

# MULTIPLICITY UPON RESTRICTION TO THE DERIVED SUBGROUP

JEFFREY D. ADLER AND DIPENDRA PRASAD

ABSTRACT. We present a conjecture on multiplicity of irreducible representations of a subgroup  $H$  contained in the irreducible representations of a group  $G$ , with  $G$  and  $H$  having the same derived groups. We point out some consequences of the conjecture, and verification of some of the consequences. We give an explicit example of multiplicity 2 upon restriction, as well as certain theorems in the context of classical groups where the multiplicity is 1.

## 1. INTRODUCTION

Suppose  $k$  is a local field,  $G$  is a connected reductive  $k$ -group,  $G'$  is a subgroup of  $G$  containing the derived group, and  $\pi$  is a smooth, irreducible, complex representation of  $G(k)$ . In an earlier work [1], the authors showed that for many choices of  $G$ , the restriction  $\text{Res}_{G'(k)}^{G(k)} \pi$  decomposes without multiplicity.

A number of years ago, in the process of identifying situations where multiplicity one did not hold, one of us discovered an example of a depth-zero supercuspidal representation  $\pi$  of  $\text{GU}(2d, 2d)$ , a  $k$ -quasisplit group, whose restriction to  $\text{SU}(2d, 2d)$  decomposes with multiplicity two, and the other formulated a conjecture in the form of a reciprocity law involving enhanced Langlands parameters. In this paper, we present both the example and the conjecture, together with some consequences of the latter, and a verification of some of those consequences. Besides these, the paper proves several results by elementary means involving classical groups where multiplicity one holds. For example, we show that all representations of  $\text{GSO}(n)$  decompose with multiplicity one upon restriction to  $\text{SO}(n)$ . (In an earlier work [1], we handled the case of restriction from  $\text{GO}(n)$  to  $\text{O}(n)$ .)

A complete analysis of decomposition of the unitary principal series for  $\text{U}(n, n)$  and its restriction to  $\text{SU}(n, n)$  was done by Keys [5], who also phrased his results in terms of “reciprocity” theorems for  $R$ -groups; in particular, he found cases of multiplicity greater than one.

After presenting our conjecture (§2), we give some of the heuristics behind it. In the formulation of the conjecture, we have considered a more general situation than that of a subgroup. We consider  $G_1$  and  $G_2$  to be two connected reductive groups over a local field  $k$ , and  $\lambda : G_1 \rightarrow G_2$  a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups, allowing us to “restrict” representations of  $G_2(k)$  to  $G_1(k)$ . Since under such a homomorphism  $\lambda$ , the image of  $G_1(k)$  is a normal subgroup of  $G_2(k)$  with abelian quotient, all the irreducible representations of  $G_1(k)$  which appear in this restriction problem for a given irreducible representation of  $G_2(k)$  appear with the same multiplicity. In §3, we verify that for our conjectural multiplicity, this relationship does indeed hold. We

show (§4) that if the conjecture is true for tempered representations, then via the Langlands classification it holds for all representations.

Our conjecture (for  $\lambda : G_1 \rightarrow G_2$  a  $k$ -homomorphism), implies multiplicity one in situations where parameters for  $G_1$  have abelian component groups. We list a few such situations in §5, and prove multiplicity one for restriction from  $\text{GSO}(n)$  to  $\text{SO}(n)$  (§6) and from  $\text{GU}(n)$  to  $\text{U}(n)$  (§7). Along the way, we prove multiplicity one in some other cases where it follows from elementary considerations. In §8, we present an example of depth-zero supercuspidal representation of quasi-split  $\text{GU}(2d, 2d)$  that decomposes with multiplicity two upon restriction to  $\text{SU}(2d, 2d)$ . Finally (§9), we give a general procedure for constructing higher multiplicities.

## 2. THE CONJECTURE ON MULTIPLICITIES

Let  $G_1^{\text{qs}}$  and  $G_2^{\text{qs}}$  be two connected quasi-split reductive groups over a local field  $k$  and  $\lambda : G_1^{\text{qs}} \rightarrow G_2^{\text{qs}}$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups. In what follows we will be twisting  $G_1^{\text{qs}}$  by a cohomology class in  $H^1(\text{Gal}(\bar{k}/k), G_1^{\text{qs}}(\bar{k}))$  to construct a pure inner form  $G_1$  of  $G_1^{\text{qs}}$ ; simultaneously, by twisting  $G_2^{\text{qs}}$  by the the image of this class under the map  $H^1(\text{Gal}(\bar{k}/k), G_1^{\text{qs}}(\bar{k})) \rightarrow H^1(\text{Gal}(\bar{k}/k), G_2^{\text{qs}}(\bar{k}))$ , we will have a pure inner form  $G_2$  of  $G_2^{\text{qs}}$ , together with map of algebraic groups that we will still call  $\lambda : G_1 \rightarrow G_2$  which will appear in considerations below, all coming from an element of  $H^1(\text{Gal}(\bar{k}/k), G_1^{\text{qs}}(\bar{k}))$ .

The map  $\lambda : G_1 \rightarrow G_2$  gives rise to a “restriction” map from representations of  $G_2(k)$  to those of  $G_1(k)$ , and from Silberger [10] one knows that the restriction of an irreducible representation of  $G_2(k)$  is a finite direct sum of irreducible representations of  $G_1(k)$ . In particular, we obtain a functor  $\lambda^* : \mathcal{R}_{\text{fin}}(G_2(k)) \rightarrow \mathcal{R}_{\text{fin}}(G_1(k))$ , where  $\mathcal{R}_{\text{fin}}(H)$  denotes the category of smooth, finite-length representations of a group  $H$ .

Let  ${}^L G_1 = \widehat{G}_1 \rtimes W'_k$  and  ${}^L G_2 = \widehat{G}_2 \rtimes W'_k$  be the  $L$ -groups associated to the quasi-split reductive groups  $G_1^{\text{qs}}$  and  $G_2^{\text{qs}}$  respectively. The map  $\lambda : G_1^{\text{qs}} \rightarrow G_2^{\text{qs}}$  also gives rise to a homomorphism of  $L$ -groups:

$${}^L \lambda : {}^L G_2 \longrightarrow {}^L G_1,$$

as well as a homomorphism of their centers:

$${}^L \lambda : Z(\widehat{G}_2)^{W_k} \longrightarrow Z(\widehat{G}_1)^{W_k}.$$

In particular, a character  $\chi_1$  of  $\pi_0(Z(\widehat{G}_1)^{W_k})$  gives rise to a character  $\chi_2$  of  $\pi_0(Z(\widehat{G}_2)^{W_k})$  which by the Kottwitz isomorphism (assuming  $k$  to be nonarchimedean at this point):

$$H^1(\text{Gal}(\bar{k}/k), G_i^{\text{qs}}(\bar{k})) \cong \text{Hom}(\pi_0(Z(\widehat{G}_i)^{W_k}), \mathbf{Q}/\mathbf{Z}),$$

constructs pure inner forms  $G_1$  of  $G_1^{\text{qs}}$  and  $G_2$  of  $G_2^{\text{qs}}$ , together with a map  $\lambda : G_1 \rightarrow G_2$  as before.

Let  $\varphi_2 : W'_k \rightarrow {}^L G_2$ , and  $\varphi_1 = {}^L \lambda \circ \varphi_2 : W'_k \rightarrow {}^L G_1$  be associated Langlands parameters, where  $W'_k = W_k \times \text{SL}_2(\mathbf{C})$ , with  $W_k$  the Weil group of  $k$ . Then  ${}^L \lambda$  gives rise to a homomorphism of centralizers of the images of the parameters  $\varphi_1$  with values in  ${}^L G_1$  and  $\varphi_2$  with values in  ${}^L G_2$ , and also a homomorphism of the

groups of connected components of their centralizers:

$$\pi_0({}^L\lambda) : \pi_0(Z_{\widehat{G}_2}(\varphi_2)) \longrightarrow \pi_0(Z_{\widehat{G}_1}(\varphi_1)).$$

This allows one to ‘restrict’ representations of  $\pi_0(Z_{\widehat{G}_1}(\varphi_1))$  to representations of  $\pi_0(Z_{\widehat{G}_2}(\varphi_2))$ , giving rise to the restriction functor

$$\lambda_* : K_0(\pi_0(Z_{\widehat{G}_1}(\varphi_1))) \rightarrow K_0(\pi_0(Z_{\widehat{G}_2}(\varphi_2))),$$

where  $K_0(H)$  denotes the Grothendieck group of finite-length representations of a group  $H$ .

The formulation of our conjecture below presumes that the local Langlands correspondence involving enhanced Langlands parameters has been achieved, giving rise to a bijection between enhanced Langlands parameters and the set of isomorphism classes of irreducible admissible representations of all pure inner forms of quasi-split groups. This will be needed for *both* of the groups  $G_1$  and  $G_2$ ; it is possible on the other hand that one could reverse this role, and use the conjectural multiplicity formula to construct an enhanced Langlands parametrization for  $G_2$ , knowing it for  $G_1$ .

**Conjecture 1.** (a) Let  $G_1$  and  $G_2$  be two connected reductive groups over a local field  $k$  and  $\lambda : G_1 \rightarrow G_2$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups. For  $i = 1, 2$ , let  $\pi_i$  be an irreducible admissible representation of  $G_i(k)$  with Langlands parameter  $\varphi_i$ . Let

$$m(\pi_2, \pi_1) := \dim \operatorname{Hom}_{G_1(k)}[\pi_1, \lambda^* \pi_2] = \dim \operatorname{Hom}_{G_1(k)}[\lambda^* \pi_2, \pi_1].$$

Then  $m(\pi_2, \pi_1) = 0$  unless  $\varphi_1 = {}^L\lambda \circ \varphi_2$ .

(b) Let  $G_1^{\text{qs}}$  and  $G_2^{\text{qs}}$  be two connected reductive quasi-split groups over a local field  $k$  and  $\lambda : G_1^{\text{qs}} \rightarrow G_2^{\text{qs}}$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups. Let  $\varphi_1$  and  $\varphi_2$  be Langlands parameters associated to the groups  $G_1^{\text{qs}}$  and  $G_2^{\text{qs}}$  with  $\varphi_1 = {}^L\lambda \circ \varphi_2$ , and let  $\chi_i$  be characters of their component groups  $\pi_0(Z_{\widehat{G}_i}(\varphi_i))$ . Then, if  $\operatorname{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_* \chi_1]$  is nonzero, the characters  $\chi_i$  define pure inner forms  $G_i$  of  $G_i^{\text{qs}}$  together with a  $k$ -homomorphism,  $\lambda : G_1 \rightarrow G_2$ , as discussed earlier. Then if  $\pi_i = \pi(\varphi_i, \chi_i)$  are the corresponding irreducible admissible representations of  $G_i(k)$ , we have

$$m(\pi_2, \pi_1) = \dim \operatorname{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_* \chi_1].$$

The main heuristic that goes in the conjectural multiplicity is the following.

(1) For any  $L$ -packet  $\{\pi\}$  on any reductive group  $G(k)$  defined by a parameter  $\varphi$ , thus  $\{\pi\} = \{\pi_{(\varphi, \chi)}\}$  where one takes those characters  $\chi$  of the component group which have a particular restriction to  $Z(\widehat{G})^{W_k}$  defining the group  $G(k)$  assumed to be a pure inner form of a fixed quasi-split group  $G^{\text{qs}}$ ,

$$\sum_{\chi} \chi(1) \Theta(\pi_{(\varphi, \chi)})$$

is a stable distribution on  $G(k)$ . Here, for any admissible representation  $\pi$  we are letting  $\Theta(\pi)$  denote its character, regarded as a distribution on  $G(k)$ .

- (2) For a homomorphism  $\lambda : G_1 \rightarrow G_2$  of reductive groups over  $k$  which is an isogeny when restricted to their derived subgroups, the pullback of a stable distribution on  $G_2(k)$  is a stable distribution on  $G_1(k)$ .
- (3) The restriction to  $G_1(k)$  of an irreducible representation  $\pi_2$  of  $G_2(k)$  is a finite-length (completely reducible) representation of  $G_1(k)$ , whose irreducible components are all in the same  $L$ -packet. This  $L$ -packet for  $G_1(k)$  depends only on the  $L$ -packet for  $G_2(k)$  containing  $\pi_2$ . If the Langlands parameter of our  $L$ -packet for  $G_2(k)$  is  $\varphi_2 : W'_k \rightarrow {}^L G_2$ , then the Langlands parameters our  $L$ -packet for  $G_1(k)$  is  $\varphi_1 := {}^L \lambda \circ \varphi_2 : W'_k \rightarrow {}^L G_1$ . (This is part (a) of the conjecture.)
- (4) If Conjecture 1 is true, then the pullback from  $G_2(k)$  to  $G_1(k)$  of the distribution

$$\sum_{\chi_2} \chi_2(1) \Theta(\pi_{(\varphi_2, \chi_2)})$$

where the sum is taken over those characters  $\chi_2$  of the component group which have a particular restriction to  $Z(\widehat{G}_2)^{W_k}$  defining the group  $G_2(k)$  assumed to be a pure inner form of a fixed quasi-split group  $G_2^{\text{qs}}(k)$ , is a stable distribution on  $G_1(k)$  as we check now.

By Conjecture 1, the pullback of the distribution  $\Theta_{\pi_2} = \Theta(\pi_{(\varphi_2, \chi_2)})$  on  $G_2(k)$  to  $G_1(k)$  is

$$\sum_{\pi_1} m(\pi_2, \pi_1) \Theta(\pi_1) = \sum_{\chi_1} \Theta(\pi_{(\varphi_1, \chi_1)}) \dim \text{Hom}_{\pi_0(Z(\varphi_2))} [\chi_2, \lambda_* \chi_1].$$

Therefore, the pullback to  $G_1(k)$  of the distribution  $\sum_{\chi_2} \chi_2(1) \Theta(\pi_{(\varphi_2, \chi_2)})$  on  $G_2(k)$  is (assuming Conjecture 1)

$$\sum_{\chi_1, \chi_2} \chi_2(1) \Theta(\pi_{(\varphi_1, \chi_1)}) \dim \text{Hom}_{\pi_0(Z(\varphi_2))} [\chi_2, \lambda_* \chi_1],$$

which is the same as

$$\sum_{\chi_1, \chi_2} \Theta(\pi_{(\varphi_1, \chi_1)}) \dim \text{Hom}_{\pi_0(Z(\varphi_2))} [\chi_2(1) \chi_2, \lambda_* \chi_1],$$

where the sum is taken over all pairs of characters  $\chi_1, \chi_2$  with particular restrictions to  $Z(\widehat{G}_1)^{W_k}$  and  $Z(\widehat{G}_2)^{W_k}$ . Observe that those characters  $\chi_2$  whose restrictions to  $Z(\widehat{G}_2)^{W_k}$  are not compatible with the restriction of  $\chi_1$  to  $Z(\widehat{G}_1)^{W_k}$  contribute 0 to the sum. Therefore, we can take the sum over all  $\chi_2$ . The sum then is the same as

$$(*) \quad \sum_{\chi_1} \Theta(\pi_{(\varphi_1, \chi_1)}) \dim \text{Hom}_{\pi_0(Z(\varphi_2))} [R, \lambda_* \chi_1],$$

where  $R = \sum \chi_2(1) \chi_2$  is the regular representation of  $\pi_0(Z(\varphi_2))$ .

By Schur orthogonality,

$$\dim \text{Hom}_{\pi_0(Z(\varphi_2))} [\chi_2, \lambda_* \chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g \in \pi_0(Z(\varphi_2))} \chi_1(\lambda^* g) \bar{\chi}_2(g),$$

where here  $\lambda^*$  denotes the map  $\pi_0({}^L \lambda) : \pi_0(Z(\varphi_2)) \rightarrow \pi_0(Z(\varphi_1))$ . So

$$\dim \text{Hom}_{\pi_0(Z(\varphi_2))} [R, \lambda_* \chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g \in \pi_0(Z(\varphi_2))} \chi_1(\lambda^* g) \chi_R(g),$$

where  $R$  is the regular representation of  $\pi_0(Z(\varphi_2))$  and  $\chi_R$  its character, thus

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \text{ is not the identity,} \\ |\pi_0(Z(\varphi_2))| & \text{if } g \text{ is the identity.} \end{cases}$$

Therefore,

$$\dim \text{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_* \chi_1] = \chi_1(1).$$

By (\*) it follows that the pullback of the distribution  $\sum_{\chi_2} \chi_2(1) \Theta(\pi_{(\varphi_2, \chi_2)})$  on  $G_2(k)$  to  $G_1(k)$  is equal to  $\sum_{\chi_1} \chi_1(1) \Theta(\pi_{(\varphi_1, \chi_1)})$ , where the sum is taken over those  $\chi_1$  with a given restriction to  $Z(\widehat{G}_1)^{W_k}$ . Thus the pullback of the distribution  $\sum_{\chi_2} \chi_2(1) \Theta(\pi_{(\varphi_2, \chi_2)})$  on  $G_2(k)$  to  $G_1(k)$  is a stable distribution on  $G_1(k)$  which is what we set out to prove.

**Remark 2.** A weaker version of our conjecture says that the pullback to  $G_1(k)$  of the stable character  $\sum_{\chi} \chi(1) \Theta_{\chi}$  on  $G_2(k)$  is  $\sum_{\mu} \mu(1) \Theta_{\mu}$  on  $G_1(k)$ , where both of the sums are over the characters of component groups defining fixed pure inner forms that are  $G_2$  and  $G_1$ , respectively.

### 3. SOME REMARKS ON THE MULTIPLICITY FORMULA

Conjecture 1 relating  $m(\pi_2, \pi_1)$  with  $\dim \text{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_* \chi_1, \chi_2]$  can be considered as a set of assertions keeping  $\pi_2$  fixed and varying  $\pi_1$ , or keeping  $\pi_1$  fixed and varying  $\pi_2$ , say inside an  $L$ -packet for  $G_2(k)$ . It is easy to see that for  $G_1$  and  $G_2$  two reductive groups over a local field  $k$ , and  $\lambda : G_1 \rightarrow G_2$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups, the image of  $G_1(k)$  inside  $G_2(k)$  is a normal subgroup, and therefore every irreducible representation of  $G_1(k)$  that appears inside a given irreducible representation  $\pi_2$  of  $G_2(k)$  does so with the same multiplicity (depending of course on  $\pi_2$ ). This section aims to prove this as a consequence of our Conjecture 1.

This section is meant to prove that  $\dim \text{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_* \chi_1, \chi_2]$  remains constant when  $\chi_2$  is a fixed character of  $\pi_0(Z(\varphi_2))$  but  $\chi_1$  varies among characters of  $\pi_0(Z(\varphi_1))$ . This is achieved by combining Corollary 4 with Lemma 5. We begin with the following lemma whose straightforward proof will be omitted.

**Lemma 3.** *Let  $N$  be a normal subgroup of a finite group  $G$  with  $A = G/N$  an abelian group. Let  $\pi$  be an irreducible representation of  $N$ . Then any two irreducible representations  $\pi_1$  and  $\pi_2$  of  $G$  containing  $\pi$  on restriction to  $N$  are twists of each other by characters of  $G/N$ , i.e.,*

$$\pi_2 \cong \pi_1 \otimes \chi,$$

for  $\chi : G/N \rightarrow \mathbf{C}^{\times}$ .

**Corollary 4.** *If  $N$  is a normal subgroup of a group  $G$  with  $A = G/N$  a finite abelian group, and  $\pi$  an irreducible representation of  $N$ , then all irreducible  $G$ -submodules of  $\text{Ind}_N^G(\pi)$  appear in it with the same multiplicity.*

**Lemma 5.** *Let  $G_1$  and  $G_2$  be two connected reductive groups over a local field  $k$ ,  $\lambda : G_1 \rightarrow G_2$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups, and giving rise to a homomorphism  ${}^L\lambda : {}^L G_2 \rightarrow {}^L G_1$  of the  $L$ -groups. Let  $\varphi_2 : W'_k \rightarrow {}^L G_2$ , and  $\varphi_1 = {}^L\lambda \circ \varphi_2 : W'_k \rightarrow {}^L G_1$  be associated Langlands parameters. Then for the associated homomorphism of finite groups  $\lambda^* : \pi_0(Z_{\widehat{G}_2}(\varphi_2)) \rightarrow \pi_0(Z_{\widehat{G}_1}(\varphi_1))$ , the image is normal with abelian cokernel.*

*Proof.* It suffices to prove the lemma separately in the two cases:

- (1)  $\lambda: G_1 \rightarrow G_2$  is injective.
- (2)  $\lambda: G_1 \rightarrow G_2$  is surjective.

We will do only the first case, the other being very similar.

Assume then that  $\lambda: G_1 \rightarrow G_2$  is injective, and thus  $\widehat{\lambda}: \widehat{G}_2 \rightarrow \widehat{G}_1$  is surjective with kernel say  $\widehat{Z}$ . Use  $\varphi_2: W'_k \rightarrow {}^L G_2$  and  $\varphi_1 = {}^L \lambda \circ \varphi_2: W'_k \rightarrow {}^L G_1$  to give  $\widehat{G}_2$  and  $\widehat{G}_1$  a  $W'_k$ -group structure such that we have an exact sequence of  $W'_k$ -groups:

$$1 \rightarrow \widehat{Z} \rightarrow \widehat{G}_2 \rightarrow \widehat{G}_1 \rightarrow 1.$$

This gives rise to a long exact sequence of  $W'_k$ -cohomology sets:

$$1 \rightarrow \widehat{Z}^{W'_k} \rightarrow \widehat{G}_2^{W'_k} \rightarrow \widehat{G}_1^{W'_k} \rightarrow H^1(W'_k, \widehat{Z}) \rightarrow \cdots.$$

Equivalently, we have the exact sequence of groups:

$$1 \rightarrow Z_{\widehat{G}_2}(\varphi_2) / \widehat{Z}^{W'_k} \rightarrow Z_{\widehat{G}_1}(\varphi_1) \rightarrow A \rightarrow 1,$$

where  $A \subset H^1(W'_k, \widehat{Z})$ , a locally compact abelian group. Taking  $\pi_0$  of the terms in the above exact sequence which all fit together in a long exact sequence of  $\pi_i$ 's (higher homotopy groups), the assertion in the lemma follows on noting that if  $E_1 \rightarrow E_2$  is a surjective map of locally compact and locally connected topological groups, then the induced map  $\pi_0(E_1) \rightarrow \pi_0(E_2)$  is also surjective.  $\square$

#### 4. REDUCTION OF THE CONJECTURE TO THE CASE OF TEMPERED REPRESENTATIONS

As before, let  $G_1$  and  $G_2$  be two reductive groups over a local field  $k$ , and  $\lambda: G_1 \rightarrow G_2$ , a  $k$ -homomorphism that is a central isogeny when restricted to their derived subgroups, giving rise to the restriction functor  $\lambda^*: \mathcal{R}_{\text{fin}}(G_2(k)) \rightarrow \mathcal{R}_{\text{fin}}(G_1(k))$ .

**Lemma 6.** *Let  $V$  be a finite-length representation of  $G_2(k)$  with maximal semi-simple quotient  $Q$ . Then  $\lambda^*Q$  is the maximal semi-simple quotient of  $\lambda^*V$ , a finite-length representation of  $G_1(k)$ .*

*Proof.* It suffices to observe that a finite-length representation of  $G_2(k)$  is semisimple if and only if its image under  $\lambda^*$  is a finite-length, semi-simple representation of  $G_1(k)$ . If  $Z(G_1(k)) \cdot G_1(k)$  is of finite index in  $G_2(k)$ , such as when  $k$  is of characteristic zero, then this is easy to see. By a theorem of Silberger [10], irreducible representations of  $G_2(k)$  remain finite-length semi-simple representations when restricted to  $G_1(k)$ , and the lemma follows in general.  $\square$

To set up the next result, let  $P_2 = M_2 N_2$  be a Levi factorization of a parabolic subgroup in  $G_2$ . If we let  $P_1 = \lambda^{-1}(P_2)$ ,  $M_1 = \lambda^{-1}(M_2)$ , and  $N_1 = \lambda^{-1}(N_2)$ , then  $P_1 = M_1 N_1$  is a Levi factorization of a parabolic subgroup in  $G_1$ . Then  $\lambda: M_1 \rightarrow M_2$  gives us a restriction functor  $\mathcal{R}_{\text{fin}}(M_2(k)) \rightarrow \mathcal{R}_{\text{fin}}(M_1(k))$  that we will also denote by  $\lambda^*$ . Since  $\lambda$  gives an isomorphism  $G_1(k)/P_1(k) \rightarrow G_2(k)/P_2(k)$ , we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}_{\text{fin}}(G_2(k)) & \xrightarrow{\lambda^*} & \mathcal{R}_{\text{fin}}(G_1(k)) \\ \text{Ind}_{P_2(k)}^{G_2(k)} \uparrow & & \uparrow \text{Ind}_{P_1(k)}^{G_1(k)} \\ \mathcal{R}_{\text{fin}}(M_2(k)) & \xrightarrow{\lambda^*} & \mathcal{R}_{\text{fin}}(M_1(k)) \end{array}$$

**Lemma 7.** *Let  $\sigma_2$  be an irreducible, essentially tempered representation of  $M_2(k)$  with strictly positive exponents along the center  $Z(M_2)(k)$  of  $M_2(k)$ . Write*

$$\lambda^* \sigma_2 = \sum_{\alpha} m_{\alpha} \sigma_{1,\alpha},$$

*a sum of irreducible, essentially tempered representations of  $M_1(k)$  with (finite) multiplicities  $m_{\alpha}$ . Let  $\pi_2$  be the Langlands quotient of the standard module  $\text{Ind}_{P_2(k)}^{G_2(k)} \sigma_2$ , and  $\pi_{1,\alpha}$  the Langlands quotients of  $\text{Ind}_{P_1(k)}^{G_1(k)} \sigma_{1,\alpha}$ . Then*

$$\lambda^* \pi_2 = \sum_{\alpha} m_{\alpha} \pi_{1,\alpha}.$$

*Proof.* Clearly,

$$\lambda^* \text{Ind}_{P_2(k)}^{G_2(k)} \sigma_2 = \text{Ind}_{P_1(k)}^{G_1(k)} \lambda^* \sigma_2 = \sum_{\alpha} m_{\alpha} \text{Ind}_{P_1(k)}^{G_1(k)} \sigma_{1,\alpha}.$$

Since “taking maximal semi-simple quotient” commutes with direct sum, our result follows from Lemma 6.  $\square$

**Corollary 8.** *If Conjecture 1 is true for tempered representations, then it is true in general.*

*Proof.* Every representation  $\pi_2$  of  $G_2(k)$  can be realized as a Langlands quotient of a standard module  $\text{Ind}_{P_2(k)}^{G_2(k)} \sigma_2$  for an essentially tempered representation  $\sigma_2$  of  $M_2(k)$ . The Langlands parameter  $\phi_2: W'_F \rightarrow {}^L G_2$  for  $\pi_2$  is the same as the Langlands parameter  $\varphi_2$  for  $\sigma_2$  considered as a map  $W'_F \xrightarrow{\varphi_2} {}^L M_2 \rightarrow {}^L G_2$ . The component groups of these parameters, and thus the representations of these component groups, correspond as discussed in [9, §5]. Therefore, our result is a consequence of Lemma 7.  $\square$

## 5. CONSEQUENCES OF THE CONJECTURE

If the group of connected components  $\pi_0(Z_{\widehat{G}_1}(\varphi_1))$  is known to be abelian, as is the case when  $G_1$  is any of the groups  $\text{SL}_n$ ,  $\text{U}_n$ ,  $\text{SO}_n$ , and  $\text{Sp}_n$ , then our conjecture predicts that for any homomorphism  $\lambda: G_1 \rightarrow G_2$  of connected reductive algebraic groups that is an isomorphism up to center (i.e.,  $\bar{\lambda}: G_1/Z_1 \rightarrow G_2/Z_2$  is an isomorphism of algebraic groups, where  $Z_i$  is the center of  $G_i$ ), any irreducible representation of  $G_2(k)$  when restricted via  $\lambda$  to  $G_1(k)$  decomposes as a sum of irreducible representations of  $G_1(k)$  with multiplicity  $\leq 1$ .

We note that by our earlier work [1], we know that multiplicity is  $\leq 1$  whenever the pair  $(G_1, G_2)$  is  $(\text{SL}_n, \text{GL}_n)$ , or (when the characteristic of  $k$  is not two) either  $(\text{O}_n, \text{GO}_n)$  or  $(\text{Sp}_n, \text{GSp}_n)$ . In the next two sections, we will see that multiplicity  $\leq 1$  also holds for  $(\text{SO}_n, \text{GSO}_n)$  and  $(\text{U}_n, \text{GU}_n)$ .

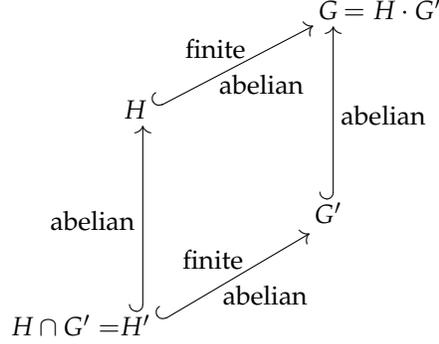
## 6. RESTRICTION FROM GSO TO SO

**Proposition 9.** *Let*

- $G$  be a locally compact, totally disconnected group;
- $G'$  a closed, normal subgroup of  $G$  with  $G/G'$  abelian;
- $H$  a normal subgroup of  $G$  such that  $G/H$  is finite abelian and  $H \cdot G' = G$ ;
- $H' = H \cap G'$ ; and

- $\tilde{\pi}$  an irreducible representation of  $G$  such that  $\text{Res}_G^G \tilde{\pi}$  is multiplicity free.

Then for every irreducible component  $\pi$  of  $\text{Res}_H^G \tilde{\pi}$ , we have that  $\text{Res}_H^H \pi$  is multiplicity free.



Before giving a proof, let us recall a well-known result that we will need to use several times.

**Lemma 10.** *Let  $G$  be a locally compact, totally disconnected group, and  $H$  a normal subgroup of  $G$  of prime index  $\ell$ . Let  $\tilde{\pi}$  be an irreducible, admissible representation of  $G$ ,  $\pi$  an irreducible representation of  $H$  contained in the restriction  $\text{Res}_H^G \tilde{\pi}$ , and  $\varepsilon$  a non-trivial character of  $G$  trivial on  $H$ . Then the following are equivalent:*

- The representation  $\pi$  of  $H$  is invariant under the action of  $G/H$ .
- $\text{Res}_H^G \tilde{\pi} \cong \pi$ .
- The representations  $\tilde{\pi} \otimes \varepsilon$  and  $\tilde{\pi}$  of  $G$  are not isomorphic.
- $\text{Ind}_H^G \pi \cong \sum_{i=0}^{\ell-1} \tilde{\pi} \otimes \varepsilon^i$ , a sum of inequivalent irreducible representations.

In the case when the above equivalent conditions do not apply, then we have

- $\text{Res}_H^G \tilde{\pi} \cong \sum_{g \in G/H} (\pi)^g$ , a sum of inequivalent irreducible representations.
- $\text{Ind}_H^G \pi \cong \tilde{\pi}$ .

*Proof.* This is a special case of [3, Lemma 2.1 and Corollary 2.2].  $\square$

*Proof of Proposition 9.* Using an inductive argument, we can reduce to the case where  $G/H \cong G'/H'$  is cyclic of prime order  $\ell$ . Assume from now on that we are in this case, and let  $\varepsilon$  denote a nontrivial character of  $G/H$ , and the corresponding character of  $G'/H'$ .

Let  $\pi$  be an irreducible component of  $\text{Res}_H^G \tilde{\pi}$ . From Lemma 10,  $\text{Ind}_H^G \pi$  must be equivalent to either  $\tilde{\pi}$  or  $\sum_{j=0}^{\ell-1} \tilde{\pi} \otimes \varepsilon^j$ , and in either case is multiplicity free.

Case I:  $\text{Ind}_H^G \pi = \tilde{\pi}$ . Then by the induction-restriction formula and the fact that  $H \cdot G' = G$ ,

$$\text{Ind}_{H'}^{G'} \text{Res}_{H'}^H \pi \cong \text{Res}_{G'}^G \text{Ind}_H^G \pi = \text{Res}_{G'}^G \tilde{\pi},$$

which is multiplicity free by hypothesis. Therefore,  $\text{Res}_{H'}^H \pi$  is multiplicity free.

Case II:  $\text{Ind}_H^G \pi \cong \sum_{j=0}^{\ell-1} \tilde{\pi} \otimes \varepsilon^j$ . Write  $\text{Res}_{H'}^H \pi = \sum n_i \pi_i$ , where the summands  $\pi_i$  are all distinct, irreducible representations of  $H'$ . We need to show that all

$n_i \leq 1$ . Let  $S$  denote the set of (equivalence classes of) irreducible representations  $\sigma$  of  $G'$  such that  $\sigma \otimes \varepsilon \cong \sigma$ . Then

$$\text{Res}_{G'}^G \tilde{\pi} = \sum_{\sigma \in S} m_\sigma \sigma \oplus \sum_{\sigma \notin S} m_\sigma \sigma$$

where each  $m_\sigma \leq 1$  by hypothesis. Therefore,

$$\text{Res}_{G'}^G \text{Ind}_H^G \pi \cong \text{Res}_{G'}^G \sum_{j=0}^{\ell-1} \tilde{\pi} \otimes \varepsilon^j = \ell \left( \sum_{\sigma \in S} m_\sigma \sigma \right) \oplus \sum_{\sigma \notin S} m_\sigma \left( \sum_{j=0}^{\ell-1} \sigma \otimes \varepsilon^j \right).$$

But by the induction-restriction formula,

$$\text{Res}_{G'}^G \text{Ind}_H^G \pi \cong \text{Ind}_{H'}^{G'} \text{Res}_{H'}^H \pi = \sum n_i \text{Ind}_{H'}^{G'} \pi_i.$$

Therefore,

$$(**) \quad \sum n_i \text{Ind}_{H'}^{G'} \pi_i = \ell \left( \sum_{\sigma \in S} m_\sigma \sigma \right) \oplus \sum_{\sigma \notin S} m_\sigma \left( \sum_{j=0}^{\ell-1} \sigma \otimes \varepsilon^j \right).$$

Fix a component  $\pi_i$  (a representation of  $H'$ ), and let  $\tilde{\pi}_i$  be a component of  $\text{Res}_{G'}^G \tilde{\pi}$  that contains  $\pi_i$  upon restriction to  $H'$ .

If  $\tilde{\pi}_i \notin S$ , then the multiplicity of  $\tilde{\pi}_i$  in the right-hand side of  $(**)$  is 1, so we must have  $n_i = 1$ .

On the other hand, if  $\tilde{\pi}_i$  is a representation of  $G'$  belonging to  $S$ , i.e.,  $\tilde{\pi}_i \otimes \varepsilon \cong \tilde{\pi}_i$ , and if  $\tilde{\pi}_i$  appears in  $\text{Ind}_{H'}^{G'} \pi_i$ , then in fact  $\ell \tilde{\pi}_i \subseteq \text{Ind}_{H'}^{G'} \pi_i$ . It follows that the multiplicity of any such representation  $\tilde{\pi}_i$  in  $n_i \text{Ind}_{H'}^{G'} \pi_i$  is a multiple of  $\ell n_i$ . Comparing with the right-hand side of  $(**)$ , and recalling that  $m_\sigma \leq 1$ , we conclude that  $n_i \leq 1$ .  $\square$

**Corollary 11.** *Assume that  $k$  is not of characteristic two. Let  $V$  be a finite-dimensional vector space over  $k$ , equipped with a nondegenerate quadratic form. Then every irreducible representation of  $\text{GSO}(V)$  restricts without multiplicity to  $\text{SO}(V)$ .*

*Proof.* From [1, Theorem 1.4], we have multiplicity one upon restriction from  $\text{GO}(V)$  to  $\text{O}(V)$ . Now apply Proposition 9, letting  $G = \text{GO}(V)$ ,  $G' = \text{O}(V)$ ,  $H = \text{GSO}(V)$ , and  $H' = \text{SO}(V)$ .  $\square$

## 7. GENERALITIES ON RESTRICTION TO UNITARY AND SPECIAL UNITARY GROUPS

Let  $E/k$  denote a separable quadratic extension of nonarchimedean local fields,  $N = N_{E/k}$  the norm map from  $E^\times$  to  $k^\times$ , and  $E_1$  the kernel of this map.

Let  $B$  denote a nondegenerate  $E/k$ -hermitian form on some  $E$ -vector space  $V$  of some dimension  $r$ . Then we can form algebraic groups  $\text{SU}(V, B)$ ,  $\text{U}(V, B)$ , and  $\text{GU}(V, B)$  whose  $k$ -points consist respectively of the elements of  $\text{SL}(r, E)$  that preserve  $B$ ; the elements of  $\text{GL}(r, E)$  that preserve  $B$ ; and the elements of  $\text{GL}(r, E)$  that preserve  $B$  up to a scalar in  $k^\times$ . The group  $\text{GU}(V, B)$  comes equipped with a map  $\mu: \text{GU}(V, B) \rightarrow \text{GL}_1$  called the *similitude character*. We will write our algebraic groups as  $\text{SU}(r)$ ,  $\text{U}(r)$ , and  $\text{GU}(r)$  when  $V$  and  $B$  are understood.

If  $G$  is a group,  $H$  is a subgroup, and  $G/Z(G)H$  is cyclic, then every irreducible representation of  $G$  restricts to  $H$  without multiplicity. How far can we exploit this fact?

**Theorem 12.** *Let  $p$  be the residual characteristic of  $k$ .*

- (a) All irreducible representations of  $\mathrm{GU}(r)(k)$  decompose without multiplicity upon restriction to  $\mathrm{U}(r)(k)$ . Such a restriction is irreducible when  $r$  is odd, and has at most two components when  $r$  is even.
- (b) All irreducible representations of  $\mathrm{U}(r)(k)$  decompose without multiplicity upon restriction to  $\mathrm{SU}(r)(k)$  when  $r$  is coprime to  $p$ , or  $k = \mathbf{Q}_p$  ( $p$  odd).
- (c) All irreducible representations of  $\mathrm{GU}(r)(k)$  decompose without multiplicity upon restriction to  $\mathrm{SU}(r)(k)$  when  $r$  is odd and coprime to  $p$ .

*Proof.* (a) Let  $\mu: \mathrm{GU}(r) \rightarrow \mathrm{GL}(1)$  denote the similitude character. Clearly the group  $\mathrm{GU}(r)$  contains the scalar matrices  $eI_r$  for all  $e \in E^\times$ , and for such matrices the similitude is  $N_{E/k}(e)$ . Therefore, the image under  $\mu$  of the center of  $\mathrm{GU}(r)(k)$  is  $N_{E/k}(E^\times)$ , so  $\mu$  thus gives an isomorphism

$$\frac{\mathrm{GU}(r)}{\mathrm{Z}(\mathrm{GU}(r))\mathrm{U}(r)} \xrightarrow{\sim} \frac{\mathrm{Im}(\mu)}{N(E^\times)}.$$

A scalar  $a \in k^\times$  is a similitude for some linear transformation  $g$  of  $V$  if and only if for all  $v, w \in V$ , we have that  $B(gv, gw) = a \cdot B(v, w)$ . That is,  $B$  and  $a \cdot B$  are equivalent Hermitian forms. It is known that two Hermitian forms over a non-archimedean local field  $k$  are equivalent if and only if their discriminants, which are elements of  $k^\times / N(E^\times)$ , are the same. Therefore,  $B$  and  $aB$  are equivalent if and only if  $\mathrm{disc} B = a^r \mathrm{disc} B$  in  $k^\times / N(E^\times) \cong \mathbf{Z}/2$ . Thus, if  $r$  is even, then  $B$  and  $aB$  are equivalent for  $a$  an arbitrary element of  $k^\times$ , but if  $r$  is odd, then  $a$  must lie in  $N(E^\times)$ . Thus,

$$\frac{\mathrm{GU}(r)}{\mathrm{Z}(\mathrm{GU}(r))\mathrm{U}(r)} \cong \mathbf{Z}/2 \quad \text{or} \quad \{1\}.$$

- (b) Let  $R_E$  and  $P_E$  denote the ring of integers and prime ideal for  $E$ . The determinant character gives us an isomorphism

$$\det: \frac{\mathrm{U}(r)(k)}{\mathrm{Z}(\mathrm{U}(r))(k)\mathrm{SU}(r)(k)} \xrightarrow{\sim} \frac{E_1}{(E_1)^r}.$$

As an abstract group,  $E_1$  inherits a direct product decomposition from  $R_E^\times \cong k_E^\times \times (1 + P_E)$ . Thus,  $E_1$  is a direct product of a cyclic group (of order coprime to  $p$ ) and a pro- $p$ -group  $A$ , implying that  $E_1/E_1^r$  is cyclic if and only if  $A/A^r$  is cyclic. But this latter quotient is trivial if  $r$  is coprime to  $p$ , and is cyclic if  $k = \mathbf{Q}_p$  ( $p$  odd).

- (c) This follows from the previous two parts of the theorem.  $\square$

## 8. AN EXAMPLE OF MULTIPLICITY UPON RESTRICTION

Let  $\omega$  be a uniformizer of  $k$ ,  $E/k$  an unramified quadratic extension,  $R_k$  and  $R_E$  the rings of integers in  $k$  and  $E$ , and  $\mathfrak{f}_k$  and  $\mathfrak{f}_E$  the residue fields. Let  $V$  be a  $4d$ -dimensional hermitian space over  $E$ , with hyperbolic basis  $\{e_1, f_1, \dots, e_{2d}, f_{2d}\}$ . Thus,  $\langle e_i, f_i \rangle = 1$  for all  $1 \leq i \leq 2d$ , and all the other products being 0. Let  $\mathrm{U}(V)$  be the corresponding unitary group. Define the lattice  $\mathcal{L}$  in  $E$  by

$$\mathcal{L} = \mathrm{span}_{R_E} \{e_1, f_1, \dots, e_d, f_d, \omega e_{d+1}, f_{d+1}, \dots, \omega e_{2d}, f_{2d}\}.$$

Clearly,  $\mathcal{L}^\vee := \{v \in V \mid \langle v, \ell \rangle \in R_E \text{ for all } \ell \in \mathcal{L}\}$  is given by

$$\mathcal{L}^\vee = \mathrm{span}_{R_E} \{e_1, f_1, \dots, e_d, f_d, e_{d+1}, \omega^{-1} f_{d+1}, \dots, e_{2d}, \omega^{-1} f_{2d}\}.$$

Observe that

$$\omega\mathcal{L}^\vee \subseteq \mathcal{L} \subseteq \mathcal{L}^\vee,$$

and  $\mathcal{L}^\vee/\mathcal{L}$  and  $\mathcal{L}/\omega\mathcal{L}^\vee$  are  $2d$ -dimensional hermitian spaces over  $\mathfrak{f}_E$  with natural hermitian structures. For example, given two elements  $\ell_1$  and  $\ell_2$  in  $\mathcal{L}^\vee$  with images  $\bar{\ell}_1$  and  $\bar{\ell}_2$  in  $\mathcal{L}^\vee/\mathcal{L}$ , the hermitian structure on  $\mathcal{L}^\vee/\mathcal{L}$  is defined by having  $\langle \bar{\ell}_1, \bar{\ell}_2 \rangle$  to be the image of  $\omega\langle \ell_1, \ell_2 \rangle$  (which belongs to  $R_E$ ) in  $\mathfrak{f}_E$ .

Define  $K = \mathrm{U}(\mathcal{L})$  to be the stabilizer of the lattice  $\mathcal{L}$  in  $\mathrm{U}(V)$ , i.e.,  $\mathrm{U}(\mathcal{L}) = \{g \in \mathrm{U}(V) \mid g\ell \in \mathcal{L} \text{ for all } \ell \in \mathcal{L}\}$ . If an element of  $\mathrm{U}(V)$  preserves  $\mathcal{L}$ , then it clearly preserves  $\mathcal{L}^\vee$  and  $\omega\mathcal{L}$ , giving a map  $\mathrm{U}(\mathcal{L}) \rightarrow \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f})$ . Similarly, we have a map  $\mathrm{SU}(\mathcal{L}) \rightarrow \mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$ .

Let  $g_0 \in \mathrm{GU}(V)$  be defined by (for  $i \leq d$ )

$$e_i \mapsto e_{d+i}, \quad f_i \mapsto \omega^{-1}f_{d+i}, \quad e_{d+i} \mapsto \omega^{-1}e_i, \quad f_{d+i} \mapsto f_i,$$

Clearly,  $g_0$  has similitude factor  $\omega^{-1}$ , and  $g_0\mathcal{L} = \mathcal{L}^\vee$ . Therefore, we have

$$g_0 \mathrm{U}(\mathcal{L}) g_0^{-1} = \mathrm{U}(\mathcal{L}^\vee).$$

Thus conjugation by  $g_0$  induces an isomorphism of  $\mathrm{U}(\mathcal{L})$  into  $\mathrm{U}(\mathcal{L}^\vee)$ , making the following diagram commute:

$$\begin{array}{ccc} \mathrm{U}(\mathcal{L}) & \xrightarrow{g_0} & \mathrm{U}(\mathcal{L}^\vee) \\ \downarrow & & \downarrow \\ \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f}) & \xrightarrow{j} & \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f}) \end{array}$$

where  $j(x, y) = (y, x)$ .

**Theorem 13.** *Let  $\rho$  be any irreducible cuspidal representation of  $\mathrm{U}(2d)(\mathfrak{f})$  such that  $\rho \not\cong \rho\chi$ , where  $\chi$  is a quadratic character of  $\mathrm{U}(2d)(\mathfrak{f})$  trivial on  $\mathrm{SU}(2d)(\mathfrak{f})$ . Let  $\sigma := \mathrm{infl}(\rho \otimes \rho\chi)$  denote the inflation of  $\rho \otimes \rho\chi$  from  $(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$  to  $\mathrm{U}(\mathcal{L})$  and let  $\pi = \mathrm{c}\text{-Ind}_{\mathrm{U}(\mathcal{L})}^{\mathrm{U}(V)} \sigma$ . Then  $\pi \oplus \pi^{g_0}$  extends to an irreducible representation  $\tilde{\pi}$  of  $\mathrm{GU}(V)$  whose restriction to  $\mathrm{SU}(V)$  decomposes with multiplicity two.*

*Proof.* From Moy-Prasad [7, Proposition 6.6],  $\pi$  is an irreducible, supercuspidal representation of  $\mathrm{U}(V)$ . Let  $\pi$  also denote one of its extensions to  $Z(\mathrm{GU}(V))\mathrm{U}(V)$ . From the last sentence of [6, Theorem 5.2],  $\pi^{g_0} \not\cong \pi$ , so the sum  $\pi \oplus \pi^{g_0}$  extends to an irreducible (also supercuspidal) representation  $\tilde{\pi}$  of  $\mathrm{GU}(V)$ . By the induction-restriction formula (observe that by the explicit description of  $\mathrm{U}(\mathcal{L})$ ,  $\det : \mathrm{U}(\mathcal{L}) \rightarrow E_1$  is surjective, and hence  $\mathrm{U}(\mathcal{L})\mathrm{SU}(V) = \mathrm{U}(V)$ ),

$$\begin{aligned} \pi|_{\mathrm{SU}(V)} &= \mathrm{c}\text{-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma|_{\mathrm{SU}(\mathcal{L})}), \\ \pi^{g_0}|_{\mathrm{SU}(V)} &= \mathrm{c}\text{-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma^{g_0}|_{\mathrm{SU}(\mathcal{L})}) \end{aligned}$$

Since  $\rho \otimes \rho\chi \cong \rho\chi \otimes \rho$  as representations of  $\mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$ , we have that  $\sigma \cong \sigma^{g_0}$  as representations of  $\mathrm{SU}(\mathcal{L})$ , so

$$\tilde{\pi}|_{\mathrm{SU}(V)} = (\pi \oplus \pi^{g_0})|_{\mathrm{SU}(V)} = 2 \cdot \mathrm{c}\text{-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma|_{\mathrm{SU}(\mathcal{L})}). \quad \square$$

In order to have an example of multiplicity at least two, it is thus sufficient to find a representation  $\rho$  of  $U(2d)(f)$  such that  $\rho \not\cong \rho\chi$ , as in the theorem. In fact, most irreducible Deligne-Lusztig cuspidal representations of  $U(2d)(f)$  will have this property, as they restrict irreducibly to  $SU(2d)(f)$ .

**Remark 14.** In a future work, we will expand upon the example given in the Theorem, whose essence is the following. Given a supercuspidal representation of  $G_2(k)$  whose restriction to  $G_1(k)$  has regular components (in the sense of Kaletha [4]), then the components occur with multiplicity one. (Nevens [8] already verified this for many cases.) If the components are not regular, then higher multiplicities can occur.

Our example begins with  $\rho$ , an irreducible cuspidal representation of  $U(2d)(f)$  that arises via Deligne-Lusztig induction from a character  $\theta$  of the group of  $f$ -points of an anisotropic torus  $T \subset U(2d)$ . Suppose also that the restriction of  $\theta$  to  $T(f) \cap SU(2d)(f)$  remains regular so that the restriction of  $\rho$  to  $SU(2d)(f)$  remains irreducible. The torus  $T \times T \subset U(2d) \times U(2d)$  lifts to give an unramified torus  $T \subset GU(V)$ , and the character  $\theta \otimes \theta\chi$  can be inflated and extended to give a character  $\Theta$  of  $T$ . The representation  $\tilde{\pi}$  of  $GU(V)$  that we have constructed in the theorem is a regular supercuspidal representation in the sense of Kaletha [4], but the irreducible components of its restriction to  $SU(V)$  are not since our character  $\Theta$  of  $T$ , when restricted to  $T \cap SU(V)$ , is not regular because of the presence of the element  $g_0 \in GU(V)$ .

For depth-zero supercuspidal representations of quasi-split unitary groups, the parahoric that we have used is the only one that can lead to higher multiplicities.

## 9. GENERALITIES ON CONSTRUCTING HIGHER MULTIPLICITIES

In this section, we discuss some generalities underlying the example of the previous section, which will be useful for constructing higher multiplicities in general.

Let  $G$  be a group, and  $N$  a normal subgroup of  $G$  such that

$$G/N \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

A good example to keep in mind is  $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ , the quaternion group of order 8, and  $N = \{\pm 1\}$ . Let  $\omega_1$  and  $\omega_2$  be two distinct, nontrivial characters of  $G$  that are trivial on  $N$ .

Suppose  $\pi$  is an irreducible representation of  $G$  such that

$$\pi \cong \pi \otimes \omega_1 \cong \pi \otimes \omega_2.$$

By [3, §2],  $\pi|_N$  must be one of the following:

- (1) a sum of four inequivalent, irreducible representations, or
- (2) a sum of two copies of an irreducible representation.

Deciding which of these two options we have is a subtle question, and this is what we wish to do here.

Let  $N_1 = \ker\{\omega_1: G \rightarrow \mathbf{Z}/2\}$ , so that  $G \supset N_1 \supset N$ . Since  $\pi \cong \pi \otimes \omega_1$ ,  $\pi|_{N_1} = \pi_1 \oplus \pi_2$ , a sum of inequivalent, irreducible representations. Further, since  $\pi \cong \pi \otimes \omega_2$ , we have

$$(\pi_1 \oplus \pi_2) \cong (\pi_1 \oplus \pi_2) \otimes \omega_{21},$$

where  $\omega_{21} = \omega_2|_{N_1}$ , a nontrivial character of  $N_1$  of order 2. Therefore, we have the following two possibilities

- (i)  $\pi_1 \cong \pi_1 \otimes \omega_{21}$ ,
- (ii)  $\pi_2 \cong \pi_1 \otimes \omega_{21}$ ,

In case (i),  $\pi_1$ , which is an irreducible representation of  $N_1$ , decomposes when restricted to  $N$  into two inequivalent irreducible representations, and therefore  $\pi$  has at least two inequivalent irreducible subrepresentations when restricted to  $N$ , hence in case (i),

$$\pi|_N = \text{a sum of 4 inequivalent, irreducible representations.}$$

In case (ii), clearly  $\pi|_N$  is twice an irreducible representation.

How does one then construct an example of an irreducible representation  $\pi$  of  $G$  for which  $\pi|_N$  is twice an irreducible representation? We start with an irreducible representation  $\pi_1$  of  $N_1$  such that the following equivalent conditions hold:

- (i)  $\pi_1$  does not extend to a representation of  $G$ ;
- (ii)  $\pi_1^g \not\cong \pi_1$  for some  $g \in G$ .

Given such a representation  $\pi_1$  of  $N_1$ , next we must ensure that

- (iii)  $\pi_1^g \cong \pi_1 \otimes \omega_{21}$  for  $g \in G \setminus N$ .

If we understand  $N_1$ , together with the action of  $G$  on the representations of  $N_1$ , then the condition

$$\pi_1^g \cong \pi_1 \otimes \omega_{21} \not\cong \pi_1$$

is checkable, constructing an irreducible representation  $\pi = \text{Ind}_{N_1}^G \pi_1$  of  $G$  such that

$$\pi|_N = 2\pi_1|_N.$$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY, 4400 MASSACHUSETTS AVE  
NW, WASHINGTON, DC 20016-8050, USA  
*E-mail address:* [jadler@american.edu](mailto:jadler@american.edu)

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD,  
COLABA, MUMBAI, INDIA

LABORATORY OF MODERN ALGEBRA AND APPLICATIONS, SAINT-PETERSBURG STATE UNIVERSITY, RUS-  
SIA  
*E-mail address:* [prasad.dipendra@gmail.com](mailto:prasad.dipendra@gmail.com)