ISRAEL JOURNAL OF MATHEMATICS **211** (2016), 257–270
DOI: 10.1007/s11856-015-1273-6

A CHARACTER RELATIONSHIP ON $\mathrm{GL}_\mathrm{n}(\mathbb{C})$

BY

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ABSTRACT

In this paper we consider the character of an irreducible finite-dimensional algebraic representation of $\mathrm{GL}_{mn}(\mathbb{C})$ restricted to a particular disconnected component of the normalizer of the Levi subgroup $\mathrm{GL}_m(\mathbb{C})^n$ of $\mathrm{GL}_{mn}(\mathbb{C})$, generalizing a theorem of Kostant on the character values at the Coxeter element.

1. Introduction

Studying representation theory of disconnected groups in the context of real and p-adic groups has been an important topic of study, finding impressive applications such as to the theory of basechange, cf. [La], which has been a key instrument in all recent proofs of reciprocity theorems in number theory, and eventually to the Fermat's last theorem!

Representation theory of disconnected algebraic groups can also be studied in a similar vein; see, for instance, the paper [KLP]. In this paper, we prove a simple result on the restriction of representations of connected reductive algebraic groups (actually we are able to handle only $\mathrm{GL}_n(\mathbb{C})$ here) to suitable disconnected algebraic subgroups whose connected component of identity has 'large enough' normalizer. Unfortuntely, we have not managed to find a general class of examples where the kind of character relationship we prove here for $\mathrm{GL}_n(\mathbb{C})$ holds.

Received March 6, 2014 and in revised form August 5, 2014

We begin with a very beautiful theorem of Kostant, Theorem 2 in [Ko], which is about the character of a finite-dimensional representation of a semi-simple algebraic group G at the Coxeter conjugacy class. Recall that one usually defines a Coxeter element—or, rather a conjugacy class—in a Weyl group (as a product of simple reflections), in this case in $N(T)/T$, where T is a maximal torus in G, with $N(T)$ its normalizer in G. An arbitrary lift of this conjugacy class in $N(T)/T$ to $N(T)$ gives a well-defined conjugacy class in G which we will denote by $c(G)$. Here is the theorem of Kostant; cf. [P] for another proof of this theorem of Kostant.

THEOREM 1: *Let G be a semi-simple algebraic group over $\mathbb{C}$, and π a finite-dimensional irreducible representation of G. Then the character Θ_π of π at the element $c(G)$ takes one of the values $1, 0, -1$.*

2. The main theorem

Before we turn to the main theorem of this paper, we note the following well-known and elementary lemma whose proof we will omit. (The first part of this lemma replaces the observation that an arbitrary lift to $N(T)$ of a Coxeter element in $N(T)/T$ belongs to a unique conjugacy class in G.)

LEMMA 1: *Let G be an arbitrary group. Consider the semi-direct product $G^n \rtimes \mathbb{Z}/n$ in which $\mathbb{Z}/n$ operates on G^n by the n-cycle:*

$$(g_1, \ldots, g_{n-1}, g_n) \longrightarrow (g_n, g_1, \ldots, g_{n-1});$$

we denote this generator of $\mathbb{Z}/n$ by σ. Then there exists a bijective correspondence between conjugacy class of elements of $G^n \rtimes \mathbb{Z}/n$ of the form $(g_1, \ldots, g_{n-1}, g_n) \rtimes \sigma$ and conjugacy class of elements of G given by the Norm mapping, $Nm : G^n \rtimes \mathbb{Z}/n \to G$, and which is defined by

$$(g_1, \ldots, g_{n-1}, g_n) \rtimes \sigma \in G^n \rtimes \mathbb{Z}/n \longrightarrow g_1 \cdots g_n \in G.$$

A finite-dimensional irreducible representation Π of $G^n \rtimes \mathbb{Z}/n$ has a nonzero character value on the coset $G^n \cdot \sigma$ only if its restriction to G^n is irreducible, and therefore Π restricted to G^n is $\pi \otimes \cdots \otimes \pi$ for an irreducible representation π of G. For such a representation Π of $G^n \rtimes \mathbb{Z}/n$,

$$\Theta_\Pi((g_1, \ldots, g_n) \cdot \sigma) = \Theta_\pi(g_1 \cdots g_n).$$

The aim of this paper is to calculate the character of irreducible representations of $\mathrm{GL}_{mn}(\mathbb{C})$ on the subgroup $\mathrm{GL}_m(\mathbb{C})^n \rtimes \mathbb{Z}/n$ which sits naturally inside $\mathrm{GL}_{mn}(\mathbb{C})$ at elements of the subgroup which have projection a fixed generator σ of $\mathbb{Z}/n$ generalizing the theorem of Kostant recalled above for $\mathrm{GL}_n(\mathbb{C})$. Since elements of $\mathrm{GL}_m(\mathbb{C})$ have n-th roots, it suffices by the previous lemma to calculate the character at very special elements, those of the form $(g, \ldots, g, g) \rtimes \sigma$. This is what we shall do here.

Consider the natural homomorphism of groups,

$$\iota : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{mn}(\mathbb{C}),$$

obtained by taking the tensor product of vector spaces over $\mathbb{C}$ of dimensions m and n. Thus, we need to calculate the characters of irreducible representations of $\mathrm{GL}_{mn}(\mathbb{C})$ at the elements of the form $\iota(t \cdot c_n)$, which we will simply write as $t \cdot c_n$, where t is any element of $\mathrm{GL}_m(\mathbb{C})$, and c_n is the conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ represented by the element in $\mathrm{GL}_n(\mathbb{C})$ which cyclically permutes the standard basis $\{e_1, e_2, \ldots, e_n\}$; the conjugacy class c_n is the Coxeter conjugacy class in $\mathrm{SL}_n(\mathbb{C})$ if n is odd, and for any n its image in $\mathrm{PGL}_n(\mathbb{C})$ is the Coxeter conjugacy class.

Theorem 2: *Let π be an irreducible finite-dimensional representation of $GL_{mn}(\mathbb{C})$ with character Θ_π and with highest weight*

$$\underline{\lambda} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{mn-1} \geq \lambda_{mn}.$$

Let $\underline{\rho}_{mn}$ be the mn-tuple of integers,

$$mn - 1 \geq mn - 2 \geq \cdots \geq 1 \geq 0.$$

Then the character $\Theta(t \cdot c_n)$ is not identically zero if and only if the mn-tuple of integers appearing in $\underline{\lambda} + \underline{\rho}_{mn}$ represents each residue class in $\mathbb{Z}/n$ exactly m-times. Assume this to be the case. Then for each residue class in $\mathbb{Z}/n$ represented by $0 \leq i < n$, let π_i be the representation of $GL_m(\mathbb{C})$ with highest weight $\underline{\mu}_i$ such that $\underline{\mu}_i + \underline{\rho}_m$ are the integers (exactly m of them by hypothesis) among

$$[\underline{\lambda} + \underline{\rho}_{mn} - i]/n.$$

Then for the irreducible representations $\pi_1, \ldots, \pi_n$ of $GL_m(\mathbb{C})$ with highest weights $\underline{\mu}_0, \underline{\mu}_1, \ldots, \underline{\mu}_{n-1}$, and with characters $\Theta_1, \ldots, \Theta_n$,

$$\Theta_\pi(t \cdot c_n) = \pm\Theta_1(t^n)\Theta_2(t^n) \cdots \Theta_n(t^n).$$

Remark: Before we begin the proof of the theorem, it may be worth pointing out that the representations π_i of $\mathrm{GL}_m(\mathbb{C})$ occurring in this theorem through the character identity

$$\Theta(t\cdot c_n) = \pm\Theta_1(t^n)\Theta_2(t^n)\cdots\Theta_n(t^n)$$

are well defined only up to permutation of the representations π_i, and twisting of the representations π_i by characters of $\mathbb{C}^\times$ via the determinant map from $\mathrm{GL}_m(\mathbb{C})$ to $\mathbb{C}^\times$ (with the only constraint that the product of these twisting characters is trivial). This character identity implies that if z_π is the central character of π (an integer by which the center of $\mathrm{GL}_{mn}(\mathbb{C})$, which is $\mathbb{C}^\times$, operates on π), and if z_i are the central characters of π_i, then we must have

$$z_\pi = z_1^n\cdot z_2^n\cdots z_n^n.$$

Since the central character of π is given by the integer $\sum_i \lambda_i$, and similarly of π_i, that the above identity of central characters holds in our case follows from the following identity:

$$\sum_{i=0}^{i=mn-1} i = m\sum_{i=0}^{i=n-1} i + n^2\sum_{i=0}^{i=m-1} i.$$

Proof of Theorem 2. We begin by proving that if the mn-tuple of integers represented by $\underline{\lambda}+\underline{\rho}_{mn}$ do not represent each residue class in $\mathbb{Z}/n$ by exactly m-integers, then the character value is identically zero at the elements of the form $t\cdot c_n$, where we take t to be the diagonal matrix in $\mathrm{GL}_m(\mathbb{C})$ with diagonal entries $(t_1,t_2,\ldots,t_m)$, and let c_n be the diagonal matrix in $\mathrm{GL}_n(\mathbb{C})$ consisting of the n-th roots of unity all with multiplicity one.

Thus $t\cdot c_n$ is the diagonal matrix in $\mathrm{GL}_{mn}(\mathbb{C})$ with entries $t_i\omega^j$ where ω is a primitive n-th root of unity in $\mathbb{C}$, and $1\le i\le m$, and $0\le j<n$.

Under the assumption on the mn-tuple of integers represented by $\underline{a}=\underline{\lambda}+\underline{\rho}_{mn}$, by re-ordering the indices, assume that the integers $a_1,a_2,\ldots,a_{m+1}$ have the same residue modulo n.

Now we write out a part of the matrix whose determinant represents the numerator of the Weyl character formula:

$$\begin{pmatrix}
t_1^{a_1} & t_2^{a_1} & \cdots & t_m^{a_1} & \omega_1 t_1^{a_1} & \cdots & \omega_1 t_m^{a_1} & \omega_2 t_1^{a_1} & \cdots & \omega_2 t_m^{a_1} & \cdots\\
t_1^{a_2} & t_2^{a_2} & \cdots & t_m^{a_2} & \omega_1 t_1^{a_2} & \cdots & \omega_1 t_m^{a_2} & \omega_2 t_1^{a_2} & \cdots & \omega_2 t_m^{a_2} & \cdots\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
t_1^{a_{m+1}} & t_2^{a_{m+1}} & \cdots & t_m^{a_{m+1}} & \omega_1 t_1^{a_{m+1}} & \cdots & \omega_1 t_m^{a_{m+1}} & \omega_2 t_1^{a_{m+1}} & \cdots & \omega_2 t_m^{a_{m+1}} & \cdots
\end{pmatrix}.$$

In this $(m+1)\times mn$-matrix, the first $(m+1)\times m$-matrix repeats itself after being scaled by ω_1, an n-th root of unity, and then again after being scaled by ω_2, an n-th root of unity, and so on. Clearly the rank of this $(m+1)\times mn$-matrix is $\leq m$, which proves that the determinant of the $mn\times mn$ matrix representing the numerator of the Weyl character is zero. The element $t\cdot c_n$ in $\mathrm{GL}_{mn}(\mathbb{C})$ is regular for a generic choice of t in $\mathrm{GL}_m(\mathbb{C})$, so the Weyl denominator is nonzero. Thus the character of the representation π at the element $t\cdot c_n$ in $\mathrm{GL}_{mn}(\mathbb{C})$ is identically zero.

Assume then that the mn-tuple of integers represented by $\underline{\lambda}+\underline{\rho}_{mn}$ represents each residue class in $\mathbb{Z}/n$ by exactly m-integers. The proof of the character relationship in the theorem is by a direct manipulation with the character formula for an irreducible highest weight module of $\mathrm{GL}_{mn}(\mathbb{C})$ given in terms of quotients of Vandermonde determinants, and depends on the observation that

$$\det\begin{pmatrix} X_1 & \omega X_1 & \cdots & \omega^{n-1}X_1 \\ X_2 & \omega^2 X_2 & \cdots & \omega^{2(n-1)}X_2 \\ \vdots & \vdots & \cdots & \vdots \\ X_n & \omega^n X_n & \cdots & \omega^{n(n-1)}X_n \end{pmatrix} = c\det(X_1)\det(X_2)\cdots\det(X_n),$$

where ω is a primitive n-th root of unity, X_i are $m\times m$ matrices, and

$$c=\prod_{i<j}(\omega^i-\omega^j)^m.$$

We will not give a more detailed proof of the theorem in general, but show the details for $m=n=2$. Since the Coxeter element c_2 for $\mathrm{GL}_2(\mathbb{C})$ is the diagonal matrix with entries $(1,-1)$, the theorem proposes to calculate the character of a representation of $\mathrm{GL}_4(\mathbb{C})$ say with highest weight $\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4$ at the diagonal elements of the form $(t_1,t_2,-t_1,-t_2)$.

The proof of the theorem will be achieved by the explicit character of the representation π given as the quotient of two Vandermonde determinants, the numerator of which is the determinant of the 4×4-matrix

$$\begin{pmatrix} t_1^{\lambda_1+3} & t_2^{\lambda_1+3} & (-t_1)^{\lambda_1+3} & (-t_2)^{\lambda_1+3} \\ t_1^{\lambda_2+2} & t_2^{\lambda_2+2} & (-t_1)^{\lambda_2+2} & (-t_2)^{\lambda_2+2} \\ t_1^{\lambda_3+1} & t_2^{\lambda_3+1} & (-t_1)^{\lambda_3+1} & (-t_2)^{\lambda_3+1} \\ t_1^{\lambda_4} & t_2^{\lambda_4} & (-t_1)^{\lambda_4} & (-t_2)^{\lambda_4} \end{pmatrix}.$$

It is easy to see that for the determinant to be nonzero, it is necessary that out of $\lambda_1+3, \lambda_2+2, \lambda_3+1, \lambda_4$ two are even and two are odd. Assume for definiteness that λ_1+3, λ_2+2 are even, and λ_3+1, λ_4 are odd, in which case the Weyl numerator becomes

$$\begin{pmatrix} t_1^{\lambda_1+3} & t_2^{\lambda_1+3} & t_1^{\lambda_1+3} & t_2^{\lambda_1+3} \\ t_1^{\lambda_2+2} & t_2^{\lambda_2+2} & t_1^{\lambda_2+2} & t_2^{\lambda_2+2} \\ t_1^{\lambda_3+1} & t_2^{\lambda_3+1} & -t_1^{\lambda_3+1} & -t_2^{\lambda_3+1} \\ t_1^{\lambda_4} & t_2^{\lambda_4} & -t_1^{\lambda_4} & -t_2^{\lambda_4} \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} a_1 & a_2 & a_1 & a_2 \\ b_1 & b_2 & b_1 & b_2 \\ c_1 & c_2 & -c_1 & -c_2 \\ d_1 & d_2 & -d_1 & -d_2 \end{pmatrix} = 4 \det \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ b_1 & b_2 & 0 & 0 \\ 0 & 0 & -c_1 & -c_2 \\ 0 & 0 & -d_1 & -d_2 \end{pmatrix},$$

we find that the Weyl numerator is the same as

$$\begin{aligned} &4 \det \begin{pmatrix} t_1^{\lambda_1+3} & t_2^{\lambda_1+3} \\ t_1^{\lambda_2+2} & t_2^{\lambda_2+2} \end{pmatrix} \cdot \det \begin{pmatrix} t_1^{\lambda_3+1} & t_2^{\lambda_3+1} \\ t_1^{\lambda_4} & t_2^{\lambda_4} \end{pmatrix} \\ =&4t_1t_2 \det \begin{pmatrix} t_1^{\lambda_1+1} & t_2^{\lambda_1+1} \\ t_1^{\lambda_2} & t_2^{\lambda_2} \end{pmatrix} \det \begin{pmatrix} t_1^{\lambda_3+2} & t_2^{\lambda_3+2} \\ t_1^{\lambda_4+1} & t_2^{\lambda_4+1} \end{pmatrix} \\ =&4t_1t_2 \det \begin{pmatrix} (t_1^2)^{\frac{\lambda_1+1}{2}} & (t_2^2)^{\frac{\lambda_1+1}{2}} \\ (t_1^2)^{\frac{\lambda_2}{2}} & (t_2^2)^{\frac{\lambda_2}{2}} \end{pmatrix} \det \begin{pmatrix} (t_1^2)^{\frac{\lambda_3+2}{2}} & (t_2^2)^{\frac{\lambda_3+2}{2}} \\ (t_1^2)^{\frac{\lambda_4+1}{2}} & (t_2^2)^{\frac{\lambda_4+1}{2}} \end{pmatrix}. \end{aligned}$$

Further, the Weyl denominator at $(t_1, t_2, -t_1, -t_2)$ can be checked to be

$$4t_1t_2(t_1^2 - t_2^2)^2.$$

Therefore,

$$\begin{aligned} \Theta(t) =& \frac{\det \begin{pmatrix} (t_1^2)^{\frac{\lambda_1+1}{2}} & (t_2^2)^{\frac{\lambda_1+1}{2}} \\ (t_1^2)^{\frac{\lambda_2}{2}} & (t_2^2)^{\frac{\lambda_2}{2}} \end{pmatrix} \cdot \det \begin{pmatrix} (t_1^2)^{\frac{\lambda_3+2}{2}} & (t_2^2)^{\frac{\lambda_3+2}{2}} \\ (t_1^2)^{\frac{\lambda_4+1}{2}} & (t_2^2)^{\frac{\lambda_4+1}{2}} \end{pmatrix}}{(t_1^2 - t_2^2)^2} \\ =&\Theta_1(t^2) \cdot \Theta_2(t^2). \qquad \blacksquare \end{aligned}$$

Remark 1: The above theorem can be re-written for general elements of

$$\mathrm{GL}_m(\mathbb{C})^n \rtimes \mathbb{Z}/n$$

(which project to the generator σ of $\mathbb{Z}/n$) as

$$(*) \qquad \Theta_\pi(g \rtimes \sigma) = \Theta'(\mathrm{Nm}\ g),$$

where Θ' is the restriction of the character of the representation $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_n$ of $\mathrm{GL}_m(\mathbb{C})^n$ to the diagonally embedded subgroup

$$\mathrm{GL}_m(\mathbb{C}) \hookrightarrow \mathrm{GL}_m(\mathbb{C})^n.$$

Remark 2: The character relationship appearing in $(*)$ is a very familiar one in the representation theory of real and p-adic groups, and was in fact at the origin of this work which was to find aspects of the character theory of the generalized Steinberg representation $St_n(\pi)$ (or the generalized Speh representation) for $\mathrm{GL}_{mn}(k)$, k a p-adic field now, which will allow one to retrieve (the character of) π from (the character of) $\mathrm{St}_n(\pi)$. It seems a worthwhile problem to find a generalization of the theorem proved here for $\mathrm{GL}_{mn}(\mathbb{C})$ to p-adic $\mathrm{GL}_{mn}(k)$. Finding an analogue of Kostant's theorem on character values at the Coxeter conjugacy class seems an interesting question for $\mathrm{GL}_m(k)$ already. For instance, it appears that the character of a supercuspidal representation of $\mathrm{GL}_m(k)$ at the Coxeter conjugacy class is always zero, say in the tame case of m coprime to the residue characteristic of k, since among other things, the Coxeter conjugacy class cannot be transported to a division algebra. The character relationship proved in the previous theorem reminds one of Jacquet modules but note that here we need only a Levi subgroup and its normalizer, rather than a parabolic, so offers more options for reductive groups over p-adic fields, such as $\mathrm{U}(m)^n \hookrightarrow \mathrm{U}(mn)$.

Example: We calculate the character of irreducible representations

$$\mathrm{Sym}^k(\mathbb{C}^{2n}) \quad \text{and} \quad \Lambda^k(\mathbb{C}^{2n})$$

at the elements of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$.

Since the character of $\mathrm{Sym}^k(\mathbb{C}^{2n})$ at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(x_1, \ldots, x_{2n})$ is given by the generating function,

$$\frac{1}{(1-tx_1)(1-tx_2)\cdots(1-tx_{2n})},$$

therefore the character of $\mathrm{Sym}^k(\mathbb{C}^{2n})$ at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$ is given by the generating function,

$$\frac{1}{(1-tx_1)\cdots(1-tx_n)(1+tx_1)\cdots(1+tx_n)} = \frac{1}{(1-t^2x_1^2)\cdots(1-t^2x_n^2)}.$$

Thus the character of $\mathrm{Sym}^k(\mathbb{C}^{2n})$ at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$ is nonzero only if k is even, say $k = 2\ell$, in which case the character at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$ is the character of $\mathrm{Sym}^\ell(\mathbb{C}^n)$ at the element of $\mathrm{GL}_n(\mathbb{C})$ with eigenvalues $(t_1^2, \ldots, t_n^2)$.

Similarly, the character of $\Lambda^k(\mathbb{C}^{2n})$ at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$ is nonzero only if k is even, say $k = 2\ell$, in which case the character at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_n, -t_1, \ldots, -t_n)$ is the character (up to a sign of $(-1)^\ell$) of $\Lambda^\ell(\mathbb{C}^n)$ at the element of $\mathrm{GL}_n(\mathbb{C})$ with eigenvalues $(t_1^2, \ldots, t_n^2)$.

In both the examples, $\mathrm{Sym}^k(\mathbb{C}^{2n})$ as well as $\Lambda^k(\mathbb{C}^{2n})$, the irreducible representation of the Levi subgroup which is $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ is trivial on one of the factors, which is of course in accordance with Theorem 2.

3. A reformulation

As we look for possible generalizations of Theorem 2, we rephrase this theorem using highest co-weights for the dual group $G^\vee$ instead of highest weights for G which we recall now.

For any integer $a \geq 1$, let $\underline{\rho}_a$ be the a-tuple of integers,

$$a - 1 \geq a - 2 \geq \cdots \geq 1 \geq 0.$$

Interchanging character and co-character groups of a torus T introduces a contravariant functor $T \to T^\vee$ such that,

$$T^\vee(\mathbb{C}) = \mathrm{Hom}[\mathbb{C}^\times, T^\vee] \otimes_{\mathbb{Z}} \mathbb{C}^\times = \mathrm{Hom}[T, \mathbb{C}^\times] \otimes_{\mathbb{Z}} \mathbb{C}^\times.$$

One can rephrase the theory of highest weight representations of $G(\mathbb{C})$ to say that there is a natural bijective correspondence between finite-dimensional irreducible representations of $G(\mathbb{C})$ and conjugacy classes of algebraic homomorphisms:

$$\phi_\lambda : \mathbb{C}^\times \longrightarrow G^\vee(\mathbb{C}).$$

It is customary to absorb the factor ρ, half the sum of positive roots, in λ itself in this Harish-Chandra–Langlands parametrization, so $\phi_\lambda = (\lambda \cdot \rho)^\vee$. This process eliminates some of the cumbersome ρ shifts, in particular in the Weyl character formula. These characters λ which index ϕ_λ are therefore strictly dominant, and ϕ_λ defines a natural Borel subgroup of $G^\vee$ (consisting of weights for $\mathbb{C}^\times$ which are non-negative for the adjoint action of $\mathbb{C}^\times$ on the Lie algebra of $G^\vee(\mathbb{C})$ via ϕ_λ). We will fix the maximal torus $T^\vee$ of $G^\vee$ as the centralizer of ϕ_λ, and also fix the Borel subgroup $B^\vee$ of $G^\vee$ containing $T^\vee$ as just defined.

We recall however that ρ, being half the sum of roots of T inside B, is not necessarily a character of T. To make sense of ρ we will have to restrict ourselves to G a semi-simple simply connected group in which case $\rho^\vee$ is a cocharacter $\rho^\vee : \mathbb{C}^\times \to T^\vee < G^\vee$. In our case of $\mathrm{GL}_a(\mathbb{C})$, where ρ does not make sense, we redefine ρ to be ρ_a as above.

Here is the reformulation of Theorem 2 in this language.

Theorem 3: *Let (t, c_n) be an element in $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ with c_n the Coxeter conjugacy class in $GL_n(\mathbb{C})$, and consider the image $t \cdot c_n$ of (t, c_n) inside $GL_{mn}(\mathbb{C})$ under the natural map $GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \to GL_{mn}(\mathbb{C})$. Then the character of a finite-dimensional irreducible representation π_λ of $GL_{mn}(\mathbb{C})$ corresponding to the cocharacter $\phi_\lambda : \mathbb{C}^\times \longrightarrow GL_{mn}(\mathbb{C})$ takes nonzero value at some element of the form $t \cdot c_n$ if and only if $\phi_\lambda(e^{2\pi i/n})$ belongs to the conjugacy class in $GL_{mn}(\mathbb{C})$ defined by $1 \cdot c_n$. Assume after conjugation that $\phi_\lambda(e^{2\pi i/n}) = 1 \cdot c_n$. If this is the case, then the image $\phi_\lambda(\mathbb{C}^\times)$ of ϕ_λ under $\phi_\lambda : \mathbb{C}^\times \longrightarrow GL_{mn}(\mathbb{C})$ commutes with $\phi_\lambda(e^{2\pi i/n}) = 1 \cdot c_n$, and therefore belongs to $GL_m(\mathbb{C})^n \hookrightarrow GL_{mn}(\mathbb{C})$ which is the commutant of $1 \cdot c_n$ in $GL_{mn}(\mathbb{C})$. Let $\phi_{0,n} = \rho_n^\vee : \mathbb{C}^\times \to GL_n(\mathbb{C})$ be the cocharacter corresponding to the trivial representation of $GL_n(\mathbb{C})$, so that $\rho_n^\vee(e^{2\pi i/n}) = c_n$. Thus, $\phi_\lambda {\rho_n^\vee}^{-1} : \mathbb{C}^\times \longrightarrow GL_m(\mathbb{C})^n \hookrightarrow GL_{mn}(\mathbb{C})$ is trivial on $e^{2\pi i/n}$, and therefore there is a cocharacter ϕ_μ with*

$$\phi_\lambda {\rho_n^\vee}^{-1} = \phi_\mu^n : \mathbb{C}^\times \longrightarrow GL_m(\mathbb{C})^n.$$

The coweight $\phi_\mu : \mathbb{C}^\times \longrightarrow GL_m(\mathbb{C})^n$ (which is regular inside $GL_m(\mathbb{C})^n$ although not necessarily in $GL_{mn}(\mathbb{C})$) defines a finite-dimensional irreducible representation π_μ of $GL_m(\mathbb{C})^n$ such that the character of π_λ at $t \cdot c_n$ is the same as the character of π_μ at the element t^n inside $\Delta GL_m(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})^n$.

4. Generalization

The author has considered several possible options for generalizing the theorem of this paper to other groups. First let us recapitulate the context and what we would like to see happen. Suppose G is a connected reductive group over $\mathbb{C}$ with M a reductive subgroup such that for the connected component M_0 of M, M/M_0 is a finite cyclic group. Write $M = M_0 \cdot \langle \theta \rangle$. Note that we have a split exact sequence,

$$1 \to \mathrm{Int}(M_0) \to \mathrm{Aut}(M_0) \to \mathrm{Out}(M_0) \to 1,$$

where $\mathrm{Aut}(M_0)$ is the group of automorphisms of M_0, and $\mathrm{Int}(M_0)$ denotes the subgroup of $\mathrm{Aut}(M_0)$ which consists of inner automorphisms from elements of M_0. This short exact sequence of groups is a split exact sequence, with splittings $\mathrm{Out}(M_0) \longrightarrow \mathrm{Aut}(M_0)$ given by fixing a pinning on M_0 (which is unique up to conjugacy by M_0 since we are over an algeraically closed field). This allows us to assume that $M = M_0 \cdot \langle\theta\rangle$ such that the automorphism induced by the action of θ on M_0 preserves a pinning on M_0; the element θ of M is unique up to conjugacy by M_0. It seems unnecessary to assume that $M = M_0 \cdot \langle\theta\rangle$ is actually a semi-direct product, as it is indeed not the case even for M the normalizer of the split torus in $\mathrm{SL}_2(\mathbb{C})$. To summarize, the group $M = M_0 \cdot \langle\theta\rangle$ has the following structure: if the action of θ on M_0 has order d as an element of $\mathrm{Out}(M_0)$, then $\theta^d = z$, for some element z in the center of M_0.

Observe that the conjugacy class of the element $m \cdot \theta$ in M is the same as the θ-conjugacy class of m in M_0 (the θ-conjugacy class of an element m in M_0 consists of $n \cdot m \cdot \theta(n^{-1})$ for $n \in M_0$).

Assume that the θ-conjugacy classes in M_0 are the same as conjugacy classes in a reductive group M_θ (this is indeed the case, cf. [KLP] for the definition of M_θ), and that there is a natural map from conjugacy classes in M_θ to conjugacy classes in M_0 (corresponding to the mapping $T_\theta = T/(1-\theta)T \longrightarrow T$ given by $t \to t\cdot\theta(t)\cdots\theta^{k-1}(t)$ where k is the order of θ in $\mathrm{Aut}(M_0)$ and T is a θ-invariant maximal torus in M_0 with T^θ of the largest possible dimension). Thus there is a map from θ-conjugacy classes in M_0 to conjugacy classes in M_0. Call this map a **Norm** map Nm from θ-conjugacy classes in M_0 to conjugacy classes in M_0. The aim then is to express the character of an irreducible representation of G at an element of M of the form $m \cdot \theta$ as the character of an irreducible representation of M_0 at $\mathrm{Nm}(m)$. In the case of $\mathrm{GL}_{mn}(\mathbb{C})$, from an irreducible representation of $\mathrm{GL}_{mn}(\mathbb{C})$ which is given by an unordered mn-tuple of characters of $\mathbb{C}^\times$, we need to construct an irreducible representation of $M_0 = \mathrm{GL}_m(\mathbb{C})^n$ (well defined only up to permutation of the factors in $\mathrm{GL}_m(\mathbb{C})^n$, and multiplication by one-dimensional characters on each factor with product trivial) which amounts to having unordered n-tuples of unordered m-tuples. This construction which we would like to emphasize is not obvious and needs to be done in some generality, and arises it seems by looking at centralizers of certain elements in the dual group of G, as in Theorem 3.

For character relations as desired in the previous paragraph to happen, it seems necessary that

(1) M is not contained in a proper connected subgroup of G,
(2) M_0 has no non-trivial normal semi-simple subgroup on which θ operates trivially. (We have already assumed that θ preserves a pinning of M_0.)

Given the theorem for $\mathrm{GL}_n(\mathbb{C})$, the most obvious next case to consider will be Levi subgroups of parabolics in reductive groups for which there are no other associate parabolics, so that the normalizer of the Levi subgroup in the ambient reductive group divided by the Levi subgroup is a Weyl group acting irreducibly on the center of the Levi subgroup modulo center of the group.

Another set of examples is provided by the centralizer of (suitable) powers of a Coxeter element in G.

Either of these two classes of examples (which coincide for $\mathrm{GL}_n(\mathbb{C})$), with the further requirement that the conditions (1), (2) are satisfied for a subgroup M of G containing these as the identity component, seem like good candidates for generalizing Theorem 2. For example, in the case of E_8, the Coxeter element is of order 30, and the various powers of it give rise to elements of order $1, 2, 3, 5, 6, 10, 15$ and 30. Since a Coxeter element c and its powers of the same order are conjugate, it follows that if the order of c^d is m, then for the cyclic group $C = \langle c^d \rangle = \mathbb{Z}/m$, the normalizer upon centralizer $N_{E_8}(C)/Z_{E_8}(C)$ contains $(\mathbb{Z}/m)^\times$. The author thanks Dick Gross for the following information on the centralizer of powers of a Coxeter element in $E_8(\mathbb{C})$ being written out just to illustrate the complexity of the examples involved; in cases (2), (3) below, we certainly have an M as desired above.

(1) For the element c^{15} of order 2,
$$Z(c^{15}) = \mathrm{Spin}(16)/\mu_2;$$
this has no outer automorphism since μ_2 involved is not left invariant under the outer automorphism of Spin(16).
(2) For the element c^{10} of order 3,
$$Z(c^{10}) = \mathrm{SL}_9(\mathbb{C})/\mu_3;$$
this group has an outer automorphism of order 2 realized in E_8 constructed earlier.

(3) For the element c^6 of order 5,

$$Z(c^6) = [\mathrm{SL}_5(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C})]/\mu_5,$$

with μ_5 sitting as (x, x^{-1}) in the diagonal center; the automorphism group $(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$ of $\mathrm{SL}_5(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C})$ preserving this μ_5 is $\mathbb{Z}/4$, and therefore, $N_{E_8}(\langle c^6 \rangle)/Z_{E_8}(\langle c^6 \rangle) = \mathbb{Z}/4$.

(4) For the elements c^2, c^3, c^5 of orders $15, 10, 6$, the semi-simple part of the centralizers are (up to isogeny) $\mathrm{SL}_2^4(\mathbb{C})$, $\mathrm{SL}_2^2(\mathbb{C}) \times \mathrm{SL}_3^2(\mathbb{C})$, and $\mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C})$ respectively.

We end with the following very specific situation.

Let $G = \mathrm{Sp}_{2mn}(\mathbb{C})$ containing $M_0 = \mathrm{GL}_m(\mathbb{C})^n$ as a Levi subgroup, sitting as the block diagonal matrices

$$\begin{pmatrix} g_1 &&&&&&& \\ & g_2 &&&&&& \\ && \ddots &&&&& \\ &&& g_n &&&& \\ &&&& g_1^* &&& \\ &&&&& g_2^* && \\ &&&&&& \ddots & \\ &&&&&&& g_n^* \end{pmatrix}$$

with $g^* = j^t g^{-1} j^{-1}$ where j is the anti-diagonal matrix with alternating 1 and -1. Define θ to be the cyclic permutation

$$\begin{pmatrix} g_1 &&&&&&& \\ & g_2 &&&&&& \\ && \ddots &&&&& \\ &&& g_n &&&& \\ &&&& g_1^* &&& \\ &&&&& g_2^* && \\ &&&&&& \ddots & \\ &&&&&&& g_n^* \end{pmatrix} \longrightarrow \begin{pmatrix} g_n^* &&&&&&& \\ & g_1 &&&&&& \\ && \ddots &&&&& \\ &&& g_{n-1} &&&& \\ &&&& g_n &&& \\ &&&&& g_1^* && \\ &&&&&& \ddots & \\ &&&&&&& g_{n-1}^* \end{pmatrix},$$

which corresponds to

$$\theta : (g_1, g_2, \ldots, g_{n-1}, g_n) \longrightarrow (g_n^*, g_1, g_2, \ldots, g_{n-1}),$$

and $M = M_0 \cdot \langle\theta\rangle$, but even this example we have not worked out. There are of course very similar examples to consider for $\mathrm{SO}_{2mn+1}(\mathbb{C})$ as well as $O_{2mn}(\mathbb{C})$.

Example: One of the simplest case of the previous example would be

$$\mathrm{GL}_n(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{C}),$$

sitting as a Levi subgroup of a Siegel parabolic. In this case, $\mathrm{GL}_n(\mathbb{C})$ has a nontrivial normalizer $N\mathrm{GL}_n(\mathbb{C})$ inside $\mathrm{Sp}_{2n}(\mathbb{C})$ with the exact sequence

$$1 \to \mathrm{GL}_n(\mathbb{C}) \to N\mathrm{GL}_n(\mathbb{C}) \to \mathbb{Z}/2 \to 1.$$

We calculate the character of $\mathrm{Sym}^k(\mathbb{C}^{2n})$—which is known to be an irreducible representation of $\mathrm{Sp}_{2n}(\mathbb{C})$—at the elements of $\mathrm{Sp}_{2n}(\mathbb{C})$ which belong to $N\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}) \cdot \langle\theta\rangle$ but not to $\mathrm{GL}_n(\mathbb{C})$, where θ is an element of $\mathrm{Sp}_{2n}(\mathbb{C})$ which operates on $\mathrm{GL}_n(\mathbb{C})$ by the outer automorphism $g \to J^t g^{-1} J^{-1}$ where J is the anti-diagonal matrix with entries which are alternatingly 1 and -1.

It can be seen that for $n = 2m$, any element of $\mathrm{GL}_n(\mathbb{C}) \cdot \theta$ can be represented as $t \cdot \theta$ where $t = (t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}) \cdot \theta$, and such elements are conjugate inside $\mathrm{GL}_{4m}(\mathbb{C})$ to $(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}, -t_1, \ldots, -t_m, -t_1^{-1}, \ldots, -t_m^{-1})$.

Since the character of $\mathrm{Sym}^k(\mathbb{C}^{2n})$ at the element of $\mathrm{GL}_{2n}(\mathbb{C})$ with eigenvalues $(x_1, \ldots, x_{2n})$ is given by the generating function

$$\frac{1}{(1 - tx_1)(1 - tx_2) \cdots (1 - tx_{2n})},$$

therefore the character of $\mathrm{Sym}^k(\mathbb{C}^{4m})$ at the element of $\mathrm{GL}_{4m}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}, -t_1, \ldots, -t_m, -t_1^{-1}, \ldots, -t_m^{-1})$ is given by the generating function

$$\frac{1}{(1 - t^2t_1^2) \cdots (1 - t^2t_m^2)(1 - t^2t_1^{-2}) \cdots (1 - t^2t_m^{-2})}.$$

Thus the character of $\mathrm{Sym}^k(\mathbb{C}^{4m})$ at the element of $\mathrm{GL}_{4m}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}, -t_1, \ldots, -t_m, -t_1^{-1}, \ldots, -t_m^{-1})$ is nonzero only if k is even, say $k = 2\ell$, in which case the character at the element of $\mathrm{GL}_{4m}(\mathbb{C})$ with eigenvalues $(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}, -t_1, \ldots, -t_m, -t_1^{-1}, \ldots, -t_m^{-1})$ is the character of $\mathrm{Sym}^\ell(\mathbb{C}^{2m})$ at the element of $\mathrm{GL}_{2m}(\mathbb{C})$ with eigenvalues $(t_1^2, \ldots, t_m^2, t_1^{-2}, \ldots, t_m^{-2})$.

Acknowledgement. The present work was conceived during the summer of 2008 while the author was enjoying the hospitality of the University of California at San Diego hosted by Wee Teck Gan, and partially supported by the Clay Math Institute. The author thanks Prof. Wee Teck Gan for his hospitality, and UCSD as well as CMI for their support.

References

[KO] B. Kostant, *On Macdonald's η-function formula, the Laplacian and generalized exponents*, Advances in Mathematics **20** (1976), 179–212.

[KLP] S. Kumar, G. Lusztig and D. Prasad, *Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups*, in *Representation Theory*, Contemporary Mathematics, Vol. 478, American Mathematical Society, Providence, RI, 2009, pp. 99–101.

[La] R. Langlands, *Base Change for GL(2)*, Annals of Mathematics Studies, Vol. 96, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1980.

[P] D. Prasad, *Half the sum of positive roots, the Coxeter element, and a theorem of Kostant*, Forum Mathematicum, to appear.