Let $F$ be $\mathbb{R}$ or a non-archimedean local field of odd residual characteristic, $\overline{F}$ a separable algebraic closure, $\text{Gal}(\overline{F}/F)$ (resp. $W_F$) the absolute Galois group (resp. Weil group) of $F$, and $\sigma : \text{Gal}(\overline{F}/F) \to \text{GL}(V)$ a continuous representation on a complex vector space $V$ of dimension $n$. Denote by $L(\sigma,s)$ and $\varepsilon(\sigma,s)$ respectively the associated $L$–function and $\varepsilon$–factor ([T]); they are also defined for virtual representations (and for representations of $W_F$). The root number of $(\sigma,V)$ is defined to be

$$W(\sigma) = \varepsilon(\sigma,1/2).$$

It is independent of all choices if $V$ has dimension zero and determinant 1, and satisfies the identity $W(\sigma) W(\sigma^\vee) = 1$, where $(\sigma^\vee,V^\vee)$ signifies the dual representation of $(\sigma,V)$. In particular we have

$$\sigma \text{ self–dual } \implies W(\sigma) = \pm 1.$$ 

Determination of the sign of $W(\sigma)$ is a basic problem. When $\sigma$ is given by the restriction of a global representation $\rho$, $W(\sigma)$ is a factor of the global root number $W(\rho)$, whose sign gives information on the vanishing of $L(\rho,s)$ at the critical center $s = 1/2$.

Assume from now on that $(\sigma,V)$ is self-dual. Then there exists a non-degenerate bilinear form $B$ on $V$ which is invariant under $\sigma(\text{Gal}(\overline{F}/F))$. One says that $(\sigma,V)$ is orthogonal (resp. symplectic) if $B$ is symmetric (resp. alternating); exactly one of these possibilities occurs when $V$ is irreducible.

Suppose $(\sigma,V)$ is a virtual sum of orthogonal representations. Then one has the associated Stiefel-Whitney classes $w_i(\sigma)$ in $H^i(F,\mathbb{Z}/2)$. Let $\tilde{w}_i(\sigma)$ denote 1 (resp. $-1$) if $w_i(\sigma)$ is trivial (resp. non-trivial). If $\sigma$ is a genuine (orthogonal) representation with determinant 1, then $w_2(\sigma)$ is simply the class in $H^2(F,\mathbb{Z}/2)$ of the extension of $\text{Gal}(\overline{F}/F)$ by $\{\pm 1\}$ obtained by pulling back via $\sigma$ the extension of $\text{SO}(V)$ defined by its double cover, namely the spin group of $V$; in this case, $\tilde{w}_2(\sigma) = 1$ iff $\sigma$ lifts to a representation of $\text{Gal}(\overline{F}/F)$ into $\text{Spin}(V)$. One has the following

**Theorem.** (Deligne [D1]) Let $(\sigma,V)$ be orthogonal of determinant 1 and dimension 0 (in the Grothendieck group). Then

$$W(\sigma) = \tilde{w}_2(\sigma).$$

Since $\text{Sp}(n,\mathbb{C})$ is simply connected, this raises the question of how one could understand symplectic representations. Our idea is to use the local Langlands correspondence to attach suitable orthogonal representations of certain compact
groups and study their Stiefel-Whitney numbers. In order to state our result (for \( n = 2 \)), we denote by \( D \) the unique quaternion division algebra over \( F \) and recall (cf. [Ku] + [JL]) that every irreducible two-dimensional representation \((\sigma,V)\) of \( W_F \) corresponds to a finite dimensional (irreducible) \( \mathbb{C} \)-representation \((\pi,X)\) of \( D^{\ast} \), such that the root numbers (and conductors) of \( \sigma \) and \( \pi \) coincide. Moreover, the central character \( \omega_\pi \) of \( \pi \) identifies with the character of \( F^{\ast} \) attached to the determinant of \( \sigma \) by class field theory. It may be seen that \( \pi \) is self-dual whenever \( \sigma \) is. We first establish the following key

**Proposition A.** For every irreducible two-dimensional symplectic representation \((\sigma,V)\), the associated representation \((\pi,X)\) of \( D^{\ast} \) is orthogonal.

Note that \( \det(\sigma) \) is trivial when \( \sigma \) is symplectic, and so \( \pi \) factors through a representation of the compact group \( D^*/F^* \). Given any virtual (orthogonal) representation \((\rho,Y)\) of a closed subgroup \( G \) of \( D^*/F^* \), one can associate \( w_2(\rho) \in H^2(G,\mathbb{Z}/2) \) and \( \bar{w}_2(\rho) \in \{\pm 1\} \) as above. When \( F \) is non-archimedean of residue field \( F_q \), \( q = 2m+1 \), denote by \( f(\pi) \) the (exponent of the) conductor of \( \pi \), and define \( s(\pi) \) to be \( mf(\pi)/2 \) (resp. \( 0 \)) when \( f(\pi) \) is even (resp. odd).

**Theorem B.** Let \( F \) be a non-archimedean local field of odd residual characteristic. Let \((\sigma,V),(\sigma',V')\) be irreducible, continuous two-dimensional symplectic representations of \( W_F \), and let \((\pi,X),(\pi',X')\) be the associated representations of \( D^*/F^* \). Assume that \( \det(\pi) = \det(\pi') \) and that \( s(\pi) \equiv s(\pi') \pmod{2} \). Then

\[
W(\sigma \oplus \sigma') = \bar{w}_2(\pi \oplus \pi').
\]

Note that this gives in particular an interpretation of the way the root number of \( \sigma \) changes when twisted by quadratic characters. It should also be remarked that \( s(\pi) \equiv s(\pi') \) (mod 2) when \( \pi \oplus \pi' \) is, for example, of dimension 0 and determinant 1 (see §5), and also when \( \sigma \) and \( \sigma' \) are both attached to characters of ramified quadratic extensions. See Prop. 5.4 for a variant describing \( W(\pi \oplus \pi') \) assuming only that \( \det(\pi \oplus \pi') = 1 \). The archimedean case is treated in Proposition 5.5.

The global implication of our results is not yet clear.

Our method is to analyze the behavior of the irreducible representations of \( D^{\ast} \) when restricted the various toric subgroups \( T \). More explicitly we consider, for each \((\pi,X)\), the representation \( \tilde{\pi} : D^{\ast}/F^{\ast} \to SO(X \oplus \mathbb{C}) \) defined by \( g \to (\pi(g),\det(\pi(g))) \), and establish criteria (in §3 and §4) for \( \tilde{\pi}|_{T/F^{\ast}} \) to lift to \( Spin(X \oplus \mathbb{C}) \). This leads to the following

**Theorem C.** Let \( F \) be non-archimedean of residue field \( \mathbb{F}_q \), \( q \) odd, and let \( \omega \) denote the unique non-trivial quadratic character of \( \mathbb{F}_q^* \). Let \( \pi \) be an irreducible representation of \( D^*/F^* \) with values in \( O(X) \) attached to a character \( \chi \) of the multiplicative group of a quadratic extension \( K \) of \( F \). Then the associated representation \( \tilde{\pi} \) lifts to \( Spin(X \oplus \mathbb{C}) \) if and only if \( \omega(-2) = -1 \) and \( \varepsilon(\pi) = \omega(-1) \) if \( K \) is ramified and \( \varepsilon(\pi) = 2f + 1 \), and \( \omega(-1)^{f-1} = -1 \) and \( \varepsilon(\pi) = -1 \) if \( K \) is unramified and the conductor of \( \varepsilon(\pi) = 2f \).

The calculations underlying the proof yield an explicit formula (see §4) for \( \bar{w}_2(\tilde{\pi}) \) in terms of \( W(\pi) \) and other extraneous factors, which simplify when we consider \( \pi \oplus \pi' \).
In §6 we indicate a geometric approach based on the cohomology of the Drinfeld coverings of $p$-adic upper half spaces and show how to deduce Proposition A for $F = \mathbb{Q}_p$, from this point of view.

We end the introduction with the following conjecture for any non-archimedean local field $F$: For any $n \geq 1$, let $D^*$ be the multiplicative group of a division algebra $D$ over $F$ of dimension $n^2$, and let $\sigma \mapsto \pi$ be the correspondence predicted by the local Langlands conjecture [La]. Then, whenever $\sigma$ is self-dual and symplectic, $\pi$ is orthogonal.

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1. Preliminaries

Let $F$ be $\mathbb{R}$ or a non-archimedean local field of odd residual characteristic. If $K/F$ is a quadratic extension, we will denote by $\omega_{K/F}$ the quadratic character of $F^*$ given by class field theory. When $F$ is non-archimedean, let $\mathfrak{O}_F$ denote the ring of integers of $F$, $\varpi$ a fixed uniformizer, $q$ the cardinality of the residue field, and $\omega$ the unique non-trivial quadratic character of $\mathbb{F}_q^*$. In this case, the Weil group $W_F$ is the subgroup of $\text{Gal}(\overline{F}/F)$ consisting of automorphisms $\tau$ which induce an integral power of the Frobenius $\phi_q : x \mapsto x^q$ on $\mathbb{F}_q$. it is thus a non-trivial extension of $\{\phi_q^n\} \simeq \mathbb{Z}$ by the inertia group $I_F$ and has dense image in $\text{Gal}(\overline{F}/F)$. When $F = \mathbb{R}$, $W_F$ can be realized as $\mathbb{C}^* \cup j\mathbb{C}^*$, where $j$ satisfies $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for all $z \in \mathbb{C}^*$; it is the unique non-trivial extension of $\text{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ by $\mathbb{C}^*$.

Fix a non-trivial (unitary) character $\psi = \psi_F$ of the additive group of $F$ and a Haar measure $dx$ on $F^*$. We refer to the articles of Deligne ([D2]) and Tate ([T]) for the definition and the basic properties of the epsilon factors $\varepsilon(\sigma, \psi, dx, s)$ of representations $(\sigma, V)$ of $\text{Gal}(\overline{F}/F)$ (or $W_F$). We note that when $V$ has dimension zero and determinant 1, $\varepsilon(\sigma, \psi, dx, s)$ is independent of $(\psi, dx)$, and we will simply write $\varepsilon(\sigma, s)$. To avoid ambiguity in general, we will take $\psi$ to have conductor $\mathfrak{O}_F$, and normalize $dx$ to be the self-dual measure relative to $\psi$. Thus our epsilon factors are those defined by Langlands. For self-dual representations, we will set: $W(\sigma) = W(\sigma, \psi) = \varepsilon(\sigma, \psi, 1/2)$.

If $\eta$ is a representation of $\text{Gal}(\overline{F}/F)$ (or $W_F$), let $f(\eta)$ denote the exponent of the conductor of $\eta$ (resp. 0) if $F$ is non-archimedean (resp. archimedean).

The following three basic results on local constants will be used in our calculations.

**Proposition 1.1.** ([D2]) Let $F$ be non-archimedean, $\sigma$ a representation of $W_F$ of dimension $n$, and $\mu$ a (quasi-) character of $F^*$ ($\simeq W_F^{ab}$). If either $\mu$ or $\sigma$ is unramified, we have

$$W(\sigma \otimes \mu) = \det(\sigma)(\varpi^{f(\mu)\mu}(\varpi^{f(\sigma)})W(\sigma)W(\mu)^n.$$ 

**Theorem 1.2.** ([D2, Lemma 4.1.6]) Let $F$ be non-archimedean, and let $\alpha, \beta$ be two (quasi-) characters of $F^*$ such that $f(\alpha) \geq 2 f(\beta)$. Choose $y \in F$ as
follows: If \( f(\alpha) \) is positive, let \( y \) be such that \( \alpha(1 + x) = \psi(xy) \) for all \( x \in F \) with \( \text{val}(x) \geq \frac{1}{2} f(\alpha) \); if the conductor of \( \alpha \) is 0, let \( y = \omega^{-\text{cond}(\psi)} \). Then
\[
W(\alpha\beta, \psi) = \beta^{-1}(y) W(\alpha, \psi).
\]

**Theorem 1.3.** (Frohlich-Queyrut [F-Q, Theorem 3]) Let \( K \) be a separable quadratic extension of a local field \( F \), and let \( \psi_K \) be the additive character of \( K \) defined by \( \psi_K(x) = \psi(tr(x)) \). Then for any character \( \chi \) of \( K^* \) which is trivial on \( F^* \), and any \( x_0 \in K^* \) with \( \text{tr}(x_0) = 0 \),
\[
W(\chi, \psi_K) = \chi(x_0).
\]

Let \( (\sigma, V) \) be an irreducible representation of \( W_F \). Then it is easy to see it must be of dimension 1 or 2; in the latter case there exists a (quasi) character \( \chi : \mathbb{C}^* \to \mathbb{C}^* \) such that \( \chi(z) \neq \chi(\overline{z}) \) and \( \sigma \simeq \text{Ind}_{K/F}^{W_F}(\chi) \) with \( \det(\sigma) = \omega_{\mathbb{C}/\mathbb{R}}(\chi|_{\mathbb{C}^*}) \).

When \( F \) is non-archimedean of odd residual characteristic, every irreducible two-dimensional representation \( \sigma \) of \( W_F \) has a similar description. It is the pull back via \( W_F \to W_{K/F} \), for some quadratic extension \( K/F \), of an induced representation \( \text{Ind}_{K/F}^{W_F}(\chi) \), where \( \chi \) is a (quasi) character of \( K^* \) such that \( \chi \neq \chi \circ \rho, \rho \) being the non-trivial automorphism of \( K/F \).

**Lemma 1.4.** Let \( F \) be non-archimedean, and let \( K/F \) be a ramified quadratic extension. Suppose \( \chi \) is a character of \( K^* \) which restricts to \( \omega_{K/F} \) on \( F^* \). Then either \( f(\chi) = 2m \), with \( m \) a positive integer, or \( f(\chi) = 1 \) and the restriction of \( \chi \) to \( \mathcal{O}_K^* \) is the inflation \( \omega_{K/F}^0 \) of the character of \( (\mathcal{O}_K/\mathcal{P}_K)\mathcal{O}_K)^* \) defined by \( \omega_{K/F} \) via its natural identification with \( (\mathcal{O}/\mathcal{P})\mathcal{O}^* \).

To see this, let \( \omega_{K/F}^0 \) denote either of the (two possible) extensions of \( \omega_{K/F}^0 \) to \( K^* \), and consider \( \mu := \chi/\omega_{K/F}^0 \). Then \( \mu \) defines a character of \( K^*/F^*(1 + \mathcal{P}) \mathcal{O}_K \). The assertion is clear if \( f(\chi) = 1 \), and when \( f(\chi) > 1 \), it is a consequence of the fact that \( F^*(1 + \mathcal{P}) \mathcal{O}_K = F^*(1 + \mathcal{P}^{2n+1}) \mathcal{O}_K \) for all positive \( n \).

Let \( D \) be the unique quaternion division algebra over the (local field) \( F \) with reduced norm \( \text{Nrd}: D^* \to F^* \). For every irreducible representation \( \pi \) of \( D^* \), one knows (cf. [JL]) how to associate an epsilon factor \( \varepsilon(\pi, \psi, s) \). One sets \( W(\pi, \psi) = \varepsilon(\pi, \psi, 1/2) \). Denote by \( f(\pi) \) the exponent of the conductor of \( \pi \), and by \( \omega_\pi \) the central character of \( \pi \). If \( \mu \) is a character of \( F^* \), we will write \( \pi \circ \mu \) for the representation \( \pi \otimes (\mu \circ \text{Nrd}) \). If \( \pi^\vee \) denotes the contragredient (dual) of \( \pi \), then it is isomorphic to \( \pi \otimes \omega_\pi^{-1} \). When \( \pi \) is self-dual, \( W(\pi, \psi) \) is \pm 1 and \( \omega_\pi \) is quadratic. The key theorem below follows by combining the main results of [J-L] and [K].

**Theorem 1.5.** There is a bijection \( \sigma \to \pi = \pi(\sigma) \) of the set of equivalence classes of continuous, irreducible 2-dimensional \( \mathbb{C} \)-representations of \( W_F \) onto the set of equivalence classes of continuous, irreducible \( \mathbb{C} \)-representations of \( D^* \) of dimension \( > 1 \) (resp. \( \geq 1 \)) if \( F \) is non-archimedean (resp. \( F = \mathbb{R} \)). This bijection satisfies

1. \( \det(\sigma) = \omega_\pi \);
2. \( f(\sigma) = f(\pi) \);
3. \( \pi(\sigma^\vee) \simeq \pi(\sigma)^\vee \); and
4. \( \varepsilon(\sigma \otimes \mu, \psi, s) = \varepsilon(\pi \otimes \mu, \psi, s) \),
for all characters μ of F*.

Note that under this bijection, σ is self-dual iff π is. Moreover, if π is the representation of D* associated to a σ induced by a character χ of a separable quadratic extension K/F, then ωπ = χ|F∗ωK/F.

The following Proposition summarizes the information we need about the characters of irreducible representations π of D*/F*, for F non-archimedean. (There is a similar, but simpler, formula for F = R, which we will not need; our treatment of that case will be more direct.) It can be deduced by combining the explicit character formulae for irreducible admissible representations π′ of PGL(2,F) (see [Si], p.50-51) with the fact (see [J-L], Prop.15.5) that there is an injection π → π′ of the irreducible representations of D*/F* into the discrete series of PGL(2,F) preserving epsilon factors such that the characters of π and π′ agree on the elliptic tori up to sign.

**Proposition 1.6.** Let F be non-archimedean, K/F a quadratic extension and π = πχ be the representation of D*/F* attached to a character χ of K*. Then we have the following table

<table>
<thead>
<tr>
<th>K/F</th>
<th>f(χ)</th>
<th>dim(π)</th>
<th>f(π)</th>
</tr>
</thead>
<tbody>
<tr>
<td>unramified</td>
<td>f</td>
<td>2qf−1</td>
<td>2f</td>
</tr>
<tr>
<td>ramified</td>
<td>2f</td>
<td>(q+1)qf−1</td>
<td>2f+1</td>
</tr>
</tbody>
</table>

Let L be any quadratic extension of F, and x the unique element of L*/F* of order 2. Denote by Θπ the character of π. Then we have:

1. If L ≠ K, Θπ(x) = 0.
2. If L = K and K/F unramified, Θπ(x) = (−1)f+12χ(x)
3. If L = K and K/F ramified,
   \[ Θπ(x) = -2Gχω(2)ω(−1)f−1χ(x), \]
   \[ Gχ = \frac{1}{\sqrt{q}} \sum_{x \in (O_F/ϖ)^*} \chi(1 + \ϖ 2f−1K x)ω(x), \]

where

We will also need the following result on the toric restriction of representations of D*.

**Proposition 1.7.** Let F be non-archimedean, K/F a (separable) quadratic extension, σ an irreducible symplectic representation of WF of dimension 2, and π the associated (irreducible) representation of D*/F*. Then

1. The restriction of π to K* is multiplicity free;
2. For a character χ of K*/F* to occur in π|K*, it is necessary and sufficient that
   \[ W(σ|W_K ⊗ χ−1, ψ ∘ tr_{K/F}) = −1. \]

For a proof of part (1), see [P1], Remark 3.5, and for part (2) (Tunnell’s formula), see [Tu].
We conclude this section by recalling some basic facts about Stiefel-Whitney classes. For any compact group $G$, let $\mathcal{C}(G, \mathbb{R})$ denote the category of real representations of $G$. A continuous representation $\sigma$ of $G$ on a finite-dimensional vector space $V$ is real iff it is realizable over $\mathbb{R}$. It is easy to see that a self-dual $\sigma$ has a real character, and it is realizable over $\mathbb{R}$ iff it is orthogonal. Denote by $R(G, \mathbb{R})$ the Grothendieck group of virtual representations in $\mathcal{C}(G, \mathbb{R})$, and by $H^*(G, \mathbb{Z}/2)$ the $\mathbb{Z}/2$-cohomology ring $\oplus_{i \geq 0} H^i(G, \mathbb{Z}/2)$ (with $G$ acting trivially on $\mathbb{Z}/2$). Then there is a Stiefel-Whitney homomorphism of groups (see [De1])

$$w_* : R(G, \mathbb{R}) \longrightarrow H^*(G, \mathbb{Z}/2)^\times,$$

which sends $\sigma$ to $\sum_{i \geq 0} w_i(\sigma)$, with $w_0(\sigma) = 1$ and $w_1(\sigma)$ being the image of $\det$ under the isomorphism $\text{Hom}(G, \mathbb{Z}/2) \cong H^1(G, \mathbb{Z}/2)$.

As in the introduction, we let $\tilde{w}_i(\sigma)$ to be $1$ or $-1$ according as $w_i(\sigma)$ is trivial or not.

If $\sigma$ is a genuine real representation with trivial determinant, then $w_2(\sigma)$ is the class of the extension of $G$ by $\{\pm 1\}$ obtained by pulling back via $\sigma$ the short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}(V) \rightarrow \text{SO}(V) \rightarrow 1,$$

where $\text{Spin}(V)$ is the Spin group of $V$, the non-trivial double cover of the special orthogonal group $\text{SO}(V)$; thus $\tilde{w}_2(\sigma)$ is trivial (in this case) iff $\sigma$ can be lifted to a representation of $G$ into the spin group.

Finally, we define, for any orthogonal $(\sigma, V)$, a homomorphism

$$\tilde{\sigma} : G \longrightarrow \text{SO}(V \oplus \mathbb{C})$$

by $g \mapsto (\sigma(g), \det(\sigma(g)))$.

2. Orthogonality of $\pi$

In view of the remark following Proposition A, it is a consequence of

**Proposition 2.1.** Every finite dimensional representation of $D^*/F^*$ is orthogonal. 

**Proof.** If $x \mapsto \bar{x}$ denote the canonical anti-automorphism of $D^*$ such that $x \cdot \bar{x} = \text{Nrd}(x)$ where $\text{Nrd}(x)$ is the reduced norm of $x$, then as an element of $D^*/F^*$, $\bar{x} = x^{-1}$. By Skolem-Noether theorem, $x$ and $\bar{x}$ are conjugate, and therefore $x$ is conjugate to $x^{-1}$ in $D^*/F^*$. This implies that every representation of $D^*/F^*$ is self-dual.

When $F = \mathbb{R}$, $D^*/F^*$ is isomorphic to $\text{SO}(3)$, and any irreducible representation $\pi$ of this group is odd dimensional. (See Remark 2.5 below for an explicit description.) Clearly, every self-dual representation of odd dimension must be orthogonal. So we may (and we will) assume henceforth that $F$ is non-archimedean.

We can find a quadratic extension $E$ of $F$ such that the trivial character of $E^*$ appears in $X$. In fact we have
Lemma 2.2. Let \( \pi \) be associated to a character \( \chi \) on a quadratic extension \( K \) of \( F \), so that the corresponding representation \( \sigma \) of \( W_F \) is \( \text{Ind}^{W_F}_{W_K} (\chi) \). Then if \( K \) is ramified, \( \pi \) contains the trivial character of the unramified quadratic extension, while if \( K \) is unramified, then \( \pi \) contains the trivial character of any ramified quadratic extension.

Indeed, for any quadratic extension \( E/F \), one knows that by part (2) of Proposition 1.7, the trivial representation of \( E^* \) occurs iff we have

\[
(2.3) \quad W(\sigma|_{W_E}, \psi \circ \text{tr}_{E/F}) = W(\pi, \psi)W(\pi \otimes \omega_{E/F}, \psi)\omega_{E/F}(-1) = -1.
\]

When \( E = K \), \( \pi \otimes \omega_{E/F} \simeq \pi \). Since \( W(\sigma, \psi) = \pm 1 \), the trivial character of \( K^* \) occurs in \( \pi \) iff \( \omega_{K/F}(-1) = -1 \), which happens iff \( K/F \) is ramified and \( q \equiv 3 \) modulo 4. If \( K/F \) is ramified, but with \( q \equiv 1 \) modulo 4, take \( E \) to be the unramified quadratic extension. Then \( \omega_{E/F} \) is an unramified character, and by Prop.1.1 (and Theorem 1.5),

\[
(2.4) \quad W(\pi \otimes \omega_{E/F}, \psi) = \omega_{E/F}(\mathcal{f}(\pi))W(\pi)\omega_{E/F}^2 = -W(\pi).
\]

The second equality comes from the fact (cf. Proposition 1.6) that \( \mathcal{f}(\pi) \) is odd when \( K/F \) is ramified. Combining (2.4) with (2.3), we see that the trivial character of \( E^* \) occurs in \( \pi \).

Finally, let \( K \) be unramified. Take \( E \) to be either of the two ramified extensions of \( F \). Then we claim that

\[
(2.5) \quad W(\pi \otimes \omega_{E/F}) = -\omega_{E/F}(-1)W(\pi).
\]

Indeed, if \( \pi \) corresponds to the representation \( \sigma = \text{Ind}^{W_F}_{W_K} (\chi) \), then the additivity and inductivity in dimension zero of the epsilon factors gives \( W(\pi \otimes \omega_{E/F}, \psi)/W(\pi, \psi) = W(\chi \nu, \psi_K)/W(\chi, \psi_K) \), where \( \mu = \omega_{E/F} \circ N_{K/F} \) is the unique nontrivial quadratic character of \( K^*/F^* \). Let \( \nu \) be the unramified character of \( K^* \) taking the value \(-1\) on any uniformizing parameter \( \varpi_K \). Then by construction, \( \nu \) restricts to \( \omega_{K/F} \) on restriction to \( F^* \); so do the characters \( \chi \) and \( \chi \mu \) as the determinant of \( \text{Ind}^{W_F}_{W_K} (\chi) \) is trivial. On the other hand, for any character \( \beta \) of \( K^* \), we have by Proposition 1.1: \( W(\beta \nu, \psi_K) = \nu(\mathcal{f}(\beta))W(\beta, \psi_K)W(\nu, \psi_K) \), where \( \psi_K = \psi \circ \text{tr}_{K/F} \). This gives \( W(\chi \nu, \psi_K)/W(\chi, \psi_K) = W(\chi \nu, \psi_K)/W(\chi, \psi_K) \), which, by Theorem 1.3, equals \( \chi \mu(t)/\chi(t) = \mu(t) \), where \( t \) is the unique element of order 2 in \( K^*/F^* \). Using the fact that we may represent \( t \) in \( K^* \) by the square-root of a unit \( \nu \) in \( \mathcal{O}_F \) which is a non-square in the residue field \( \mathbb{F}_q \), one sees that \( \mu(t) = -1 \) (resp. 1) when \( q \) is 1 (resp. 3) modulo 4, which is the negative of the value of \( \omega_{E/F} \) at \(-1\). Hence the claim. The Lemma now follows by combining (2.3) and (2.5).

Let \( E/F \) be as in the Lemma. Then, since the restriction of \( \pi \) to \( E^* \) is multiplicity-free (see Proposition 1.7, part (1)), the eigenspace of \( X \) corresponding to the trivial character of \( E^* \) must in particular be one-dimensional. The unique non-degenerate bilinear form on \( X \) must be non-zero on this one-dimensional subspace, and therefore the bilinear form must be symmetric, whence the Proposition. \( \square \)
Remark 2.6. It should be noted that self-dual representations $\pi$ of $D^*$ not factoring through $D^*/F^*$ need not be orthogonal. To see this let $F = \mathbb{R}$ and $\pi = \rho \otimes \det(\rho)^{-1/2}$, where $\rho$ is the standard two-dimensional $\mathbb{C}$-representation of $D^*$. Then the symmetric square of $\pi$ is irreducible.

The proof of Proposition 2.1 shows more generally that a self-dual irreducible representation $(\eta,Y)$ of a group $G$ must be orthogonal if we can find a subgroup $H$ such that the restriction to $H$ is completely reducible and contains the trivial representation of $H$ with multiplicity one. One gets the following

**Proposition 2.7.** Every irreducible, admissible, self-dual, generic representation $(\eta,Y)$ of $GL(n,F)$, $F$ non-archimedean, is orthogonal for any $n \geq 1$.

Indeed, the theory of new vectors for generic representations of $GL(n,F)$ (cf. [J-PS-S]) gives the existence of an open compact subgroup $C$ such that the space of $C$-invariant vectors in $Y$ is one-dimensional. The restriction of $\eta$ to $C$ is completely reducible by admissibility.

Note that since every discrete series representation is generic, this Proposition applies in particular to any representation of $GL(2,F)$ associated to an irreducible representation $(\pi,X)$ of $D^*$ by the Jacquet-Langlands correspondence.

3. Criteria for Liftability

Let $\Sigma(F)$ denote the set of quadratic extensions of $F$ in $\overline{F}$. The object of this section is to show that for any virtual sum $\sigma$ of orthogonal representations of $D^*/F^*$, the second Stiefel-Whitney number $\tilde{w}_2(\sigma)$ is 1 iff it is so when restricted to $K^*/F^*$, for every $K \in \Sigma(F)$. In fact we have

**Proposition 3.1.** The natural homomorphism (given by restriction)

$$H^2(D^*/F^*,\mathbb{Z}/2) \longrightarrow \bigoplus_{K \in \Sigma(F)} H^2(K^*/F^*,\mathbb{Z}/2)$$

is injective.

**Proof.** First consider the case $F = \mathbb{R}$. One has a natural isomorphism $H^2(D^*/\mathbb{R}^*,\mathbb{Z}/2) \simeq \text{Hom}(\pi_1(D^*/\mathbb{R}^*),\mathbb{Z}/2)$. Hence it suffices to establish the surjectivity at the fundamental group level, i.e., $\pi_1(C^*/\mathbb{R}^*) \longrightarrow \pi_1(D^*/\mathbb{R}^*)$. This is clear via the identifications of $C^*/\mathbb{R}^*$ and $D^*/\mathbb{R}^*$ with $SO(2)$ and $SO(3)$ respectively.

Now let $F$ be non-archimedean, and denote by $U^1_D$ the image in $D^*/F^*$ of the first congruence subgroup of $D^*$ under the standard filtration. Then since the residue characteristic of $F$ is odd, $H^i(U_D^1,\mathbb{Z}/2) = 0$ if $i > 0$. It follows that $H^2(D^*/F^*,\mathbb{Z}/2) = H^2(D^*/F^*U_D^1,\mathbb{Z}/2)$. Now $D^*/F^*U_D^1$ is a dihedral group defined by the extension

$$0 \rightarrow \mathbb{F}_q^* / \mathbb{F}_q^* \rightarrow D^*/F^*U_D^1 \rightarrow \mathbb{Z}/2 \rightarrow 0,$$

where $\mathbb{F}_q$ is the residue field of $F$.

Let $D_r$ be the quotient of $D^*/F^*U_D^1$ by its maximal subgroup of odd order. Then $H^2(D^*/F^*,\mathbb{Z}/2)$ is the same as $H^2(D_r,\mathbb{Z}/2)$, and $D_r$ is a dihedral 2-group given by

$$0 \rightarrow \mathbb{Z}/2^r \rightarrow D_r \rightarrow \mathbb{Z}/2 \rightarrow 0.$$
Clearly $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subseteq D_r$, and one sees that $H^2(D_r, \mathbb{Z}/2) \cong H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2)$, and an element of $H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2)$ is zero if and only if its restriction to all the three $\mathbb{Z}/2$’s in $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is zero. These three $\mathbb{Z}/2$’s come from the three distinct quadratic extensions, whence the proposition.

**Lemma 3.2.** Let $SO(2n + 1, \mathbb{C})$ correspond to the quadratic form $q = x_1x_2 + \cdots + x_{2n-1}x_{2n} + x_{2n+1}^2$, and $T$ the associated maximal torus. For characters $(\chi_1, \ldots, \chi_n)$ of an abelian group $G$, let $\phi$ be the representation of $G$ with values in $SO(2n + 1, \mathbb{C})$ given by $x \mapsto (\chi_1(x), \chi_1^{-1}(x), \chi_2(x), \chi_2^{-1}(x), \ldots, \chi_n(x), \chi_n^{-1}(x), 1)$. Then the representation $\phi$ of $G$ lifts to $\text{Spin}(2n + 1, \mathbb{C})$ if and only if $\prod_{i=1}^n \chi_i = \mu^2$

for some character $\mu$ of $G$, i.e. if and only if $\prod_{i=1}^n \chi_i$ is trivial on the subgroup $G[2] = \{g \in G \mid 2g = 1\}$.

**Proof.** The assertion is a direct consequence of the fact that the spin covering of $SO(2n + 1, \mathbb{C})$ when restricted to the maximal torus $T = \{(z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}, 1) \mid z_i \in \mathbb{C}^*\}$ is the two-fold cover of $T$ obtained by attaching $\sqrt{\prod z_i}$.

4. Formulae for toric restrictions

In this section $F$ will always be non-archimedean, and $(\pi, X)$ an irreducible representation of $D^*$, trivial on $F^*$, of dimension $> 1$ and character $\Theta_{\pi}$.

Let $E = F(x)$ be a separable quadratic extension with $x^2 \in F^*$. Clearly $x = x(E)$ is the unique element of $E^*/F^*$ of order 2.

By Skolem-Noether theorem, there is an element $g$ in $D^*/F^*$ such that the inner conjugation action of $g$ on $K^*$ preserves $K^*$ and induces the non-trivial Galois action on it. It follows that whenever a character $\mu$ of $K^*$ appears in $\pi$, so does $\mu^{-1}$. The (multiplicity-free) restriction of $\pi$ to $E^*$ may then be decomposed as

$$X = \sum_{\mu \in S} \mu \oplus \sum_{\mu \in S} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \quad (i)$$

where $a = a(E)$ and $b = b(E)$ are integers $0 \leq a, b \leq 1$, $\nu = \nu(E)$ is the unique character of $E^*/F^*$ of order 2, and $S = S(E)$ a finite set of characters of $E^*/F^*$ of order $\geq 3$. Since the dimension of $X$ is even (cf. Prop. 1.6), $a(E) = b(E)$. Recall that $\nu(x) = -1$ except when $E$ is a quadratic unramified extension of $F$ with $q \equiv 3 \pmod{4}$, in which case $\nu(x) = 1$.

**Lemma 4.1.** Let $s = s(E)$ denote the number of characters $\mu$ in $S(E)$ which take the value $-1$ on $x$. Define $\delta_q = \delta_q(E)$ to be 1 if $E$ is unramified with $q$ congruent to 3 modulo 4, and to be 0 otherwise. Then

$$\dim(X) = 4s + 2a(1 - 2\delta_q) + \Theta_{\pi}(x). \quad (ii)$$

Moreover, $a(E)$ is $> 0$ in exactly the following cases:

1. $E$ is unramified and $\pi$ is not associated to $E$;
2. $E$ is ramified and $\pi$ is unramified;
3. $E$ is ramified, $\pi$ is associated to $E$, and $q$ is congruent to 3 modulo 4; and
4. $E$ is ramified, $\pi$ is ramified, but not associated to $E$, and $q \equiv 1 \pmod{4}$. 


Proof. Let \( r = r(E) \) denote the number of characters \( \mu \) in \( S(E) \) which take the value 1 at \( x \). (Since \( x \) has order 2 in \( E^*/F^* \), every character of \( E^* \) trivial on \( F^* \) is \( \pm 1 \) at \( x \).) Evaluating the trace of \( \pi|_E/F^* \) at 1 and \( x \) respectively, we get
\[
\dim(X) = 2(r + s) + 2a
\]
and
\[
\Theta_s(x) = 2(r - s) + 2a\delta_q,
\]
whence the assertion on the dimension of \( X \).

Since \( a = a(E) \) is the multiplicity of the trivial representation in the restriction of \( \pi \) to \( E^* \), the occurrence of cases (1), (2) and (3) is contained in Lemma 2.2. Suppose we are in case (4) with \( \pi \) and \( \omega \) associated to a quadratic extension \( L \), say. Then \( \omega_{E/F}(1) = 1 \), and we have to show that \( W(\pi) = -W(\pi \otimes \omega_{E/F}) \).

Noting that the pull back of \( \omega_{E/F} \) to \( L^* \) is the unique character \( \nu = \nu(L) \) of order 2, which is unramified, and that \( f(\chi) \) is even (see Prop. 1.6), we get
\[
W(\pi \otimes \omega_{E/F}, \psi)/W(\pi, \psi) = W(\text{Ind}_{\pi}^{W(L)}(\chi \nu \otimes \chi), \psi_L) = W(\chi \nu, \psi_L)/W(\chi, \psi_L) = W(\nu, \psi_L) = \nu(x(L)), \quad \text{which is indeed } -1.
\]
It is also now evident that there are no further cases when the trivial representation occurs. \( \square \)

Proposition 4.2. Let \( \pi \) be an irreducible representation of \( D^* \) with values in \( O(X) \) associated to a quadratic extension \( K \) of \( F \), and let \( L \) be a quadratic extension of \( F \) different from \( K \). Let \( f \) denote \( f(\pi) \). Then the restriction to \( L^*/F^* \) of the associated representation \( \pi \) with values in \( SO(X \otimes \mathbb{C}) \) lifts to \( \text{Spin}(X \otimes \mathbb{C}) \) if and only if \( \omega(-2) = -1 \) if \( K \) is a ramified extension, and \( \omega(1)^{-1} = -1 \) if \( K \) is the unramified extension.

Proof. By Lemma 3.2, the restriction of \( \pi \) to \( L^*/F^* \) lifts to \( \text{Spin}(X \otimes \mathbb{C}) \) if and only if
\[
\left( \nu^a \prod_{\mu \in X} \mu \right)(x) = 1 \quad \text{(iii)}
\]
As \( x \) has order 2 in \( L^*/F^* \), all the characters of \( L^*/F^* \) take the value \( \pm 1 \) on \( x \). Let \( r \) be the number of characters \( \mu \) from \( X \) such that \( \mu(x) = 1 \), and let \( s \) be the number of characters \( \mu \) from \( X \) such that \( \mu(x) = -1 \). By Proposition 1.4, the character of \( \pi \) at \( x \) is zero. Therefore we get (by Lemma 4.1)
\[
\dim(X) = 4s + 2(a - \delta_q) \quad \text{(iv)}
\]
We also have
\[
\tilde{w}_2(\widetilde{\pi}|_{L^*/F^*}) = (-1)^a \nu(x)^a \quad \text{(v)}
\]

Proposition 4.3. We have the following table for \( L \neq K \) and \( q = 2m + 1 \):

| \( K/F \) | \( \tilde{w}_2(\widetilde{\pi}|_{L^*/F^*}) \) |
|----------|----------------|
| unramified | \((-1)^{1+m(f-1)}\) |
| ramified   | \((-1)^{[1+\frac{m}{2}]}\) |
where \([t]\) denotes, for any \(t \in \mathbb{R}\), the integral part of \(t\).

**Proof.** By Proposition 1.6, the dimension of \(\pi\) is \(2(q-1)\) (resp. \((q+1)(q-1)\)) when \(K\) is unramified (resp. ramified), where \(f\) (resp. \(2f\)) denotes \(f(\chi)\). Let us first consider the unramified case. Then \(\delta_q\) is zero as \(L\) is ramified, and by equation (iv) above, \(4a + 2 = 2q^{-1}\). Since \(a \in \{0, 1\}\), we must have \(a = 1\). (This also follows from Lemma 4.1.) Moreover,

\[
s = \frac{1}{2}((2m+1)/(q-1)-1) = \frac{1}{2} \sum_{j=1}^{f-1} (f-1)!(2m-j)! \equiv m(f-1) \mod 2
\]

which yields the assertion, thanks to equation (v).

Now let \(K/F\) be ramified. Then we know by Lemma 4.1 that \(a = 1\) if either \(L\) is ramified or if \(L\) is ramified with \(q = 2m + 1\) congruent to \(1\) modulo \(4\); it is \(0\) otherwise. Thus when \(m\) is even (resp. odd), \(a(1 - \delta_q) = 1\) (resp. \(0\)), and by equation (iv), \(s\) is congruent modulo \(2\) to \(\frac{m}{2}\) (resp. \(\frac{m+1}{2}\)). Furthermore, since \(a = 0\) when \(m\) is odd, \(\nu(x)^{m+1} = 1\) in that case. Applying (v) we see that \(\hat{\omega}(\pi_{L,F}^{(x)})\) equals \((-1)^{\frac{m+1}{2}}\) (resp. \((-1)^{\frac{m}{2}}\)) when \(m\) is even (resp. odd). This finishes the proof of Proposition 4.3.

Proposition 4.2 follows directly from this once we note that \(\omega(1) = (-1)^{m}\) and that \(\omega(-2) = 1\) iff \(q = 2m + 1\) is congruent modulo \(8\) to \(5\) or \(7\).

We next consider the lifting problem for a representation \(\pi\) of \(D^*/F^*\) associated to a quadratic field \(K\) when restricted back to \(K^*/F^*\). In this case the obstruction to lifting is related to the epsilon factor of \(\pi\).

**Proposition 4.4.** Let \(\pi\) be an irreducible representation of \(D^*/F^*\) with values in \(O(X)\) associated to a character \(\chi\) of \(K^*\), where \(K\) is a (separable) quadratic extension of \(F\). Then the restriction to \(K^*/F^*\) of the associated representation \(\vec{\pi}\) with values in \(SO(X \oplus \mathbb{C})\) lifts to \(Spin(X \oplus \mathbb{C})\) if and only if \(\nu(\pi) = \omega(2)\) if \(K\) is ramified and the conductor of the representation \(\pi\) is \(2f + 1\), and \(\nu(\pi) = \omega(1)^{-1}\) if \(K\) is unramified and the conductor of \(\pi\) is \(2f\).

**Proof.** The reasoning here is very similar to that above, and the assertion is an immediate consequence of the following

**Proposition 4.5.** Write \(q = 2m + 1\). Then we have the following table:

<table>
<thead>
<tr>
<th>(K/F)</th>
<th>(\hat{\omega}(\pi_{K^<em>/F^</em>}^{(x)}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>unramified</td>
<td>((-1)^{m(f-1)}(1-W(\pi)))</td>
</tr>
<tr>
<td>ramified</td>
<td>((-1)^{(m+1)/2}(1+W(\pi)))</td>
</tr>
</tbody>
</table>

**Proof.** Again, let \(x\) denote the unique element of order \(2\) in \(K^*/F^*\). When \(K\) is ramified, \(x\) is the image of a uniformizing parameter \(\varpi_K\) such that \(\varpi = \varpi_K^2 \in F^*\). Recall that by Proposition 1.4, we have

\[
\Theta_\varpi(x) = -2G_\chi \omega(2) \omega(1)^{-1} \chi(x),
\]

when \(K/F\) is ramified, and

\[
\Theta_\varpi(x) = (-1)^{-1} 2\chi(x),
\]
when $K/F$ is unramified. Substituting the formula for $\dim(X)$, we get by Lemma 4.1, 
\[ 4s + 2a = (q + 1)q^{f-1} + 2G\chi_\omega(2)\omega(-1)^{f-1}\chi(x) \]  
when $K/F$ is ramified, and 
\[ 4s = 2q^{f-1} + (-1)^{f-1}2\chi(x) \]
when $K/F$ is unramified.

To establish Proposition 4.5, we now need to relate the character value at $x$ to the epsilon factor. To begin, since the representation $\sigma$ of $W_F$ associated to $\pi$ is induced from the character $\chi$ of $K^*$, the additivity of epsilon factors gives us the following:

\[ \epsilon(\pi) = \epsilon(\text{Ind}_{K^*}^{W_f}(\chi, \psi_F)) = \epsilon(\text{Ind}_{K^*}^{W_f}(\chi^{-1}, \psi_F) \cdot \epsilon(\text{Ind}_{K^*}^{W_f}(1, \psi_F)) \]

We get by the inductivity (in dimension 0) of epsilon factors,

\[ W(\pi, \psi) = W(\chi, \psi_F)W(\omega_{K/F}, \psi). \]  
\[ \text{(viii)} \]

Choose a (quasi) character $\beta$ of $K^*$ which extends $\omega_{K/F}$. Then $\chi\beta$ is trivial on $F^*$, and we get by Theorems 1.2 and 1.3,

\[ W(\pi, \psi) = W(\chi\beta, \psi_F)W(\omega_{K/F}, \psi) = (\chi\beta(x)\beta(y)W(\omega_{K/F}, \psi), \]  
where $y$ is the element of $K^*$ with the property:

\[ \chi\beta(1 + x) = \psi_K(xy) \quad \text{for all } x \text{ with } \text{val}(x) \geq \frac{1}{2}f(\chi). \]

**Lemma 4.6.** We have the following table:

<table>
<thead>
<tr>
<th>$K/F$</th>
<th>$W(\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>unramified</td>
<td>$(-1)^f\chi(x)$</td>
</tr>
<tr>
<td>ramified</td>
<td>$\omega(2)\omega(-1)^{f+1}G\chi(x)$</td>
</tr>
</tbody>
</table>

**Proof of Lemma.** First we make the choice of $\beta$ explicit as follows. When $K$ is unramified, take $\beta$ to be the unramified character of $K^*$ which takes the value $-1$ at any uniformizer (say $\varpi_K$) of $K$. When $K$ is ramified, take $\beta$ to be the character $\omega_{K/F}$ defined in the proof of Lemma 1.4. Note that $f(\beta)$ is 0 when $K$ is unramified, and equals 1 when $K$ is ramified.

First consider the unramified case. Then, since $\omega_{K/F}$ is unramified, our choice of $\psi$ in §1 implies that $W(\omega_{K/F})$ is trivial, and thus (iii) gives $W(\pi, \psi) = W(\chi, \psi_K)$. Moreover, since $\beta$ is unramified, we get (by Prop.1.1 or by using (ix))

\[ W(\chi\beta, \psi_K) = W(\chi, \psi_K)\beta(\varpi_K)W(\beta, \psi_K), \]

which equals $(-1)^fW(\chi, \psi_K)$ as $\beta(\varpi_K) = -1$. Thus $W(\pi, \psi) = (-1)^fW(\chi\beta, \psi_K)$. But the theorem of Frohlich and Queyrut (Thm.1.3) says that
$W(\chi, \psi_K)$ is $\chi(1)$ Clearly, $\chi(x) = 1$ as $x$ is a unit and $\beta$ unramified. The assertion of the Lemma follows for $K/F$ unramified.

Next we consider the ramified case, which is more subtle. Let $y$ be as in equation (ix). Then we can write: $y = \varpi^{-2f-1}a_0(\chi)$ higher order terms. It follows that

$$\chi((1 + \varpi^{-2f-1}a_0(\chi))x).$$

From the definition of epsilon factors ([T]),

$$\sum_{x \in (\mathcal{O}_F/\mathcal{O})^*} \omega(x)\psi_F(\varpi^{-1}x) = \sqrt{q}\omega_{K/F}(\varpi) W(\omega_{K/F}, \psi),$$

and therefore

$$\sum_{x \in (\mathcal{O}_F/\mathcal{O})^*} \omega(x)\chi((1 + \varpi^{-1}a_0(\chi))x) = \sqrt{q}\omega_{K/F}(2a_0(\chi)\varpi) W(\omega_{K/F}, \psi).$$

Comparing with the definition of $G_\chi$, we get

$$G_\chi = \omega_{K/F}(2a_0(\chi)\varpi) W(\omega_{K/F}, \psi).$$

Thus by equation (ix), $W(\pi, \psi)$ is given by

$$(\chi(\varpi_K)\beta(\varpi_K^{-2f-1}a_0(\chi)) W(\omega_{K/F}, \psi) = \chi(\varpi_K) W(\omega_{K/F}, \psi),$$

which equals $\chi(\varpi_K)\omega(2)\omega(-1)^{f+1}G_\chi$ as $\omega_{K/F}(\varpi) = \omega(-1)$. Hence the Lemma. \(\square\)

Proof of Prop. 4.5 (contd.) Combining this Lemma with equation (vi), we get, for $K/F$ ramified,

$$4(s + a) = (q + 1)q^{f-1} + 2a + 2W(\pi, \psi).$$

As $K$ is ramified, $\nu(x) = -1$, and we have (by (v)) $\tilde{\omega}_2(\tilde{\pi}_{K/F}) = (-1)^{s+a}$. We know by Lemma 4.1 that, for $K$ ramified, $m$ is even iff $a = 0$. Suppose $m = 2k$, for some $k \in \mathbb{N}$. Then

$$s = k(4 + 1)^{f-1} + \frac{1}{2} \left\{ \sum_{j=0}^{f-1} \binom{f-1}{j}(4k)^j + W(\pi) \right\},$$

which is congruent modulo 2 to $k + \frac{1}{2}(1 + W(\pi))$. If $m = 2k - 1$, then we see that

$$s = k(4k - 1)^{f-1} + \frac{1}{2}(1 + W(\pi)),$$

which is congruent modulo 2 to $k + \frac{1}{2}(1 + W(\pi))$. This proves Proposition 4.5 in the ramified case. Suppose $K$ is unramified. Then we have by equation (vii) and Lemma 4.6,

$$2s = q^{f-1} - 2W(\pi).$$

Setting $q = 2m + 1$, we get

$$s = \frac{1}{2} \sum_{j=1}^{f-1} \binom{f-1}{j}(2m)^j + (1 - W(\pi)),$$

which is congruent modulo 2 to $(f - 1)m + \frac{1}{2}(1 - W(\pi))$. Applying equation (vi) and the fact that $a = 0$ (cf. Lemma 4.1), we see that $\tilde{\omega}_2(\tilde{\pi}_{K/F})$ equals $W(\pi)$ or $-W(\pi)$ depending on whether $(f - 1)m$ is even or odd. \(\square\)
5. The main result

Propositions 4.2 and 4.4 can now be combined to give Theorem C.

We now begin the proof of Theorem B. First note that, since the virtual representation \( \pi \ominus \pi' \) has by hypothesis determinant 1, we have by (1.8),

\[
w_2(\tilde{\pi}) = w_2(\tilde{\pi} \ominus \tilde{\pi}') = w_2(\pi) + w_2(\pi').
\]

This implies, by Proposition 3.1, that

\[
(5.1) \quad \tilde{w}_2(\pi \ominus \pi') = 1 \iff w_2(\tilde{\pi})|_{E^*} = w_2(\tilde{\pi}')|_{E^*}, \quad \forall E \in \Sigma(F).
\]

Lemma 5.2. Let \( F \) be non-archimedean and \( \pi \) an irreducible representation of \( D^* \) of dimension \( > 1 \) and trivial central character. Then we have

\[
\det(\pi) = \omega_{L/F} \circ \operatorname{Nrd},
\]

where \( L \) is a separable quadratic extension chosen as follows: \( L = K \) if \( K/F \) unramified or if \( F \) is non-archimedean with \( q \equiv 1 \) modulo 4. If \( F \) is non-archimedean with \( q \equiv 3 \) modulo 4 and \( K/F \) ramified, \( L \) is the other ramified quadratic extension. In particular, any such \( \pi \) is associated to a unique (separable) quadratic extension.

Proof. Since the kernel of \( \operatorname{Nrd} \) is the commutator subgroup of \( D^* \), we can write \( \det(\pi) = \mu \circ \operatorname{Nrd} \), for a character \( \mu \) of \( F^* \). Since \( \pi \) is self-dual, its determinant has order dividing 2, and by class field theory, \( \mu \) is either trivial or \( \omega_{E/F} \), for a quadratic extension \( E/F \). Let \( K \) be a (separable) quadratic extension of \( F \) such that \( \pi \) is associated to a character \( \chi \) of \( K^* \). Let us first consider the non-archimedean case. For any \( E \in \Sigma(F) \), the decomposition of \( (\pi, X) \) given by (i) (of §4) implies that

\[
\det(\pi|_{E^*/F^*}) = 1 \iff a(E) = 0,
\]

in which case \( \mu \) is trivial on the norm subgroup \( NE^* \). Suppose \( K \) is unramified or ramified with \( q \) congruent to 1 modulo 4. Then by Lemma 4.1, \( a(K) = 0 \), and \( a(L) = 1 \) for either of the remaining extensions \( L \) in \( \Sigma(F) \). So \( \mu \) is trivial on \( NK^* \), but not on \( NL^* \). Hence \( \mu = \omega_{K/F} \). Similarly, when \( K \) is ramified with \( q \) congruent to 3 modulo 4, \( a(E) = 0 \) for the other ramified quadratic extension \( E \), and \( a(L) = 1 \) for \( L \) different from \( E \). Thus \( \beta = \omega_{E/F} \) as claimed.

The expression for the determinant immediately gives the assertion about the uniqueness of \( K \) given such a \( \pi \). (This can also be seen directly by first showing that if \( \pi \) is associated to more than one quadratic extension of \( F \), then it is associated to all the three quadratic extensions, which contradicts the formula for \( \dim(\pi) \) (Proposition 1.6). Note that \( \pi \) can be associated to more than one extension if either \( \omega_{\pi} \) is non-trivial or if the residual characteristic is 2.) \( \square \)

Let \( \pi, \pi' \) be as in Theorem B. Then, since \( \det(\pi) \) equals \( \det(\pi') \), we see by the Lemma above that there is a unique \( K \in \Sigma(F) \) such that \( \pi \) and \( \pi' \) are both associated to characters \( \chi \) and \( \chi' \) of \( K^* \).

Combining Propositions 4.3 and 4.5, we get
Proposition 5.3. Let \( \pi, \pi' \) be as above, associated to characters \( \chi, \chi' \) of a quadratic extension \( K/F \). Then we have the following table for \( F \) non-archimedean with residue field of \( q = 2m + 1 \) elements, and for \( L \neq K \) quadratic over \( F \):

\[
\begin{array}{c|c|c}
K/F & \tilde{w}_2(\tilde{\pi}|_{K^*/F^*})/\tilde{w}_2(\tilde{\pi'}|_{K^*/F^*}) & \tilde{w}_2(\tilde{\pi}|_{L^*/F^*})/\tilde{w}_2(\tilde{\pi'}|_{L^*/F^*}) \\
\hline
\text{unramified} & (-1)^{m(\ell(x')-\ell(x))} & (-1)^{m(\ell(x)-\ell(x'))} \\
\text{ramified} & (-1)^{\frac{1}{2}(W(\pi)-W(\pi'))} & 1 \\
\end{array}
\]

Proof. The assertion is an immediate consequence of Propositions 4.3 and 4.5. \( \square \)

When \( K \) is ramified, then \( f(\pi) \) and \( f(\pi') \) are both odd (see Prop.1.5), and \( s(\pi) = s(\pi') = 0 \). When \( K \) is unramified, \( f(\pi) = 2f(\chi) \) and \( f(\pi') = 2f(\chi') \); thus \( s(\pi) = m\ell(\chi) \) and \( s(\pi') = m\ell(\chi') \). Since \( s(\pi) \) and \( s(\pi') \) are assumed to have the same parity, Theorem B now follows (for \( F \) non-archimedean) by appealing to (5.1). Note also that, when \( \pi \cap \pi' \) has determinant 1, the conductors of \( \pi \) and \( \pi' \) are both even or both odd as they are associated to the same quadratic extension \( K \); when they are even, then \( K \) must be unramified. If moreover, \( \pi \cap \pi' \) has dimension zero (modulo 2), then as claimed in the introduction, \( s(\pi) \) and \( s(\pi') \) have the same parity. \( \square \)

An immediate consequence of Prop. 5.3 and (5.1) is the following variant of Theorem B:

Proposition 5.4. Let \( F \) be a non-archimedean local field of odd residual characteristic. Suppose \( \sigma, \sigma' \) are two-dimensional, irreducible, symplectic representations of \( W_F \) such that the associated representations \( \pi, \pi' \) of \( D^* \) have the same determinant. Then

\[
W(\sigma \circ \sigma') = (-1)^{s(\pi) - s(\pi')} \prod_{E \in \Sigma(F)} \tilde{w}_2((\pi \circ \pi')|_{E^*/F^*}).
\]

It remains to treat the case \( F = \mathbb{R} \). As noted before, \( D^*/\mathbb{R}^* \) identifies with \( \text{SO}(3) \), and its irreducible representations are parametrized by their dimension, which must be odd. For every integer \( k \geq 0 \), the unique irreducible \( \pi_{2k+1} \), say, of dimension \( 2k + 1 \) corresponds to the symmetric \( 2k \)-th power representation of the standard representation of \( \text{SU}(2) \). It is easy to see that the image of \( \pi_{2k+1} \) lands in \( \text{SO}(2k+1) \). Define

\[
\lambda : \mathbb{Z}/4 \times \mathbb{Z}/4 \to \{ \pm 1 \}
\]

by sending \((a, b)\) to 1 iff either \((a, b)\) or \((b, a)\) is of one of the following types: (i) \((a, a)\); (ii) \((1, 2)\); and (iii) \((3, 0)\). If \( k, \ell \) are integers with images \( \overline{k}, \overline{\ell} \) in \( \mathbb{Z}/4 \), we will write \( \lambda(k, \ell) \) for \( \lambda(\overline{k}, \overline{\ell}) \).

Proposition 5.5. Let \( \sigma, \sigma' \) be irreducible, 2-dimensional, symplectic representations of \( W_\mathbb{R} \) such that the associated representations \( \pi, \pi' \) of \( D^* \) have dimensions \( 2k + 1, 2\ell + 1 \) respectively. Then we have

\[
W(\sigma \circ \sigma') = \lambda(k, \ell) \tilde{w}_2(\pi \circ \pi').
\]
Proof. Every irreducible two-dimensional self-dual representation of \( W_\mathbb{R} \) is of the form \( \sigma_m = Ind_{\mathbb{Z}_q}^{\mathbb{Z}_q[x]}(\chi_m) \), with \( \chi_m(z) = (z/|z|)^m \), for \( m \geq 1 \). Since \( \det(\sigma_m) = \text{sgn}^{n+1} \), it is symplectic iff \( m \) is odd. The Langlands correspondence pairs \( \sigma_{2k+1} \) with \( \pi_{2k+1} \). So we get the following, by the additivity and inductivity in dimension zero of epsilon factors:

\[
W(\sigma_{2k+1} \otimes \sigma_{2l+1}, \psi) = W(\chi_{2k+1}, \psi) W(\chi_{2l+1}, \psi).
\]

But for any \( m \), \( W(\chi_m, \psi) \) equals \( i^{-m} \) (cf. [T]). So we get

\[
\varepsilon(\pi \otimes \pi') = (-1)^{k-\ell}.
\]

On the other hand, we know by Proposition 3.1 that \( \pi_{2k+1} \) lifts to the spin group iff its restriction to \( \mathbb{C}^* \) does. It is easy to see that \( \pi_{2k+1}|_{\mathbb{C}^*} \) can be decomposed as \( \mathbb{Z} \otimes \mathbb{Z} \mathbb{Z}^{2k-1} \mathbb{Z} \mathbb{Z}^{2k-1} \mathbb{Z}^{2k} \). Applying Lemma 3.2 and noting that \( i \) defines the unique non-trivial element of \( (\mathbb{C}^*/\mathbb{R}^*)[2] \), we see that \( \tilde{w}_2(\pi_{2k+1}) \) is trivial iff \( k \) is 0 or 3 modulo 4. The Proposition now follows easily. \( \square \)

6. A geometric approach

In this section we will indicate a geometric approach to prove Proposition A, which incidentally works for even residual characteristic as well. First we need some preliminaries. Fix a non-archimedean local field \( F \) of characteristic zero, with ring of integers \( \mathcal{O} \), uniformizer \( \varpi \), residue field \( F_q \) and a separable algebraic closure \( \overline{F} \). Denote by \( F_{\text{ur}} \) maximal unramified extension in \( F \) with completion \( \hat{F}_{\text{ur}} \subset \overline{F} \). Let \( \mathcal{O}_{\text{ur}} \subset \mathcal{O} \) and \( \hat{\mathcal{O}}_{\text{ur}} \subset \hat{\mathcal{O}} \) be the corresponding inclusions of rings of integers. For every \( n \geq 1 \), let \( \Omega_{\mathcal{O}} \) denote the complement of the union of all the rational hyperplanes in the projective space \( P^{n-1}_F \), equipped with the rigid analytic structure defined by Drinfeld, and \( X \) the universal family of formal groups associated to the corresponding formal \( \mathcal{O} \)-scheme \( \Omega_{\mathcal{O}} \). For every \( m \geq 1 \), the \( \varpi^m \)-division subgroup \( \Gamma_m \) of \( X \) define rigid étale covers \( \hat{\Gamma}_m = \Gamma_m \otimes_{\mathcal{O}_{\text{ur}}} \hat{F}_{\text{ur}} \)

\[ \Sigma^{n,m} = \hat{\Gamma}_m - \hat{\Gamma}_{m-1} \]

and

\[ \Sigma_0^{n,m} = \text{Res}^{\hat{F}_{\text{ur}}/F}(\Sigma^{n,m}) \]

where \( \text{Res}^\tau \) denotes the descent to \( F \) of the disjoint union of \( \Sigma_0^{n,m} \otimes_{\hat{F}_{\text{ur}}} \hat{F}_{\text{ur}} \), with \( \tau \) running over all the integral powers of the Frobenius \( \phi_q \). Let \( \Sigma_0^{n,m} \) be the projective limit of \( \{ \Sigma_0^{n,m} \} m \geq 1 \).

Fix a prime \( \ell \). Denoting by \( H^\ast \) the rigid étale cohomology (cf. V. Berkovich, “Étale cohomology for non-archimedean analytic spaces”, to appear in Publ. Math. IHES), we set:

\[ H^{n-1} = H^{n-1}(\Sigma_0^{n,m} \otimes_{F} \overline{F}, \overline{Q}_l) \]

This space admits simultaneous commuting actions of \( W_F, \text{GL}(n,F) \) and of \( D^\ast \), where \( D \) is the unique division algebra of dimension \( n^2 \) over \( F \) of invariant \( 1/n \), but the action of \( \text{GL}(n,F) \) is not smooth. In fact, \( H^{n-1} \) is the linear dual of an
admissible representation of $\text{GL}(n, F)$. Let us define $H_{n-1}^{n-1}$ to be the admissible subspace of $H^{n-1}$ (for the action of $\text{GL}(n, F)$). It decomposes as a direct sum

$$H_{n-1}^{n-1} \simeq \bigoplus m_{\sigma \otimes \pi^j \otimes \pi} \sigma \otimes \pi^j \otimes \pi,$$

where $\sigma$, $\pi'$ and $\pi$ run over certain irreducible representations of $W_F$, $\text{GL}(n, F)$ and $D^*$ respectively, with $m_{\sigma \otimes \pi^j \otimes \pi}$ denoting the multiplicity of $\sigma \otimes \pi^j \otimes \pi$. The expectation (see [Ca], §3.3) is that this should give a geometric model of the local Langlands conjecture giving rise to a trijection $\sigma \mapsto \pi'$, at least when restricted to supercuspidal $\pi'$. In particular, $m_{\sigma \otimes \pi^j \otimes \pi}$ should be 1. This conjecture is known to be true for $n = 2$ by Carayol ([Ca]) (once we take care to use $H_{n-1}^1$ instead of $H^1$). The facts on the rigid cohomology assumed in [Ca] have now been provided by the work of Berkovich (loc. cit., §6.3), one actually knows by M. Harris ("Supercuspidal representations in the cohomology of Drinfeld upper upper half spaces; elaboration of Carayol’s program", preprint) of the conjecture for any $n$, every supercuspidal representation of $\text{GL}(n, F)$ occurs in the linear dual of $H^{n-1}$. There is also now a purely local proof due to Faltings (The trace formula and Drinfeld’s upper half plane, preprint) of the conjecture for $n = 2$, giving a description of the rigid $H^1$ with compact supports of $\Sigma_0 \otimes_F \bar{F}$, where all the supercuspidals occur.

Now we give a second proof of Proposition A, which says that an irreducible representation $\pi$ of $D^*$ is orthogonal when the corresponding representation $\sigma$ of $W_F$ is symplectic. Indeed, fix such a pair $(\sigma, \pi)$ and consider also the (supercuspidal) representation $\pi'$ of $\text{GL}(2, F)$ given by the Jacquet-Langlands correspondence. Then $\pi' \simeq \pi^\vee$, and $\sigma \otimes \pi' \otimes \pi$ occurs in $H_{n-1}^1$ by the results above. Moreover, thanks to Berkovich (loc. cit., §6.3), one has Poincaré duality relating $H^1$ with $H_{n-1}^2$, which gives us a non-degenerate pairing defined by cup product:

$$<,> : H^1 \times H_{n-1}^1 \rightarrow H_{n-1}^2.$$

Carayol’s description of the connected components in [Ca], §4.3, describes $H_{n-1}^2$ as a module under $W_F \times \text{GL}(2, F) \times D^*$. Moreover, the pairing $<,>$ can be seen to be equivalent for the action of this product group. Consequently we find using the self-duality of $\sigma$, $\pi$ and $\pi'$, multiplicity-freeness of $H_{n-1}^1$, and the fact that supercuspidals do not intertwine with other representations, that $<,>$ defines a non-degenerate bilinear form $B$ on (the space of) $\sigma \otimes \pi' \otimes \pi$ with values in a one-dimensional subspace of $H^2(\Sigma_0 \otimes_F \bar{F}, \mathbb{Q}_l)$, on which all three groups act by the trivial representation, as det $(\sigma)$ is trivial. It is evident that this invariant form $B$ must be skew-symmetric; thus the tensor product $\sigma \otimes \pi' \otimes \pi$ is symplectic. But we have already proved (see Proposition 2.7) that every supercuspidal, even generic, self-dual representation of $\text{GL}(2, F)$ is orthogonal. This implies that $\pi$ must be orthogonal, as $\sigma$ and $\sigma \otimes \pi' \otimes \pi$ are both symplectic.

For general $n > 2$, note that $\sigma$ can be symplectic only when $n$ is even, in which case the cup product pairing on the self-dual part of $H_{n-1}^{n-1}$ will still be skew-symmetric. So once one knows the truth of the conjectural decomposition of $H_{n-1}^{n-1}$, our reasoning above will show that the corresponding representation $\pi$ of $D^*$ is orthogonal, which justifies the conjecture made at the end of the introduction.

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