x \cap U(\bar{k})$, $U_x(\bar{k}) \cdot [U, U]$, contains a simple root space of U , then there is no irreducible admissible representation of $G(k)$ distinguished by $K(k)$ which is generic.*

Proof. Suppose there is an irreducible admissible representation of $G(k)$ distinguished by $K(k)$ which is generic. By Proposition 5, there exists an orbit of $U(k)$ passing through a point $x \in K(k) \setminus G(k)$, and an abstract Whittaker character $\ell : U(k) \rightarrow k$ such that ℓ restricted to $U(k) \cap K(k)$ is trivial, but $\ell|_{U_\alpha} \neq 0$ for root spaces U_α corresponding to all simple roots α in U . Being algebraic (in fact a linear form on a finite dimensional vector space over k), such an ℓ defines an abstract Whittaker character $\bar{\ell} : U(\bar{k}) \rightarrow \bar{k}$ such that $\bar{\ell}$ restricted to $U(\bar{k}) \cap K(\bar{k})$ is trivial (this we prove in Lemma 4 below), but $\bar{\ell}|_{U_\beta} \neq 0$ for each simple root space U_β in $U(\bar{k})$ against the hypothesis in the proposition. \square

In the next section we find that the geometric condition on an *algebraic Whittaker character* on U being nontrivial on each U_x can be nicely interpreted which will then prove Theorem 1.

Remark 5. Observe that in Proposition 6 we can deal with all quasi-split groups which become isomorphic over \bar{k} at the same time. For instance, it gives (after we have proved the required statements on the orbits of $U(\bar{k})$ on $K(\bar{k}) \setminus G(\bar{k})$) the analogue of Matringe's theorem on non-existence of generic representations of $GL_{m+n}(k)$ distinguished by $GL_m(k) \times GL_n(k)$ if $|m - n| > 1$ to unitary groups: there are no generic representations of $U(V + W)$ (assumed to be quasi-split) distinguished by $U(V) \times U(W)$ if $|\dim V - \dim W| > 1$.

Remark 6. This section is written for a symmetric space (G, θ) over k together with a given unipotent subgroup $U(k)$ of $G(k)$. The involution θ plays no role in this section, and the section remains valid for an arbitrary homogeneous space $K \setminus G$. (The group U^θ appearing in this section is then $U \cap K$.) In particular, this section is valid in characteristic 2 except that Lemma 4 uses the involution θ crucially and we have not found its analogue in characteristic 2 which will prevent us from proving the analogue of our main theorem (Theorem 1) in characteristic 2 where there is only a subgroup K and not the involution θ .

Proposition 7. *Let k be either a finite field, or a non-archimedean local field. Let (G, K) be a symmetric space with G quasi-split over k . If for any point $x \in K(\bar{k}) \setminus G(\bar{k})$, for $U_x(\bar{k}) = x^{-1}K(\bar{k})x \cap U(\bar{k})$, $U_x(\bar{k}) \cdot [U, U]$, contains a simple root space of U , then there is no generic irreducible admissible representation of $G(k)$ distinguished by $K^1(k)$ for $K^1 = [K^0, K^0]$ where K^0 is the connected component of identity of K .*

Proof. As pointed out in Remark 6, the considerations in this section also hold good for $K^1 \setminus G$. The condition " $U_x(\bar{k}) \cdot [U, U]$ contains a simple root space of U ," is the same for K^1 as for K . For this we note that the algebraic groups $x^{-1}K(\bar{k})x \cap U(\bar{k})$ and $x^{-1}K^1(\bar{k})x \cap U(\bar{k})$ have the same connected component of

identity, and therefore the condition that “ $U_x(\bar{k}) \cdot [U, U]$ contains a simple root space” (which is a connected group) is the same for K as for K^1 . \square

Lemma 4. *With the notation as before, if a linear form $\ell : U(k) \rightarrow k$ is trivial when restricted to $U^\theta(k) = U(k) \cap K(k)$, then $\bar{\ell} : U(\bar{k}) \rightarrow \bar{k}$ is also trivial when restricted to $U^\theta(\bar{k}) = U(\bar{k}) \cap K(\bar{k})$.*

Proof. The subtlety in the lemma arises from the fact that a priori we only know that $U \cap K$ is a unipotent group, and a subgroup of U , and in +ve characteristic, subgroups of unipotent groups do not have any simple minded structure (even for $\mathbb{G}_a^n \cong U/[U, U]$ which is where our analysis is done). But our unipotent subgroups ($U \cap K$ of U) are not the pathological ones as we analyze now.

Let B be the Borel subgroup of G containing U , and containing a θ -invariant maximal torus T (this is possible by Lemma 1). By generalities around Bruhat decomposition, we know that the intersection of any two maximal unipotent subgroups of G , in particular $V = U \cap \theta(U)$ is a connected unipotent subgroup of G generated by their common root spaces, i.e.,

$$V = U \cap \theta(U) = \prod_{\alpha > 0, \theta(\alpha) > 0} U_\alpha,$$

where the product is taken in any order (it is useful to note that the “coordinates” corresponding to simple roots are independent of the ordering).

We will prove that the image of V^θ under the natural group homomorphism from V^θ to $U/[U, U] \cong \prod_{\alpha \text{ simple}} U_\alpha \cong \mathbb{G}_a^d$ is a linear subspace, which will prove the lemma. Clearly, the image of V^θ in $\prod_{\alpha \text{ simple}} U_\alpha$ is contained in $\prod_{\alpha \in S} U_\alpha$ where S is the set of simple roots α with the property that $\theta(\alpha) > 0$. For simple roots α with $\theta(\alpha) > 0$, there are three options:

- (1) $\alpha = \theta(\alpha)$ in which case θ preserves the root space $U_\alpha = \mathbb{G}_a^{d_\alpha}$ (for some +ve integer d_α), and being an involution, one can decompose $U_\alpha = \mathbb{G}_a^{d_\alpha^+} + \mathbb{G}_a^{d_\alpha^-}$ with $d_\alpha^+ + d_\alpha^- = d_\alpha$ such that θ acts as identity on $\mathbb{G}_a^{d_\alpha^+}$ and as -1 on $\mathbb{G}_a^{d_\alpha^-}$ (using that we are over a field of characteristic not 2).
- (2) $\alpha \neq \theta(\alpha)$ but both simple. In this case, the image of V^θ lands inside a linear subspace (the diagonal subgroup consisting of element $(u_\alpha, \theta(u_\alpha))$) of $U_\alpha \times U_{\theta(\alpha)}$.
- (3) $\theta(\alpha)$ is +ve but not simple. In this case, we prove that the image of V^θ inside U_α is all of U_α .

For this observe that for $u_\alpha \in U_\alpha$, $v = u_\alpha \cdot \theta(u_\alpha)$ belongs to V . If $\alpha + \theta(\alpha)$ is not a root, then the root spaces U_α and $U_{\theta(\alpha)}$ commute, and $\theta(v) = v$, hence it belongs to V^θ with image $u_\alpha \in U_\alpha$, so the image of V^θ inside U_α is U_α .

Next, assume that $\alpha + \theta(\alpha)$ is a root. In this case since α and $\theta(\alpha)$ have the same norm, by properties of root systems, the only possible roots among $i\alpha + j\theta(\alpha)$ for $i > 0, j > 0$ is $\alpha + \theta(\alpha)$, hence by Chevalley

commutation relation, $v\theta(v)^{-1} = [u_\alpha, \theta(u_\alpha)] \in U_{\alpha+\theta(\alpha)}$ which is a θ -stable linear space over a field of characteristic not 2. Since multiplication by 2 is an isomorphism on $U_{\alpha+\theta(\alpha)}$, $H^1(\langle \theta \rangle, U_{\alpha+\theta(\alpha)}) = 0$. It follows that $v\theta(v)^{-1} = z^{-1}\theta(z)$ for some $z \in U_{\alpha+\theta(\alpha)}$, hence $zu_\alpha\theta(u_\alpha) \in V^\theta$, proving once again that the image of V^θ inside U_α is U_α .

Our linear form $\ell : U(k) \rightarrow k$ arises from a linear form of vector spaces $\ell : (U/[U, U])(k) \rightarrow k$ trivial on the image of U^θ inside $(U/[U, U])(k)$ which is a linear subspace of the k -vector space $(U/[U, U])(k)$. The conclusion of the lemma now follows. \square

4. THE MAIN THEOREM

In this section we work with an arbitrary algebraically closed field E of characteristic not 2. The following proposition is a special case of the main theorem (Theorem 2) of this section which is proved using this special case.

Proposition 8. *Let G be a connected reductive algebraic group over E , T a maximal torus in G contained in a Borel subgroup B of G . Let θ be the involution on G which is conjugation by an element $t_0 \in T$ which acts by -1 on all simple root spaces of T in B . Then the symmetric space (G, θ) is quasi-split, i.e., there exists a Borel subgroup B' of G for which $\theta(B')$ is opposite of B' .*

Proof. For the proof of the proposition it suffices to assume that G is an adjoint simple group. In fact if two symmetric spaces (G, θ) and (G', θ') are related by a homomorphism $\phi : G \rightarrow G'$ with $\theta' \circ \phi = \phi \circ \theta$ such that $\ker(\phi)$ is central in G , and the image of G under ϕ is normal in G' , with $G'/\phi(G)$ a torus, then the proposition is true for (G, θ) if and only if it is true for (G', θ') . The proof of the proposition will be divided into 3 cases. The proposition amounts to saying that the element $t_0 \in T$ has a conjugate in G which takes B to an opposite Borel which is what we will prove below.

Case 1 (Assuming the Jacobson-Morozov theorem):

Let T_0 be the diagonal torus in $SL_2(E)$, B_0^\pm the group of upper-triangular and lower-triangular matrices in $SL_2(E)$. Assume first that there is a $j : SL_2(E) \rightarrow G$, the Jacobson-Morozov homomorphism corresponding to a regular unipotent element in B with $j(T_0) \subset T$. The Jacobson-Morozov homomorphism is known to exist if either $p = 0$, or $p > h$, where h is the Coxeter number of G (see [22], Prop. 2, and other references in the bibliography of this paper). The element $j_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL_2(E)$ with $i = \sqrt{-1}$, acts by -1 on the simple root space of $SL_2(E)$ contained in B_0^+ , therefore its image under j also acts by -1 on all simple root spaces of T in B .

Observe that the matrix $j_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $SL_2(E)$ normalizes T_0 , and acts on it by $t \rightarrow t^{-1}$. It follows that the conjugate of B by $j(j_2)$ is opposite to B .

The matrices $j_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $j_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\mathrm{SL}_2(E)$ are conjugate in $\mathrm{SL}_2(E)$, and therefore their images under j are conjugate in $G(E)$. Therefore, the proof of the proposition is completed whenever we have the Jacobson-Morozov homomorphism corresponding to a regular unipotent element in B .

Case 2 (Classical groups):

Although Jacobson-Morozov theorem is not available when $p < h$, for classical groups, $\mathrm{GL}_n(E), \mathrm{SL}_n(E), \mathrm{Sp}_n(E), \mathrm{SO}_n(E)$ (all subgroups of $\mathrm{GL}_n(E)$), there is a natural homomorphism of $\mathrm{SL}_2(E)$ to $G(E)$ given by $\mathrm{Sym}^{n-1}(E + E)$ in each of these cases except for $\mathrm{SO}_{2n}(E)$ in which case the representation of $\mathrm{SL}_2(E)$ is $\mathrm{Sym}^{2n-2}(E + E) + E$. These representations of $\mathrm{SL}_2(E)$ gives rise to maps $j : \mathrm{SL}_2(E) \rightarrow G(E)$ such that T_0 goes into the standard diagonal torus of these classical groups, with T_0 acting by the same character on all simple roots of T inside B . Thus the image of j_1 in G acts on each simple root space of T in B by -1 , and the image of j_2 takes B to an opposite Borel, therefore the proof of the proposition is complete in all *odd* characteristics for Classical groups. (For $p < h$, the image of the upper triangular unipotent matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under $j : \mathrm{SL}_2(E) \rightarrow G$ is *not* a regular unipotent element in G .)

Case 3 (Exceptional groups):

The first observation to make is that if w_0 is a longest element in the Weyl group of an adjoint group, then by Lemma 5.4 of [2], w_0 has a lift to G , say \tilde{w}_0 , with $\tilde{w}_0^2 = 1$.

Therefore to prove the proposition for adjoint simple Exceptional group, it suffices to prove that the involutions t_0 and \tilde{w}_0 in G are conjugate. However, for exceptional groups there are very few conjugacy classes of elements of order 2 in G (cf. [11], chapter X, table V for $E = \mathbb{C}$, and therefore also for all algebraically closed fields E of characteristic not 2):

- (1) G_2 has only 1;
- (2) F_4 has only 2;
- (3) E_6 has only 2 (with dimension of G^θ being 38, 46 (EII and EIII in Chapter X, table V of [11]), so the quasi-split symmetric space has $\dim(G^\theta) = 38$);
- (4) E_7 has only 2;
- (5) E_8 has only 2.

Since the dimension of the fixed points subgroups for both the involutions t_0 and w_0 can be easily estimated (see Remark 3 for w_0 which defines a quasi-split symmetric space), the proof of the proposition follows. \square

Remark 7. In the proof of Proposition 8, we have used a homomorphism $j : \mathrm{SL}_2(E) \rightarrow G(E)$ such that the image of the diagonal torus T_0 in $\mathrm{SL}_2(E)$ has as its centralizer in G a maximal torus T in G contained a Borel subgroup B of G such that T_0 acts on each simple root space of B by the same character. Our proof did

space (G, θ) is quasi-split, i.e., there exists a Borel subgroup B' of G such that $\theta(B')$ and B' are opposite, i.e., $B' \cap \theta(B')$ is a maximal torus of B' .

Proof. Assume that $T = HA$ on which θ operates as identity on H and as $t \rightarrow t^{-1}$ on A .

If $\text{rank}(A) = 0$, then T is a maximal torus of G on which θ operates trivially. In this case, we know by Lemma 2 that such an automorphism of G is an inner-conjugation by an element, say t_0 , of T . We are furthermore given that $U^\theta \cdot [U, U]$ has no simple roots of T inside U . Since θ is an involution on G induced by $t_0 \in T$, its action on each simple root space of T in B is by 1 or -1 . Since $U^\theta \cdot [U, U]$ has no simple roots, we find that t_0 operates by -1 on all simple roots, and therefore we are in the context of Proposition 5, which proves the theorem in this case.

If $\text{rank}(A) > 0$ consider the subgroup $Z_G(A)$ of G with the same maximal torus $T = HA$ contained now in the Borel subgroup $B \cap Z_G(A)$ of $Z_G(A)$. Since A is a central subgroup in $Z_G(A)$, $Z_G(A)/A$ has the maximal torus $T/A = H/(H \cap A)$ on which θ operates trivially. By Lemma 2, the restriction of θ to $Z_G(A)/A$ is an inner-conjugation by an element $s_0 \in Z_G(A)$. Given the hypothesis in this theorem that “no simple root of B with respect to T is contained in $U^\theta \cdot [U, U]$ ”, by Lemma 5, θ acts on any simple root space of $[B \cap Z_G(A)]/A$ by -1 . Therefore by Proposition 5, $Z_G(A)/A$ — and therefore $Z_G(A)$ — is θ -quasi-split. Now by Lemma 3, G itself is θ -quasi-split, proving the theorem. \square

We also note the following corollary of this theorem.

Corollary 1. *Let (G, θ) be a symmetric space over E with $K = G^\theta$. Then there exists a Borel subgroup B of G with unipotent radical U , such that $U^\theta = U \cap K = \{e\}$ if and only if the symmetric space (G, θ) is quasi-split.*

Proof. If the symmetric space (G, θ) is quasi-split, let B be a Borel subgroup of G with B and $\theta(B)$ opposite. If U is the unipotent radical of B , clearly $U^\theta = U \cap K \subset U \cap \theta(U) = \{e\}$, proving one implication in the corollary.

Conversely, assume that $U^\theta = 1$. Theorem 2 applies, proving that the symmetric space (G, θ) is quasi-split. \square

5. PROOF OF LEMMA 5

In this section we work with an arbitrary algebraically closed field E of characteristic not 2 and prove Lemma 5 from last section (recalled again here) which plays a crucial role in the descent argument (from G to $Z_G(A)$) of the previous section.

The following lemma from Bourbaki [7], Ch. VI, §1, Corollary 3(a) in Section 6 will play a role in the proof of Lemma 5 below.

Lemma 6. *Let R be an irreducible root system with $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ a set of simple roots in R . For a root $\alpha = \sum_i n_i \alpha_i$ in R , let $\Delta(\alpha)$ be the support of α consisting of*

those simple roots α_i in Δ for which $n_i \neq 0$. Then $\Delta(\alpha)$ gives rise to a connected subset of the Dynkin diagram of the root system R .

Lemma 5. *Let $T = HA$ be a maximal torus for G left invariant by θ which operates as identity on H and as $t \rightarrow t^{-1}$ on A . Let $B = TU$ be a Borel subgroup of G containing T . Assume that no simple root space of B with respect to T is contained in $U^\theta \cdot [U, U]$. Then θ acts on any simple root space of $B \cap Z_G(A)$ with respect to T by -1 .*

Proof. It suffices to prove the lemma assuming that G is an adjoint group which is then a product of simple adjoint groups $G = G_1 \times G_2 \times \cdots \times G_k$. An involution on such a product group G is built out of involutions on G_i and involutions on $G_j \times G_j$ which is permuting the two co-ordinates. The lemma is obvious for the latter involution (in fact, in this case, $Z_G(A) = T$, so the lemma is vacuously true), so we are reduced to assuming G to be an adjoint simple group, which we assume is the case in the rest of the proof.

Observe that a root $\alpha : T \rightarrow E^\times$ for G is a root for $Z_G(A)$ if and only if $\alpha|_A = 1$, equivalently, $\theta(\alpha) = \alpha$, i.e., α is an imaginary root. Let $\langle X_\alpha \rangle$ be the corresponding root space. If $\theta(\alpha) = \alpha$, then since θ is an involution, $\theta(X_\alpha) = \pm X_\alpha$. If X_α generates a simple root in B , then by the hypothesis that “no simple root space of B is contained in $U^\theta \cdot [U, U]$ ”, we find that for imaginary simple roots of B , $\theta(X_\alpha) = -X_\alpha$. The subtlety in the lemma arises from the fact that simple roots in $Z_G(A)$ may not be simple roots in G which is what we deal with in the proof that follows.

Note that under the assumption that “no simple root space generated by an element X_α in the Lie algebra of B is contained in $U^\theta \cdot [U, U]$ ”, for any complex simple root α , either $\theta(\alpha) < 0$ or $\theta(\alpha)$ is simple. To prove this, assume the contrary, and let $\theta(\alpha) > 0$ and not simple. We give a proof of this which is valid in all odd characteristics. For this, begin by observing that $V = U \cap U^\theta$ is a θ -invariant and T -invariant unipotent subgroup of U . Being T -invariant, V is filtered by \mathbb{G}_a . If u_α is an element of U in the root space α , then we claim that:

- (1) If $x = u_\alpha \cdot \theta(u_\alpha)$, then x belongs to $U^\theta \cdot [U, U]$;
- (2) $\theta(u_\alpha)$ belongs to $[U, U]$.

By (1) and (2) above, it follows that $u_\alpha \in U^\theta \cdot [U, U]$, contrary to our assumption that “no simple root space generated by an element X_α in the Lie algebra of B is contained in $U^\theta \cdot [U, U]$ ”.

The proof of (2) is clear since by assumption, $\theta(\alpha) > 0$ and not simple. For the proof of (1), note that $x = \theta(x)$ up to $[V, V]$. Since θ is an involution on V preserving $[V, V]$ which is filtered by \mathbb{G}_a on which multiplication by 2 is an isomorphism, hence $H^1(\theta, [V, V]) = 0$, therefore a θ -invariant in $V/[V, V]$ can be lifted to one in V .

Now, suppose α is a simple root for $B \cap Z_G(A)$ but is not a simple root for B . Let $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the set of simple roots of T on B . Write α as a sum

of simple roots for G :

$$\alpha = \sum_i n_i \alpha_i.$$

Applying θ to this equality, we have:

$$\theta(\alpha) = \alpha = \sum_i n_i \theta(\alpha_i).$$

The set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of simple roots of T on B consists of imaginary roots, real roots and complex roots. The only possibility for a simple root α_i which is taken to a positive root under θ is when α_i is either imaginary or complex, and in either case if $\theta(\alpha_i)$ is positive, it is simple. Therefore, the only way $\theta(\alpha) = \alpha$, the nonzero n_i must correspond to either imaginary roots or to pairs of complex simple roots $\{\alpha_i, \theta(\alpha_i)\}$ with equal coefficients. This follows by considering $h(\alpha)$, the height of a root $\alpha = \sum_i n_i \alpha_i$ defined by $h(\alpha) = \sum_i n_i$ and observing that by the remark above $h(\theta(\alpha)) \leq h(\alpha)$ with equality if and only if the nonzero n_i correspond to either imaginary roots or to pairs of complex simple roots $\{\alpha_i, \theta(\alpha_i)\}$.

Thus we can write any imaginary root as:

$$\alpha = \sum_{i \in I} n_i \alpha_i + \sum_{i \in J} n_i [\alpha_i + \theta(\alpha_i)],$$

with α_i simple imaginary roots for $i \in I$, and complex roots for $i \in J$.

By Lemma 6, the support $\Delta(\alpha)$ of α is a connected subset of the Dynkin diagram of the root system R associated to the group G .

Any connected subset of the Dynkin diagram of G is the Dynkin diagram of a reductive subgroup of G sharing a maximal torus and with a simple adjoint group. In our case, the Dynkin diagram associated to $\Delta(\alpha)$ comes equipped with the involution θ for which the simple imaginary roots in I are the fixed points of θ , and simple complex roots have orbits under θ of cardinality 2.

For our goal of proving that θ acts on any simple root space of $B \cap Z_G(A)$ by -1 , a conclusion we have already made for imaginary simple roots, we can assume that for the simple root $\alpha = \sum_{i \in I} n_i \alpha_i + \sum_{i \in J} n_i [\alpha_i + \theta(\alpha_i)]$, $\Delta(\alpha)$ has non-empty set of complex roots. Thus the involution θ on the connected Dynkin diagram associated to $\Delta(\alpha)$ is non-trivial. Since we know all connected Dynkin diagrams with a non-trivial involution, here are all the possibilities, together with a check that in each case $\theta(X_\alpha) = -X_\alpha$.

- (1) $\Delta(\alpha) = \{\beta_1, \beta_2, \dots, \beta_{2n}\}$ is a root system of type A_{2n} , with the unique involution on this root system θ (thus the set I in this case is empty), and the root α whose support is all of $\Delta(\alpha)$ must be

$$\alpha = \beta_1 + \beta_2 + \dots + \beta_{2n} = \lambda + \theta(\lambda),$$

where $\lambda = \beta_1 + \beta_2 + \dots + \beta_n$ which is a root in A_{2n} , hence in G .

Therefore, we can assume that

$$X_\alpha = [X_\lambda, \theta(X_\lambda)],$$

for which clearly

$$\theta(X_\alpha) = -X_\alpha,$$

proving the lemma.

- (2) $\Delta(\alpha) = \{\beta_1, \beta_2, \dots, \beta_{2n+1}\}$ is a root system of type A_{2n+1} , $n \geq 1$, with the unique involution on this root system θ (thus the set I in this case has exactly one element β_{n+1}), and the root α whose support is all of $\Delta(\alpha)$ must be

$$\alpha = \beta_1 + \beta_2 + \dots + \beta_{2n+1} = \lambda + \beta_{n+1} + \theta(\lambda),$$

where $\lambda = \beta_1 + \beta_2 + \dots + \beta_n$ as well as $\lambda + \beta_{n+1}$ is a root in A_{2n+1} , hence in G . Therefore, we can assume that

$$X_\alpha = [[X_\lambda, X_{\beta_{n+1}}], \theta(X_\lambda)].$$

Using the Jacobi identity, we can write $\theta(X_\alpha)$ as:

$$\begin{aligned} \theta(X_\alpha) &= [[\theta(X_\lambda), \theta(X_{\beta_{n+1}})], X_\lambda] \\ &\stackrel{(*)}{=} -[[X_\lambda, \theta(X_\lambda)], \theta(X_{\beta_{n+1}})] - [[\theta(X_{\beta_{n+1}}), X_\lambda], \theta(X_\lambda)]. \end{aligned}$$

Now we note that

$$\lambda + \theta(\lambda) = \beta_1 + \beta_2 + \dots + \beta_n + \beta_{n+2} + \beta_{n+3} + \dots + \beta_{2n+1},$$

is not a root in A_{2n+1} , and hence $[X_\lambda, \theta(X_\lambda)] = 0$. Further, since β_{n+1} is a fixed point of θ , it is a simple imaginary root, therefore $\theta(X_{\beta_{n+1}}) = -X_{\beta_{n+1}}$. Therefore, from the equation (*) above,

$$\theta(X_\alpha) = -X_\alpha,$$

proving the lemma in this case.

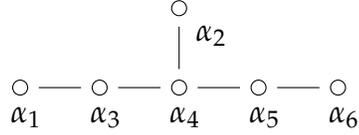
- (3) $\Delta(\alpha) = \{\beta_1, \beta_2, \dots, \beta_{n-2}, \beta_{n-1}, \beta_n\}$ is a root system of type D_n , $n \geq 4$, with the unique involution on this root system θ , thus the set I in this case is $I = \{\beta_1, \beta_2, \dots, \beta_{n-2}\}$, with $\theta(\beta_{n-1}) = \beta_n$. In the standard coordinates, one has $\beta_i = e_i - e_{i+1}$ for $i \leq n-1$, and $\beta_n = e_{n-1} + e_n$. Since the root α has support all of $\Delta(\alpha)$, one can see that the only possible options for α are:

$$\alpha = \beta_1 + \beta_2 + \dots + \beta_i + 2(\beta_{i+1} + \dots + \beta_{n-2}) + \beta_{n-1} + \beta_n = e_1 + e_{i+1},$$

for some $1 < i \leq n-2$.

Note that one of the necessary conditions for two *distinct* roots $\{\alpha, \beta\}$ to be simple is that $(\alpha, \beta) \leq 0$. Since we are given that the fixed points of the involution i.e., $\beta_j, j \leq n-2$ are simple roots, we must have $(\alpha, \beta_j) \leq 0$ for all $j \leq n-2$. For the possible α as above, we have $(\alpha, \beta_1) = (e_1 + e_{i+1}, e_1 - e_2) = 1$. So $\Delta(\alpha)$ cannot be a root system of type $D_n, n \geq 4$.

(4) $\Delta(\alpha)$ is a root system of type E_6 :



This also can be ruled out as in case (3) as we argue now. Following Bourbaki's [7], Plate V on E_6 , the only positive roots of E_6 with all coefficients positive, and for which the coefficients for α_i and $\theta(\alpha_i)$ are the same, are:

$$\begin{aligned}
 \beta_1 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\
 \beta_2 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
 \beta_3 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\
 \beta_4 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
 \beta_5 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.
 \end{aligned}$$

The fixed points of the involution are α_2 and α_4 which are anyway simple roots for $\Delta(\alpha)$. In the standard normalization, each of α_i has $(\alpha_i, \alpha_i) = 2$, and nonzero $(\alpha_i, \alpha_j) = -1$. This allows one to make following calculations:

$$\begin{aligned}
 (\beta_1, \alpha_2) &= 1, \\
 (\beta_2, \alpha_4) &= 1, \\
 (\beta_3, \alpha_4) &= 1, \\
 (\beta_4, \alpha_4) &= 1, \\
 (\beta_5, \alpha_2) &= 1,
 \end{aligned}$$

proving that $\Delta(\alpha)$ cannot be a root system of type E_6 .

□

6. PROOF OF PROPOSITION 1

In this section we give a proof of Proposition 1. We begin by recalling that a connected reductive algebraic group G over an algebraically closed field E is given by a based root datum $\Psi_G = (X, R, S, X^\vee, R^\vee, S^\vee)$ where X, X^\vee are finitely generated free abelian groups in perfect duality under a bilinear map $X \times X^\vee \rightarrow \mathbb{Z}$, and R, R^\vee are finite subsets of X, X^\vee satisfying certain axioms, see e.g. [23], and S is a set of simple roots in R . We let $W = W_G$ be the Weyl group of the group or of the root system. This implies that if G is a reductive algebraic group over one algebraically closed field \bar{k}_1 , it makes sense to use the same letter G to denote the corresponding reductive algebraic group over any other algebraically closed field \bar{k}_2 . The group G can be constructed from the based root datum $\Psi_G = (X, R, S, X^\vee, R^\vee, S^\vee)$ in an explicit way starting with the maximal torus $T(E) = X^\vee \otimes E^\times$. This means that the 2-torsion subgroup

$T(E)[2] = X^\vee \otimes \mathbb{Z}/2 = X^\vee/2X^\vee$, and the set of conjugacy classes of 2-torsion elements in G which is $T(E)[2]/W$ depends only on G and is independent of the algebraically closed field E (as long as it is not of characteristic 2). Note also that if θ is an automorphism of X^\vee , it gives rise to an automorphism of $T(E)$ for any algebraically closed field E .

For any connected reductive group G , say over an algebraically closed field E , a *pinning* on G is a triple $(B, T, \{X_\alpha\})$ where T is a maximal torus in G contained in a Borel subgroup B of G , and X_α are nonzero elements in the simple root spaces of T contained in B . Automorphisms of G fixing a pinning will be said to be *diagram automorphisms* of G (thus for instance if G is a torus, then any automorphism of G will be called a diagram automorphism). If the center of G is Z , the group of diagram automorphisms of G is isomorphic to $\text{Out}(G) = \text{Aut}(G)/(G/Z)$, which can be read from the root datum, in fact it is isomorphic to $\text{Aut}(X, R, S, X^\vee, R^\vee, S^\vee) = \text{Aut}(X, R, X^\vee, R^\vee)/W_G$ where W_G is the Weyl group of the group or of the root system, and which is a normal subgroup of $\text{Aut}(X, R, X^\vee, R^\vee)$.

We have the following short exact sequence of algebraic groups which is split by fixing a pinning on G ,

$$1 \rightarrow \text{Int}(G) = G/Z \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \cong \text{Aut}(G, B, T, \{X_\alpha\}) \rightarrow 1.$$

We now come to the proof of Proposition 1 of the introduction which we again recall here.

Proposition 1. *If \bar{k}_1 and \bar{k}_2 are any two algebraically closed fields of characteristic not 2, then for any connected reductive algebraic group G , there exists a canonical identification of finite sets*

$$\text{Aut}(G)(\bar{k}_1)[2]/G(\bar{k}_1) \longleftrightarrow \text{Aut}(G)(\bar{k}_2)[2]/G(\bar{k}_2).$$

Under this identification of conjugacy classes of involutions, if $\theta_1 \leftrightarrow \theta_2$, then in particular,

- (1) *the connected components of identity $(G^{\theta_1})^0$ and $(G^{\theta_2})^0$ are reductive algebraic groups, and correspond to each other in the sense defined earlier.*
- (2) *the symmetric space $(G(\bar{k}_1), \theta_1)$ is quasi-split if and only if $(G(\bar{k}_2), \theta_2)$ is quasi-split.*

Proof. Observe that any element of $\text{Aut}(G)(\bar{k}_1)[2]$ gives rise to an element of $\text{Out}(G)[2]$, so to prove the proposition, it suffices to prove that for any $\theta \in \text{Out}(G)[2]$, the set of elements in $\text{Aut}(G)(\bar{k}_1)[2]/G(\bar{k}_1)$ giving rise to this θ is independent of the algebraically closed field \bar{k}_1 .

If the element θ is the trivial element, then we are considering elements of $(G/Z)(\bar{k}_1)[2]$ up to conjugation by $G(\bar{k}_1)$, which is nothing but $(T/Z)(\bar{k}_1)[2]/W_G$. Since we are dealing with fields of characteristic $\neq 2$, the structure of $(T/Z)(\bar{k}_1)[2]$ is independent of the algebraically closed field \bar{k}_1 , and the action of W_G on it also is independent of \bar{k}_1 , so the proof of the proposition for such elements is completed. The proof in general is a variant of this proof.

Now, take any $\theta \in \text{Out}(G)[2]$, and let $(G/Z)(\bar{k}_1) \cdot \theta_0$ be the set of elements in $\text{Aut}(G)(\bar{k}_1)$ which project to this element θ in $\text{Out}(G)[2]$ where θ_0 is the unique element of $\text{Aut}(G)$ fixing the pinning chosen earlier and giving rise to the element θ of $\text{Out}(G)[2]$.

By a theorem due to Gantmacher, cf. [6], elements of order 2 in $(G/Z)(\bar{k}_1) \cdot \theta_0$ can be conjugated, using $G(\bar{k}_1)$, to lie inside $(T/Z)(\bar{k}_1) \cdot \theta_0$. An element say $t \cdot \theta_0 \in (T/Z)(\bar{k}_1) \cdot \theta_0$ is of order 2 if and only if $(t \cdot \theta_0)^2 = t \cdot \theta_0(t)$ is central. This gives an identification of the set of elements of order 2 in $(G/Z)(\bar{k}_1) \cdot \theta_0$ up to conjugation by $G(\bar{k}_1)$ to $S[2]/W^\theta$ where S is the largest quotient of T/Z on which θ_0 operates trivially, i.e., $S = T/(Z \cdot \{\theta_0(t) \cdot t^{-1} | t \in T\})$, and W^θ is the fixed points of θ on W_G . (Observe that $t \cdot \theta_0(t) \in Z$ if and only if $t \cdot \theta_0(t) = \theta_0(t) \cdot t^{-1} \cdot t^2 \in Z$, i.e., the image of t in S belongs to $S[2]$.)

This proves that the set of elements of order 2 in $(G/Z)(\bar{k}_1) \cdot \theta_0$ up to conjugation by $G(\bar{k}_1)$ has a structure which is independent of the algebraically closed field \bar{k}_1 , proving the canonical identification of finite sets

$$\text{Aut}(G)(\bar{k}_1)[2]/G(\bar{k}_1) \longleftrightarrow \text{Aut}(G)(\bar{k}_2)[2]/G(\bar{k}_2),$$

as desired.

The involutions we have constructed above belong to $T(\bar{k}_1) \cdot \theta_0$ with θ_0 a diagram automorphism of G , thus these involutions preserve (T, B) ; for such involutions, Steinberg in Theorem 8.2 of [25] proves that the identity component of G^{θ_0} is a reductive group, and describes G^{θ_0} explicitly in terms of root datum, proving that this group is independent of the algebraically closed field \bar{k}_1 . (Actually the theorem of Steinberg is only for simply connected groups, but it works well for general reductive groups too to describe the connected component of identity of G^{θ_0} .)

Finally, we prove the last assertion in the proposition on quasi-split symmetric spaces using a result from [1] recalled below according to which if (G, θ) is a quasi-split symmetric space over an algebraically closed field, we can fix a θ -stable pair $B \supset T$ (a Borel subgroup and a maximal torus) such that every simple root is either complex or noncompact imaginary. If an automorphism of G stabilizes a pair $B \supset T$, it is of the form $t \cdot \theta_0$ as before, and now the fact that for this involution, all simple roots are complex or noncompact imaginary is independent of the field. \square

The following proposition is part of the equivalence of parts (b) and (g) of Proposition 6.24 of [1] proved there for complex groups, but the proof is valid for general algebraically closed fields. (One part of the proposition is a consequence of Theorem 2.)

Proposition 9. *A symmetric space (G, θ) over an algebraically closed field E is quasi-split if and only if there exists a θ -stable pair $B \supset T$ (a Borel subgroup and a maximal torus) such that every simple root is either complex or noncompact imaginary.*

7. SYMMETRIC SPACES IN CHARACTERISTIC 2

In most literature on symmetric spaces (G, θ) , it is traditional to assume that one is dealing with fields of characteristic not 2, for example the article of Springer [24], as well as the article of Lusztig [16] assumes this is the case. For applications to representation theory, one would prefer not to make this assumption. For instance, a basic example of a symmetric space is $G(k) = \mathrm{GL}_{m+n}(k)$ with the involution $\theta : g \rightarrow \theta_{m,n} g \theta_{m,n}$ where $\theta_{m,n}$ is the diagonal matrix in $\mathrm{GL}_{m+n}(k)$ with the first m entries 1, and the last n entries -1 . In this case, the fixed points of the involution is $K(k) = \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$, which makes good sense in all characteristics even though the involution itself does not in characteristic 2. In fact, as is well-known, the subgroup G^θ of G uniquely determines θ in characteristic zero, say by a Lie algebra argument, thus using fixed points of an involution seems a good enough replacement for the involution itself till we realize we have lost the main anchor for the arguments with symmetric spaces.

Thus the first order of business is to define what's meant by a symmetric space in characteristic 2, which we now take as a pair (G, K) with K a 'symmetric' subgroup of G , which we will presently define. It may be remarked that one reason for circumspection regarding involutions (and therefore symmetric spaces) in characteristic 2 is that their fixed point subgroups need not be reductive. Our definition below will continue to assume reductiveness for the subgroup although it seems to be useful not to insist on it such as the example of Shalika subgroup (centralizer of the unipotent element $u = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ of order 2 inside $\mathrm{GL}_{2n}(F)$ where I_n is the $n \times n$ identity matrix) shows.

Definition 5. (a) *(Symmetric subgroup)* Let (G, K) be a pair, consisting of a connected reductive algebraic group G , and a possibly disconnected reductive subgroup K of G over a field k . Let R be a discrete valuation ring with residue field $R/\mathfrak{m} = k$, and quotient field L (of characteristic 0). Let G_R be a (split) connected reductive algebraic group scheme over R which is G over $k = R/\mathfrak{m}$. The subgroup K of G is said to be a symmetric subgroup of G if one can spread these to split reductive group schemes $K_R \subset G_R$ such that $(G(L), K(L))$ is a symmetric space in the usual sense (over a field of characteristic zero now) defined by an involution on G_R .

(b) *(Symmetric space)* A pair of reductive groups (G, K) over a field k , with K a symmetric subgroup over k as defined in (a), will be said to be a symmetric space over k .

(c) *(Quasi-split symmetric space)* A symmetric space (G, K) over a field k of characteristic 2 will be said to be quasi-split if the corresponding symmetric space (G_L, K_L) over L is quasi-split.

Proposition 10. *With the notation as in the definition above, if a symmetric space (G, K) over k , a finite or non-archimedean local field, has property $[G]$, i.e., there exists a Borel subgroup B of G with unipotent radical U such that $(U \cap K) \cdot [U, U]$ contains no*

simple root, then the corresponding symmetric space over L also has this property, hence is quasi-split by Theorem 2.

Proof. It may be conceptually simpler to fix U , and vary K (up to conjugacy by $G(k)$), allowing us to use a fixed unipotent group scheme U_R and a maximal torus T_R normalizing U_R together with its root system. Since $(U \cap K) \cdot [U, U]$ is the reduction modulo \mathfrak{m} of the unipotent group U_R in G_R , this is clear. \square

Remark 8. In the absence of Lemma 4 in characteristic 2, we are not able to prove that Theorem 1 remains valid for symmetric spaces over k , a finite or non-archimedean local field, in characteristic 2; in more detail, we are not able to prove that if G is quasi-split over k , and if $G(k)$ has a generic representation distinguished by $K^1(k)$ for $K^1 = [K^0, K^0]$ where K^0 is the connected component of identity of K , then the symmetric space (G, K) is quasi-split.

8. EXAMPLES

This paper was conceived to explain many examples of symmetric spaces (G, θ) for which it was known that there are no generic representations of $G(k)$ distinguished by $G^\theta(k)$. Usually such theorems are known only in the context of $G = \mathrm{GL}_n$, and proved by different methods (Gelfand pairs, mirabolic subgroups and theory of derivatives...). Here are some of these examples, all consequence of our Theorem 1.

- (1) Let $G(k) = \mathrm{GL}_{m+n}(k)$, $\theta = \theta_{m,n}$ the involution $g \rightarrow \theta_{m,n} g \theta_{m,n}$ where $\theta_{m,n}$ is the diagonal matrix in $\mathrm{GL}_{m+n}(k)$ with first m entries 1, and last n entries -1 . In this case, the real reductive group G_θ is the group $\mathrm{U}(m, n)$ which is quasi-split over \mathbb{R} if and only if $|m - n| \leq 1$. It is a theorem due to Matringe, Theorem 3.2 in [17], that if there is a generic distinguished representation of $\mathrm{GL}_{m+n}(k)$ then $|m - n| \leq 1$.
- (2) Let $G(k) = \mathrm{GL}_{2n}(k)$, θ the involution on $\mathrm{GL}_{2n}(k)$ given by $g \rightarrow J^t g^{-1} J^{-1}$ with J any skew-symmetric matrix in $\mathrm{GL}_{2n}(k)$. The fixed point set of θ is the symplectic group $\mathrm{Sp}_{2n}(k)$. In this case, the real reductive group G_θ is the group $\mathrm{GL}_n(\mathbb{H})$ where \mathbb{H} is the quaternion division algebra over \mathbb{R} . The real reductive group $\mathrm{GL}_n(\mathbb{H})$ is not quasi-split, and it is known that there are no generic representations of $\mathrm{GL}_{2n}(k)$ distinguished by $\mathrm{Sp}_{2n}(k)$, a theorem due to Heumos-Rallis [13].
- (3) Let $G(k) = \mathrm{GL}_n(k)$, θ the involution on $\mathrm{GL}_n(k)$ given by $g \rightarrow J^t g^{-1} J^{-1}$ with J any symmetric matrix in $\mathrm{GL}_n(k)$. The fixed point set of θ is the orthogonal group $\mathrm{O}_n(k)$. In this case, the real reductive group G_θ is the group $\mathrm{GL}_n(\mathbb{R})$ which is split. It is known that there are generic representations of $\mathrm{GL}_n(k)$ distinguished by $\mathrm{O}_n(k)$.

Twisted analogues of examples in (1), (2), (3), such as:

- (4) $\mathrm{U}(V + W)$ (assumed to be quasi-split) containing $\mathrm{U}(V) \times \mathrm{U}(W)$ as fixed point of an involution. The group G_θ is the same as in (1), so not quasi-split if $|\dim V - \dim W| > 1$; we are not certain if one knew before that

there are no generic representations of $U(V + W)$ (assumed to be quasi-split) distinguished by $U(V) \times U(W)$ if $|\dim V - \dim W| > 1$.

- (5) $U(V \otimes E)$ is a quasi-split unitary group over F with E/F quadratic, V a symplectic space over F with $V \otimes E$ the corresponding skew-hermitian space over E , then $\mathrm{Sp}(V) \subset U(V \otimes E)$, and analogous to the work of Heumos-Rallis [13], there are no generic representations of $U(V \otimes E)$ distinguished by $\mathrm{Sp}(V)$. This result is due to [15], Theorem 4.4.

Here are the classical groups.

- (6) $\mathrm{SO}(V + W)$ (assumed to be quasi-split) containing $\mathrm{S}[\mathrm{O}(V) \times \mathrm{O}(W)]$ as the fixed points of an involution. Assume that $\dim V = m$, $\dim W = n$. Then the group $G_\theta = \mathrm{SO}(m, n)(\mathbb{R})$ which is not quasi-split if $|\dim V - \dim W| > 2$. We are not certain if one knew before that there are no generic representations of $\mathrm{SO}(V + W)$ (assumed to be quasi-split) distinguished by $\mathrm{S}[\mathrm{O}(V) \times \mathrm{O}(W)]$ if $|\dim V - \dim W| > 2$.
- (7) $\mathrm{Sp}(V + W)$ containing $\mathrm{Sp}(V) \times \mathrm{Sp}(W)$ as the fixed points of an involution. Assume that $\dim V = m$, $\dim W = n$. In this case the group $G_\theta = \mathrm{Sp}(m, n)(\mathbb{R})$ which is never quasi-split. We are not certain if one knew before that there are no generic representations of $\mathrm{Sp}(V + W)$ distinguished by $\mathrm{Sp}(V) \times \mathrm{Sp}(W)$. These are vanishing pairs of [3], i.e., there are no cuspidal representations of $\mathrm{Sp}(V + W)$ distinguished by $\mathrm{Sp}(V) \times \mathrm{Sp}(W)$.

There is also the twisted analogue of this example $G = \mathrm{Sp}_{4n}(F) \supset \mathrm{Sp}_{2n}(E) = H$. Here also by our theorem there are no generic representations of G distinguished by H .

- (8) Besides the involutions used in the previous two examples, there is another kind of involution for the groups $\mathrm{SO}_{2n}(F)$ as well as $\mathrm{Sp}_{2n}(F)$. For this suppose $V = X + X^\vee$ is a complete polarization on a $2n$ -dimensional vector space over F which may be orthogonal or symplectic. Define j_0 to be the involution on the corresponding classical group $G(V)$ obtained by inner conjugation of the element $j \in \mathrm{PGL}(V)(F)$ which acts on X by multiplication by $i = \sqrt{-1}$, and on X^\vee by multiplication by $-i$. These involutions define $G_\theta = \mathrm{SO}^*(2n, \mathbb{R}), \mathrm{Sp}(2n, \mathbb{R})$ in the orthogonal and symplectic cases respectively with maximal compact $\mathrm{U}(n, \mathbb{R})$ in both cases. The group $\mathrm{SO}^*(2n)$ is not quasi-split, whereas $\mathrm{Sp}(2n, \mathbb{R})$ is. Therefore, by our theorem, there are no generic representations of $\mathrm{SO}_{2n}(F)$ which are distinguished by $\mathrm{GL}_n(F)$, whereas there should be generic representations of $\mathrm{Sp}_{2n}(F)$ distinguished by $\mathrm{GL}_n(F)$ (Theorem 1 only rules out generic distinguished representations and does not construct one when it allows one which we take up in the next section). We are not certain if one knew before that there are no generic representations of $G(V) = \mathrm{SO}_{2n}(F)$ distinguished by $\mathrm{GL}_n(F)$. It is a vanishing pair of [3] (they assert this only for n odd whereas we do not distinguish between n even and n odd).

9. THE CONVERSE AND CONCLUDING REMARKS

In this paper we have proved that if for a symmetric space (G, θ) over a finite or a non-archimedean local field k , there is an irreducible generic representation of $G(k)$ distinguished by $G^\theta(k)$, then the symmetric space over \bar{k} is quasi-split. A nice part of this theorem is worth mentioning: it used knowledge of G as well as θ only over \bar{k} .

If the symmetric space is quasi-split over k , we have the following converse. The author thanks Matringe for his help with the proof of this proposition especially by suggesting the use of Proposition 7.2 of [18] in this proof.

Proposition 11. *Let (G, θ) be a symmetric space over a finite or a non-archimedean local field k which is quasi-split over k , thus there is a Borel subgroup B of G over k with $B \cap \theta(B) = T$, a maximal torus of G over k . If k is finite, assume that its cardinality is large enough (for a given G). Then there is an irreducible generic unitary principal series representation of $G(k)$ distinguished by $G^\theta(k)$.*

Proof. It is easy to see in the non-archimedean local case by a well-known Lie algebra argument that $K(k) \cdot B(k)$ is an open subset of $G(k)$ giving rise to an open subset $H(k) \backslash K(k) \subset B(k) \backslash G(k)$ where $H(k) = K(k) \cap B(k)$ is the k -points of a torus inside K contained in a maximal torus T of B . Thus we have the inclusion $\mathcal{S}(H(k) \backslash K(k)) \subset \mathcal{S}(B(k) \backslash G(k))$ of compactly supported functions. If $Ps(\chi)$ is the principal series representation of $G(k)$ induced using a character $\chi : T(k) \rightarrow \mathbb{C}^\times$ for which $\chi|_{H(k)} = 1$, then have the inclusion $\mathcal{S}(H(k) \backslash K(k)) \subset Ps(\chi)$; it is important at this point to note that this inclusion remains true for normalized induction because of lemma 7 below.

By Frobenius reciprocity, the subspace $\mathcal{S}(H(k) \backslash K(k)) \subset Ps(\chi)$ carries an $H(k)$ -invariant linear form if $\chi|_{H(k)} = 1$. It is a result of Blanc-Delorme, cf. Theorem 2.8 of [5], see Proposition 7.2 of [18] for a precise statement in our specific context, that if $\chi|_{H(k)} = 1$, then indeed the principal series representation $Ps(\chi)$ is distinguished.

It suffices then to construct characters χ of $T(k)$ with $\chi|_{H(k)} = 1$ such that the principal series representation $Ps(\chi)$ is irreducible. This can be done for k a finite field under the hypothesis that it has large enough cardinality (for a given G) by choosing a regular character of $(T/H)(k)$, i.e., one for which $\chi^w \neq \chi$ for any $w \neq 1$ in W_G .

For k a non-archimedean local field, note that by Lemma 7 if δ_B is the modulus function of B , then δ_B restricted to $H(k)$ is trivial. Therefore, for any nonzero real number t , $\chi = \delta_B^{it}$ is a unitary character of T , trivial on $H(k)$, and is not invariant by any nontrivial element of the Weyl group of G . By well-known results of Bruhat, such principal series representations of G are irreducible, completing the proof of the proposition. \square

Lemma 7. *With the notation as above, if δ_B is the modulus function of B , then δ_B restricted to $H(k)$ is trivial.*

Proof. Observe that $\delta_{\theta(B)}(\theta(t)) = \delta_B(t)$. Therefore if $\theta(t) = t$ as is the case for elements in $H(k)$, we have $\delta_{\theta(B)}(t) = \delta_B(t)$. On the other hand, $\theta(B)$ being the opposite Borel, $\delta_{\theta(B)}(t) = \delta_B(t^{-1})$. This completes the proof of the lemma. \square

Our paper has dealt with quasi-split symmetric spaces, but if the symmetric space (G, θ) is not quasi-split, question arises as to what are the ‘largest’ representations which contribute to the spectral decomposition of $L^2(K(k)\backslash G(k))$? There is a natural $\mathrm{SL}_2(\mathbb{C})$ inside the L -group of G which controls this, and which can be constructed as follows. Let $T = HA$ be a maximal torus in G on which θ operates by identity on H , and A is a maximal θ -split torus in G . The centralizer $M = Z_G(A)$ of A in G is a Levi subgroup of G , hence it associates a Levi subgroup \widehat{M} in the dual group \widehat{G} too. The $\mathrm{SL}_2(\mathbb{C})$ associated to the regular unipotent conjugacy class in \widehat{M} plays an important role for $L^2(K(k)\backslash G(k))$ whose spectral analysis is dictated by the centralizer of this (Arthur) $\mathrm{SL}_2(\mathbb{C})$ in \widehat{G} . For example, for the symmetric space $(\mathrm{GL}_{2n}(k), \mathrm{Sp}_{2n}(k))$, $M = \mathrm{GL}_2(k)^n$, and we will be looking at the representation of $\mathrm{SL}_2(\mathbb{C})$ inside $\mathrm{GL}_{2n}(\mathbb{C})$ which is $[2] + \cdots + [2]$ where $[2]$ is the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$, with centralizer $\mathrm{GL}_n(\mathbb{C})$ which is well-known to control the spectral decomposition of $L^2(\mathrm{Sp}_{2n}(k)\backslash \mathrm{GL}_{2n}(k))$.

Our previous analysis done in the presence of a θ -split Borel subgroup can be done with a minimal θ -split parabolic $P = MN$ with $M = Z_G(A) = Z_K(A) \cdot A$, and now involve principal series representations $P_S(\pi)$ induced from an irreducible representation π of M but which are trivial on $Z_K(A)$ (which contains the derived subgroup of M) to describe $K(k)$ -distinguished (principal series) representations of G . Because for these principal series representations $P_S(\pi)$, π is trivial on the derived subgroup of M , π is one dimensional, giving some substance to the suggestion in the previous paragraph.

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