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DISTINGUISHED REPRESENTATIONS FOR SL(n)

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Abstract. For E/F a quadratic extension of local fields, and π an irreducible admissible generic representation of $\mathrm{SL}_n(E)$, we calculate the dimension of $\mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ and relate it to fibers of the base change map corresponding to base change of representations of $\mathrm{SU}_n(F)$ to $\mathrm{SL}_n(E)$ as suggested in [16]. We also deal with finite fields.

1. Introduction

The paper [16] formulates a general conjecture - in terms of Langlands parameters, more specifically in terms of fibers of a certain base change map - on the dimension of the space $\mathrm{Hom}_{G(F)}[\pi, \mathbb{C}]$ for an irreducible admissible representation π of $G(E)$ where G is a general reductive group over a local field F, and E/F is a quadratic extension of fields. In this paper, we consider the case of $G = \mathrm{SL}_n$. The main theorem of this paper, Theorem 5.6, computes $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ for an irreducible admissible generic representation π of $\mathrm{SL}_n(E)$ in terms of the fiber of the base change map from $\mathrm{SU}(n)$ to $\mathrm{SL}_n(E)$, and thus confirms the general conjecture in [16] for $G = \mathrm{SL}_n$. The dimension of $\mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ was computed earlier in [4] when $n = 2$ and in [1] when n is odd. This paper could be considered a natural sequel to these two earlier works, but now considered more from the point of view of base change from unitary groups.

The symmetric space $(\mathrm{SL}_2(E), \mathrm{SL}_2(F))$ studied in [4] was the first example in the literature which is not a supercuspidal Gelfand pair, that is to say the symmetric space affords irreducible supercuspidal representations with multiplicity > 1. In contrast with the $n = 2$ case, when n is odd, it was proved in [1] that the symmetric space $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ is a Gelfand pair, i.e., for any irreducible admissible representation π of $\mathrm{SL}_n(E)$, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 1$. In this paper we reconsider the multiplicity one theorem of [1] for $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ as well as go a little further for n even.

The pair $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ is much simpler than the general pair $(G(E), G(F))$ among other things because the adjoint group of $\mathrm{SL}_n(E)$, i.e., $\mathrm{PGL}_n(E)$, operates transitively on an L-packet of $\mathrm{SL}_n(E)$, and in fact $\mathrm{PGL}_n(F)$ operates transitively on those representations of $\mathrm{SL}_n(E)$ in a given generic L-packet of $\mathrm{SL}_n(E)$ for which $\mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \neq 0$, and clearly $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ is the same for all representations of $\mathrm{SL}_n(E)$ which are conjugate under $\mathrm{PGL}_n(F)$.

Before we end the introduction, let us briefly describe the main ingredients in this work. There are two non-obvious inputs in our work. First, a recent work of Matringe describes exactly which generic representations of $\mathrm{GL}_n(E)$ are distinguished by $\mathrm{GL}_n(F)$ [13]. This allows one to make some headway into understanding $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ where π is an irreducible, admissible generic representation of

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$\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$ and is contained in an irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ distinguished by $\mathrm{GL}_n(F)$. Second, we are able to say that the set of irreducible, admissible representations of $\mathrm{SL}_n(E)$ contained inside $\widetilde{\pi}$ and which are distinguished by $\mathrm{SL}_n(F)$, are in a single orbit under the action of $\mathrm{GL}_n(F)$. This follows from a more precise result according to which an irreducible, admissible generic representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$ must have a Whittaker model for a non-degenerate character of $N(E)/N(F)$ where N is the group of upper-triangular unipotent matrices. This is a consequence of some recent work of the first author with Matringe [3], for which we have given a more direct proof but one which is valid only for tempered representations, or more generally unitary representations.

Most of the paper is written both for p-adic as well as for finite fields since methods are essentially uniform. It might be mentioned that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ for F a finite field which we prove in this paper to be bounded by 2 (and bounded by 1 if n is odd, or p is even) was not known in any precise way in the literature. Throughout the paper, when dealing with p-adic fields, we will assume that they have characteristic zero unless otherwise mentioned.

2. Preliminaries

In this paper, E will be a quadratic extension of either a p-adic or a finite field F. Let $\widetilde{G} = \mathrm{GL}_n(E)$, $\widetilde{H} = \mathrm{GL}_n(F)$, $G = \mathrm{SL}_n(E)$, and $H = \mathrm{SL}_n(F)$. An irreducible admissible representation of $\widetilde{G}$ will usually be denoted by $\widetilde{\pi}$ and that of G by π. Let σ be the non-trivial element of the Galois group $\mathrm{Gal}(E/F)$. Let $\mathbb{N}m : E^\times \to F^\times$ be the norm map. If F is p-adic, the quadratic character of $F^\times/N_{E/F}(E^\times)$ is denoted by $\omega_{E/F}$; if F is finite, we let $\omega_{E/F} = 1$.

For a p-adic field k, let $W_k' = W_k \times \mathrm{SL}_2(\mathbb{C})$ be its Weil-Deligne group where W_k is the Weil group of k. A Langlands parameter of W_k' valued in $\mathrm{GL}(n, \mathbb{C})$, for some n, will typically be denoted by $\widetilde{\rho}$ and a Langlands parameter of W_k' valued in $\mathrm{PGL}_n(\mathbb{C})$ will typically be denoted by ρ.

For a representation τ of a group, $\tau^\vee$ stands for the contragredient representation, and ω_τ denotes its central character (if it has one). For a representation τ of $\widetilde{G}$ or G, τ^σ is the Galois conjugate representation given by $\tau^\sigma(g) = \tau(g^\sigma)$. Similarly for a Langlands parameter τ of W_E', its Galois conjugate is given by $\tau^\sigma(g) = \tau(\sigma^{-1} g \sigma)$. A representation π of $\mathrm{GL}_n(E)$ (or its Langlands parameter) is said to be conjugate self-dual if $\pi^\sigma \cong \pi^\vee$. Conjugate self-dual representations of $\mathrm{GL}_n(E)$ (or its Langlands parameter) come in two flavors: conjugate orthogonal and conjugate symplectic (to be referred to as having parity 1 and -1 respectively); we refer to [10] for the definition. Note that a conjugate self-dual representation may be neither conjugate orthogonal nor conjugate symplectic, and that these two options are not mutually exclusive either. This paper will deal exclusively with representations/parameters which are either conjugate orthogonal or conjugate symplectic since they are the only ones relevant for distinction by $\mathrm{GL}_n(F)$.

For a character α of $F^\times$, an irreducible admissible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is said to be α-distinguished if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}[\widetilde{\pi}, \alpha] \neq 0;$$

here, as elsewhere in the paper, we identify a character α of $F^\times$ to a character of $\mathrm{GL}_n(F)$ via the determinant map $\det : \mathrm{GL}_n(F) \to F^\times$. If $\alpha = 1$, an α-distinguished representation is also said to be distinguished by $\mathrm{GL}_n(F)$.

The most basic result about distinguished representations for $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ is the following result due to Flicker which is proved by the well-known Gelfand-Kazhdan method [8, Propositions 11 & 12].

Proposition 2.1. *If $\widetilde{\pi}$ is an irreducible admissible representation of $\mathrm{GL}_n(E)$ which is $\mathrm{GL}_n(F)$-distinguished, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}[\widetilde{\pi}, \mathbb{C}] = 1,$$

and furthermore, $\widetilde{\pi}^\vee \cong \widetilde{\pi}^\sigma$ (and also $\omega_{\widetilde{\pi}}|_{F^\times} = 1$).

The following theorem due to Matringe [13, Theorem 5.2] is much more precise (which builds on the earlier works on discrete series representations [12, 2]).

Proposition 2.2. *Let $\widetilde{\pi}$ be an irreducible admissible generic representation of $\mathrm{GL}_n(E)$ which is conjugate self-dual. Write*

$$\widetilde{\pi} \cong \Delta_1 \times \cdots \times \Delta_t$$

as the representation parabolically induced from irreducible essentially square integrable representations Δ_i of $\mathrm{GL}_{n_i}(E)$, with $n = n_1 + \cdots + n_t$, and where the segments associated to Δ_i's are not linked (in the sense of Bernstein-Zelevinsky). Then, $\widetilde{\pi}$ is distinguished by $\mathrm{GL}_n(F)$ if and only if after a reordering of the indices if necessary,

(1) $\Delta_{i+1}^\sigma \cong \Delta_i^\vee$, for $i = 1, 3, \ldots, 2r-1$, for some integer $r \geq 0$, and
(2) $\Delta_i^\sigma \cong \Delta_i^\vee$, for $2r < i \leq t$, and the discrete series representations Δ_i of $\mathrm{GL}_{n_i}(E)$ are distinguished by $\mathrm{GL}_{n_i}(F)$.

The following result was known as the Flicker-Rallis conjecture, and is now a theorem by combining [14, Lemma 2.2.1] (see also [10, Theorem 8.1]) and [13, Theorem 5.2].

Theorem 2.3. *An irreducible admissible generic representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is distinguished by $\mathrm{GL}_n(F)$ if n is odd, and $\omega_{E/F}$-distinguished if n is even, if and only if its Langlands parameter is in the image of the restriction map*

$$\Phi : H^1(W_F', \widehat{\mathrm{U}}_n) \to H^1(W_E', \mathrm{GL}_n(\mathbb{C})),$$

where $\widehat{\mathrm{U}}_n$ is the Langlands dual group of a unitary group defined by a hermitian space of dimension n over E (which comes equipped with an action of W_F'). Equivalently, an irreducible admissible generic representation of $\mathrm{GL}_n(E)$ is distinguished by $\mathrm{GL}_n(F)$ precisely when it is a conjugate orthogonal representation.

For the case of finite fields, we will need to use the following result due to Gow [11, Theorem 3.6] (see also [15]).

Proposition 2.4. *For E/F a quadratic extension of finite fields, an irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is distinguished by $\mathrm{GL}_n(F)$ if and only if $\widetilde{\pi}^\sigma \cong \widetilde{\pi}^\vee$.*

As this paper deals with representations of $\mathrm{SL}_n(E)$ through restriction of representations from $\mathrm{GL}_n(E)$ to $\mathrm{SL}_n(E)$, and similarly deals with representations of special unitary groups through restriction of representations from unitary groups, we will need to use twisting representations of $\mathrm{GL}_n(E)$, or parameters of them, by characters of $E^\times$, or in the case of unitary groups, by characters of $E^1 = E^\times/F^\times$.

This motivates us to introduce *Strong and Weak Equivalences* among representations of $\mathrm{GL}_n(E)$, or parameters of them.

Two Langlands parameters of W'_E with values in $\mathrm{GL}_n(\mathbb{C})$ will be said to be weakly equivalent if they are twists of each other by a character of $E^\times$, and they will be said to be strongly equivalent if they are twists of each other by a character of $E^\times/F^\times$, i.e.,

$$\widetilde{\rho}_2 \sim_w \widetilde{\rho}_1 \iff \widetilde{\rho}_2 \cong \widetilde{\rho}_1 \otimes \chi \text{ for } \chi : E^\times \to \mathbb{C}^\times, \tag{1}$$

and

$$\widetilde{\rho}_2 \sim_s \widetilde{\rho}_1 \iff \widetilde{\rho}_2 \cong \widetilde{\rho}_1 \otimes \chi \text{ for } \chi : E^\times/F^\times \to \mathbb{C}^\times. \tag{2}$$

We denote the weak (resp. strong) equivalence class by $[\cdot]_w$ (resp. $[\cdot]_s$), and the set of strong equivalence classes in the weak equivalence class containing a representation $\widetilde{\pi}$ by $[\widetilde{\pi}]_w/\sim_s$.

In this paper, we will use these equivalence relations among representations of the same parity $= (-1)^{n-1}$. If $\widetilde{\rho}$ is one such representation, the number of strong equivalence classes in the weak equivalence class of $\widetilde{\rho}$ (among representations of the same parity as $\widetilde{\rho}$) will be denoted by $q(\widetilde{\rho})$.

Clearly, the same notions can be defined on the class of irreducible admissible conjugate self-dual representations $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ of a given parity, and as for parameters, we will denote by $q(\widetilde{\pi})$ the number of strong equivalence classes in the weak equivalence class of $\widetilde{\pi}$ (among conjugate self-dual representations of $\mathrm{GL}_n(E)$ of the same parity as $\widetilde{\pi}$).

We remark that *Strong and Weak Equivalences* among representations of $\mathrm{GL}_n(E)$ was first introduced in [1].

3. Distinction for $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$

The subgroup of $\mathrm{GL}_n(E)$ defined by

$$\mathrm{GL}_n(E)^+ = \{g \in \mathrm{GL}_n(E) \mid \det g \in F^\times\} = \mathrm{GL}_n(F)\mathrm{SL}_n(E),$$

will play an important role in our analysis as we will consider the restriction of an irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ to $\mathrm{SL}_n(E)$ in two stages. First we restrict $\widetilde{\pi}$ to $\mathrm{GL}_n(E)^+$ and write it as a direct sum of irreducible representations, and then we look at the restriction of each of these direct summands to $\mathrm{SL}_n(E)$. This was indeed the strategy employed in [4]. In the paper [4], we had added the center $E^\times$ too in the definition of $\mathrm{GL}_n(E)^+$, but since center acts by a scalar, this makes no difference, whereas for a later argument in this paper, where we will have to make an induction on n, the present definition is better.

Note the following simple lemma:

Lemma 3.1. *All the irreducible constituents of the restriction of a representation of* $\mathrm{GL}_n(E)^+$ *to* $\mathrm{SL}_n(E)$ *admit the same number of linearly independent* $\mathrm{SL}_n(F)$*-invariant functionals.*

Proof. Since $\mathrm{GL}_n(F)\mathrm{SL}_n(E) = \mathrm{GL}_n(E)^+$, all the irreducible constituents of the restriction of a representation of $\mathrm{GL}_n(E)^+$ to $\mathrm{SL}_n(E)$ are conjugates to one another under the inner conjugation action of $\mathrm{GL}_n(F)$ on $\mathrm{SL}_n(F)$, proving the lemma. □

For an irreducible, admissible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$, define the sets $X_{\widetilde{\pi}}$, $X'_{\widetilde{\pi}}$, $Y_{\widetilde{\pi}}$, $Z_{\widetilde{\pi}}$ as follows:

(1) $X_{\widetilde{\pi}} = \{\alpha \in \widehat{F^\times} \mid \widetilde{\pi} \text{ is } \alpha\text{-distinguished}\}$,
(2) $X'_{\widetilde{\pi}} = \{\alpha \in \widehat{F^\times} \mid \widetilde{\pi} \text{ is } \alpha\text{-distinguished or } \alpha \cdot \omega_{E/F}\text{-distinguished}\}$,
(3) $Z_{\widetilde{\pi}} = \{\chi \in \widehat{E^\times} \mid \widetilde{\pi} \otimes \chi \cong \widetilde{\pi}\}$,
(4) $Y_{\widetilde{\pi}} = \{\chi \in \widehat{E^\times} \mid \widetilde{\pi} \otimes \chi \cong \widetilde{\pi}, \chi|_{F^\times} = 1\}$.

Observe that $Z_{\widetilde{\pi}}, Y_{\widetilde{\pi}}$ are abelian groups, whereas $X_{\widetilde{\pi}}, X'_{\widetilde{\pi}}$ are just sets, and that characters of $E^\times$ in $Z_{\widetilde{\pi}}$ when restricted to $F^\times$ act on the sets $X_{\widetilde{\pi}}, X'_{\widetilde{\pi}}$ by translation, giving rise to a faithful action of $Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$ on the sets $X_{\widetilde{\pi}}, X'_{\widetilde{\pi}}$. Characters in $Z_{\widetilde{\pi}}$ are said to be self-twists of $\widetilde{\pi}$.

Lemma 3.2. *Let E be a quadratic extension of either a finite or a p-adic field F which if it is of positive characteristic we assume p does not divide n. Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$. Then there exists an irreducible admissible representation of $\mathrm{GL}_n(E)$ containing π upon restriction to $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{GL}_n(F)$.*

Proof. Let $\widetilde{\pi}$ be an irreducible admissible representation of $\mathrm{GL}_n(E)$ containing π upon restriction to $\mathrm{SL}_n(E)$. Consider the vector space

$$V = \mathrm{Hom}_{\mathrm{SL}_n(F)}[\widetilde{\pi}, \mathbb{C}]$$

of $\mathrm{SL}_n(F)$-invariant linear functionals on $\widetilde{\pi}$. The group $\mathrm{GL}_n(F)$ operates on V via

$$(g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v),$$

for $v \in \widetilde{\pi}$. Observe that $\mathrm{SL}_n(F)$ acts trivially on V by the definition of V, and $F^\times < \mathrm{GL}_n(F)$ acts by a character – the central character of $\widetilde{\pi}$ restricted to $F^\times$ – on V. Since,

$$\mathrm{GL}_n(F)/F^\times\mathrm{SL}_n(F) \cong F^\times/F^{\times n},$$

a finite abelian group (here we are using that if F is p-adic, the characteristic of F does not divide n), it follows that V is a finite direct sum of characters of $F^\times$ whose restriction to the n-th roots of unity $\mu_n(F)$ in $F^\times$ is trivial (since π is distinguished by $\mathrm{SL}_n(F)$). Such characters have an n-th root, i.e., $\alpha = \beta^n$ for some character β of $F^\times$ as follows from the exact sequence:

$$1 \to \mu_n(F) \to F^\times \xrightarrow{n} F^\times.$$

If

$$V = \bigoplus_{\alpha \in \widehat{F^\times}} m_\alpha \alpha,$$

take any character $\alpha = \beta^n$ appearing in this sum. Clearly, the representation $\widetilde{\pi} \otimes \tilde{\beta}$, where $\tilde{\beta}$ is any extension of β to $E^\times$, is an irreducible admissible representation of $\mathrm{GL}_n(E)$ containing π upon restriction to $\mathrm{SL}_n(E)$, and which is distinguished by $\mathrm{GL}_n(F)$, completing the proof of the lemma. □

Proposition 3.3. *Let E be a quadratic extension of either a finite or a p-adic field F. Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$, and let $\widetilde{\pi}$ be an irreducible admissible representation of $\mathrm{GL}_n(E)$ which is distinguished by $\mathrm{GL}_n(F)$ with $\widetilde{\pi}$ containing π upon restriction to $\mathrm{SL}_n(E)$. Then*

(1) If n is odd,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 1.$$

(2) If n is even,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq |\{\alpha \in \widehat{F^\times} \mid \widetilde{\pi} \otimes (\alpha \circ \mathbb{N}m) \cong \widetilde{\pi}, \alpha^2 = 1\}| \leq |F^\times / F^{\times 2}|;$$

in particular, for F a finite field, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 1$ *if F has characteristic 2, and* $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 2$ *in odd characteristics.*

Proof. Let us begin by considering the vector space

$$V = \mathrm{Hom}_{\mathrm{SL}_n(F)}[\widetilde{\pi}, \mathbb{C}]$$

of $\mathrm{SL}_n(F)$-invariant linear functionals on $\widetilde{\pi}$ as in the last lemma. The group $\mathrm{GL}_n(F)$ operates on V and we have the decomposition of V as a direct sum of characters of $F^\times$. If

$$V = \bigoplus_{\alpha \in \widehat{F^\times}} m_\alpha \alpha,$$

then $\widetilde{\pi}$ is α^{-1}-distinguished with respect to $\mathrm{GL}_n(F)$ for any character α of $F^\times$ with $m_\alpha \neq 0$. Notice also that $m_\alpha \leq 1$ for each $\alpha \in \widehat{F^\times}$, since

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}[\widetilde{\pi}, \alpha] \leq 1,$$

by the first part of Proposition 2.1. Therefore,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\widetilde{\pi}, \mathbb{C}] = |X_{\widetilde{\pi}}|\,.$$

Note that if $\widetilde{\pi}$ is α^{-1}-distinguished for a character $\alpha : F^\times \to \mathbb{C}^\times$, then if $\widetilde{\alpha}$ denotes any extension of α to $E^\times$, by Proposition 2.1 we must have,

$$(\widetilde{\pi} \otimes \widetilde{\alpha})^\sigma \cong (\widetilde{\pi} \otimes \widetilde{\alpha})^\vee.$$

This combined with the isomorphism $\widetilde{\pi}^\sigma \cong \widetilde{\pi}^\vee$ (because $\widetilde{\pi}$ is $\mathrm{GL}_n(F)$-distinguished), implies that:

$$\widetilde{\pi} \otimes (\widetilde{\alpha}^\sigma \cdot \widetilde{\alpha}) \cong \widetilde{\pi},$$

or $\alpha \circ \mathbb{N}m \in Z_{\widetilde{\pi}}$.

Sending a character α of $F^\times$ to the character $\alpha \circ \mathbb{N}m$ of $E^\times$, defines a homomorphism, call it $\mathbb{N}m$ from $\widehat{F^\times}$ to $\widehat{E^\times}$, whose restriction to $X_{\widetilde{\pi}}$ will also be denoted by the same symbol $\mathbb{N}m$,

$$\mathbb{N}m : X_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}} / Y_{\widetilde{\pi}}.$$

Note that $X_{\widetilde{\pi}}$ being only a set, the map $\mathbb{N}m$ on it is only a set theoretic map, but being the restriction of a group homomorphism, the fibers of this map are contained in translates of any particular element in the fiber by 'the kernel of the map' which consists of those characters α of $F^\times$ for which $\alpha \circ \mathbb{N}m \in Y_{\widetilde{\pi}}$, i.e., $\pi \otimes (\alpha \circ \mathbb{N}m) \cong \pi$ and $\alpha \circ \mathbb{N}m|_{F^\times} = \alpha^2 = 1$. By central character considerations, we already know that if χ and $\chi \cdot \alpha$ both belong to $X_{\widetilde{\pi}}$, then $\alpha^n = 1$. Therefore if n is odd, the map of sets $\mathbb{N}m : X_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}} / Y_{\widetilde{\pi}}$, is injective, and if n is even, any fiber of this map has order at most the number of characters α of $F^\times$ with $\pi \otimes (\alpha \circ \mathbb{N}m) \cong \pi$ and $\alpha^2 = 1$.

It is clear that an irreducible representation of

$$\mathrm{GL}_n(F)\mathrm{SL}_n(E) = \mathrm{GL}_n(E)^+$$

when restricted to $\mathrm{SL}_n(E)$ has $|Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}|$ many irreducible components, and since $\mathrm{GL}_n(F)$ acts transitively on these irreducible representations of $\mathrm{SL}_n(E)$, the number of $\mathrm{SL}_n(F)$-invariant linear forms on $\widetilde{\pi}$ contributed by that irreducible representation of $\mathrm{GL}_n(E)^+$ which contains π equals $|Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}| \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$. On the other hand, the space of $\mathrm{SL}_n(F)$-invariant linear forms on $\widetilde{\pi}$ has dimension equal to $|X_{\widetilde{\pi}}|$. Thus, we get the obvious inequality:

$$|Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}| \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq |X_{\widetilde{\pi}}|.$$

Now, the properties of the mapping $\mathbb{N}m : X_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$ discussed earlier proves parts (1) and (2) of the proposition. $\square$

Remark 1. The multiplicity one property for n odd was already proved in [1] by a similar method as above. It is not clear to the authors if this multiplicity one property is a consequence of 'Gelfand's trick'.

The following proposition refines the earlier proposition when $X'_{\widetilde{\pi}}$ is known to be a group, for example, when F is a finite field, or when F is a p-adic field, and $\widetilde{\pi}$ is a discrete series representation.

Proposition 3.4. *Let E/F be a quadratic extension of either finite or p-adic fields. Let π be an irreducible admissible discrete series representation of $\mathrm{SL}_n(E)$ if F is p-adic, and any irreducible representation of $\mathrm{SL}_n(E)$ if F is a finite field. Assume π is distinguished by $\mathrm{SL}_n(F)$ and is contained in an irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ distinguished by $\mathrm{GL}_n(F)$. Let $\mathbb{N}m : X'_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$ be the norm map defined earlier. Let $c(F) = 2$ if F is a p-adic field, and $c(F) = 1$ if F is a finite field. Then for n an even integer,*

$$\begin{aligned} c(F) \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] &= \frac{|X'_{\widetilde{\pi}}|}{|Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}|} \\ &= \frac{|Ker\ \mathbb{N}m|}{|Coker\ \mathbb{N}m|} \\ &= \frac{|\{\chi \in \widehat{F^\times} \mid \widetilde{\pi} \otimes (\chi \circ \mathbb{N}m) \cong \widetilde{\pi}, \chi^2 = 1\}|}{|\{\lambda|_{F^\times} \text{ such that } \lambda \text{ is a self-twist of } \tilde{\pi}\}/2X'_{\widetilde{\pi}}|}. \end{aligned}$$

Proof. The proof of this proposition follows the same strategy which was used in the proof of the previous proposition by using the following additional inputs:

(1) A discrete series representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is $\omega_{E/F}$-distinguished or distinguished if and only if

$$\widetilde{\pi}^\sigma \cong \widetilde{\pi}^\vee;$$

furthermore, such a representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is either distinguished or $\omega_{E/F}$-distinguished, with exactly one possibility. This follows from [12, Theorem 7] and [2, Corollary 1.6] if F is a p-adic field, and a consequence of Proposition 2.4 if F is finite (where $\omega_{E/F} = 1$). This implies in particular that $X'_{\widetilde{\pi}}$ is a group and the map

$$\mathbb{N}m : X'_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}},$$

is now a group homomorphism, whose kernel is

$$\{\chi \in \widehat{F^\times} \mid \widetilde{\pi} \otimes (\chi \circ \mathbb{N}m) \cong \widetilde{\pi}, \chi^2 = 1\}.$$

(2) The restriction of $\widetilde{\pi}$ to $\mathrm{GL}_n(E)^+$ has exactly one irreducible representation - the one which carries a Whittaker functional for a character of $N(E)/N(F)$ - which is distinguished by $\mathrm{SL}_n(F)$; this is the content of the next section.

The first two equalities in the statement of the proposition follows from these. For the last equality in the statement of the proposition, observe that

(a) The natural map $j : Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}} \to X_{\widetilde{\pi}}$ is injective, and
(b) the composition of the maps: $X'_{\widetilde{\pi}} \xrightarrow{\mathbb{N}m} Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}} \xrightarrow{j} X'_{\widetilde{\pi}}$ is multiplication by 2.

□

Corollary 3.5. *Assume that F is a finite field, n is even, and the representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is distinguished by $\mathrm{GL}_n(F)$. Then if $\widetilde{\pi}$ does not have a self-twist by the unique character of $E^\times$ order 2 (such a character of $E^\times$ comes from $F^\times$ through the norm map),*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 1.$$

If $\widetilde{\pi}$ has a self-twist by the unique character of $E^\times$ of order 2, and also by a character χ with $\chi(-1) = -1$, then also

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \leq 1.$$

Proof. Observe that the image of the map $\mathbb{N}m : X_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$, consists of those characters on $F^\times$ whose value on $-1 \in F^\times$ is 1. Therefore, if there is a self-twist of π by a character χ of $E^\times$ with $\chi(-1) \neq 1$, then the map $\mathbb{N}m : X_{\widetilde{\pi}} \longrightarrow Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$, could not be surjective. This allows one to prove the corollary. □

Remark 2. Proposition 3.4 allows us to calculate $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ (which we already know is ≤ 2) in all cases for F a finite field even if in the above corollary, we have not handled all the cases. We just want to add the observation – without proof – that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$ for π an irreducible representation of $\mathrm{SL}_n(E)$ as well as $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}[\widetilde{\pi}, \mathbb{C}]$ for $\widetilde{\pi}$ an irreducible representation of $\mathrm{GL}_n(E)$, depends only on the *semisimple part of the Jordan decomposition* of $\pi, \widetilde{\pi}$ (in the sense of Lusztig).

The next proposition follows from the method of proof of Proposition 3.3 (using that a generic distinguished representation of $\mathrm{SL}_n(E)$ is generic for a character of $N(E)/N(F)$ for which we refer to the next section). For $n = 2$, this proposition is [4, Theorem 1.4] and for a tempered representation π for any n, this is [1, Theorem 4.3]).

Proposition 3.6. *Let π be an irreducible admissible generic representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$ and contained in an irreducible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ distinguished by $\mathrm{GL}_n(F)$. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] = \frac{|X_{\widetilde{\pi}}|}{|Z_{\widetilde{\pi}}|/|Y_{\widetilde{\pi}}|}.$$

Remark 3. That the right hand side of the identity in Proposition 3.6 is indeed a positive integer can be observed independently. Indeed, the group $Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$ acts freely on $X_{\widetilde{\pi}}$, and hence it is the number of orbits under this action.

Our next result relates distinction for the symmetric space $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ to the notions of strong and weak equivalence defined at the end of §2.

Proposition 3.7. *Let π be an irreducible admissible generic representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$. Let $\widetilde{\pi}$ be an irreducible admissible generic representation of $\mathrm{GL}_n(E)$ which contains π on restriction to $\mathrm{SL}_n(E)$, and is distinguished by $\mathrm{GL}_n(F)$. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] = q(\widetilde{\pi}),$$

where $q(\widetilde{\pi})$ is the number of strong equivalence classes in the weak equivalence class of $\widetilde{\pi}$, i.e., the cardinality of the set $[\widetilde{\pi}]_w/\sim_s$ (inside conjugate orthogonal representations of $\mathrm{GL}_n(E)$).

Proof. If α is a character of $F^\times$ in $X_{\widetilde{\pi}}$ and if $\widetilde{\alpha}$ is any extension of α to $E^\times$, then by the definition of $X_{\widetilde{\pi}}$, $\widetilde{\pi}[\widetilde{\alpha}] = \widetilde{\pi} \otimes \widetilde{\alpha}^{-1}$ is distinguished by $\mathrm{GL}_n(F)$, hence by Theorem 2.3, it is a conjugate orthogonal representation, therefore $\widetilde{\pi}[\widetilde{\alpha}] \in [\widetilde{\pi}]_w$; different extensions $\widetilde{\alpha}$ of α give rise to elements in a given strong equivalence class, thus $\widetilde{\pi}[\widetilde{\alpha}]$ as an element of $[\widetilde{\pi}]_w/\sim_s$ depends only on α. Since by Theorem 2.3, conjugate orthogonal generic representations are precisely the irreducible admissible generic representations of $\mathrm{GL}_n(E)$ that are distinguished by $\mathrm{GL}_n(F)$, the mapping $\alpha \to \widetilde{\pi}[\widetilde{\alpha}]$ is surjective onto $[\widetilde{\pi}]_w/\sim_s$.

Under the natural action of $Z_{\widetilde{\pi}}/Y_{\widetilde{\pi}}$ on $X_{\widetilde{\pi}}$, it is clear that $\widetilde{\pi}[\widetilde{\beta}\widetilde{\alpha}] = \widetilde{\pi}[\widetilde{\alpha}]$ as an element of $[\widetilde{\pi}]_w/\sim_s$ for $\widetilde{\beta} \in Z_{\widetilde{\pi}}$.

Conversely, if $\widetilde{\pi}[\widetilde{\alpha}] = \widetilde{\pi}[\widetilde{\beta}]$ as an element of $[\widetilde{\pi}]_w/\sim_s$, then $\widetilde{\pi} \otimes \widetilde{\beta}^{-1} \cong \widetilde{\pi} \otimes \widetilde{\alpha}^{-1}\chi$ for some character χ of $E^\times/F^\times$. This condition is equivalent to saying that $\widetilde{\beta}\widetilde{\alpha}^{-1}\chi \in Z_{\widetilde{\pi}}$. Therefore $\widetilde{\alpha}$ and $\widetilde{\beta}$ differ by an element of $Z_{\widetilde{\pi}}$. □

4. Distinction by $\mathrm{SL}_n(F)$ and Whittaker models

In [4] using the explicit realization of a $\mathrm{GL}_2(F)$-invariant linear form on the Kirillov model of a representation π of $\mathrm{GL}_2(E)$ due to Jeff Hakim, it was proved that any irreducible admissible generic representation of $\mathrm{SL}_2(E)$ which is distinguished by $\mathrm{SL}_2(F)$ has a Whittaker model for a character $\psi : E/F \to \mathbb{C}^\times$. This non-trivial result was among the most important ingredients to our work in [4]. Its analogue for $\mathrm{SL}_n(E)$ will be similarly crucial to us in this paper.

In a recent work of the first author with Matringe [3], it has been proved that for an irreducible generic representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$, the linear form

$$\ell(W) = \int_{N_n(F)\backslash P_n(F)} W(p)dp$$

defined on the Whittaker space $\mathcal{W}(\widetilde{\pi}, \psi)$ of $\widetilde{\pi}$ (absolutely convergent integral for $\widetilde{\pi}$ unitary [7, Lemma 4], and defined by regularization in general [3, §7]), is up to multiplication by scalars, the unique non-zero element in $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\widetilde{\pi}, 1)$. This allows one to conclude as in [4] that any irreducible generic representation of $\mathrm{SL}_n(E)$ which is distinguished by $\mathrm{SL}_n(F)$ has a Whittaker model for a non-degenerate character $\psi : N(E)/N(F) \to \mathbb{C}^\times$.

In this section, we offer a 'pure thought' argument based on Clifford theory with the 'mirabolic' subgroup of $\mathrm{GL}_n(E)$, the subgroup of $\mathrm{GL}_n(E)$ with last row $(0, \cdots, 0, 1)$, first for $\mathrm{SL}_2(E)$, and then for $\mathrm{SL}_n(E)$ in general but only for tempered representations.

Our proof for $\mathrm{SL}_2(E)$ works for finite fields, but the proof for $\mathrm{SL}_n(E)$, when E is finite, works only for cuspidal representations.

Lemma 4.1. *Let π be an irreducible generic representation of $\mathrm{SL}_2(E)$. Then if π is distinguished by $\mathrm{SL}_2(F)$, π must have a Whittaker model for a character $\psi : E/F \to \mathbb{C}^\times$.*

Proof. Since π is distinguished by $\mathrm{SL}_2(F)$, the largest quotient of π on which $\mathrm{SL}_2(F)$ operates trivially is non-zero. As a consequence, the largest quotient π_F of π on which $N(F) = F$ operates trivially is non-zero. Clearly π_F is a smooth module for $N(E)/N(F) = E/F$. Thus there are two options:

(1) $N(E)/N(F)$ does not operate trivially on π_F, in which case it is easy to prove that for some non-trivial character $\psi : N(E)/N(F) \to \mathbb{C}^\times$, $\pi_\psi \neq 0$.

(2) $N(E)/N(F)$ operates trivially on π_F, in which case in particular $N(E)$ will operate trivially on the linear form $\ell : \pi \to \mathbb{C}$ which is $\mathrm{SL}_2(F)$-invariant. Thus this linear form will be invariant under $\mathrm{SL}_2(F)$ as well as $N(E)$, and therefore by the group generated by $\mathrm{SL}_2(F)$ and $N(E)$. It is easy to see that the group generated by $\mathrm{SL}_2(F)$ and $N(E)$ is $\mathrm{SL}_2(E)$. Thus $\ell : \pi \to \mathbb{C}$ is invariant under $\mathrm{SL}_2(E)$, so π must be one dimensional, a contradiction to its being generic.

This completes the proof of the lemma. □

Proposition 4.2. *Let π be an irreducible admissible representation of $\mathrm{SL}_n(E)$ which is tempered. Then if π is distinguished by $\mathrm{SL}_n(F)$, π must have a Whittaker model for a non-degenerate character $\psi : N(E)/N(F) \to \mathbb{C}^\times$.*

We will prove this proposition in the following equivalent form.

Proposition 4.3. *Let π be an irreducible admissible tempered representation of the group $\mathrm{GL}_n^+(E) = \mathrm{GL}_n(F)\mathrm{SL}_n(E)$. Then if π is distinguished by $\mathrm{GL}_n(F)$, π must have a Whittaker model for a non-degenerate character $\psi : N(E)/N(F) \to \mathbb{C}^\times$.*

The proof of this proposition depends on the following lemma which allows an inductive procedure to prove the previous proposition.

In what follows, for any $k \geq 0$, we let ν be the character $\nu(g) = |\det g|$ on $\mathrm{GL}_k(F)$, and any of its subgroups.

Lemma 4.4. *Let $P_k^+(E)$ be the mirabolic subgroup of $\mathrm{GL}_k^+(E)$, thus with $P_k^+(E) = \mathrm{GL}_{k-1}^+(E) \rtimes N_k(E) = \mathrm{GL}_{k-1}^+(E) \rtimes E^{k-1}$. Let ${}_k\Delta$ be a smooth representation of $P_k^+(E)$ having a Whittaker model. Fix a non-trivial character $\psi_0 : E/F \to \mathbb{C}^\times$, and let $\psi_{k-1} = \psi_0 \circ p_{k-1} : E^{k-1} \to \mathbb{C}^\times$ be the character on E^{k-1} where $p_{k-1} : E^{k-1} \to E$ is the projection to the last co-ordinate. Then if ${}_k\Delta$ is distinguished by $P_k(F)$, but the (un-normalized) Jacquet module ${}_k\Delta_{N(E)}$, a representation of $\mathrm{GL}_{k-1}^+(E)$ is not distinguished by $\mathrm{GL}_{k-1}(F)$, the smooth representation (un-normalized twisted Jacquet module) $\Delta_{N_k(E),\psi_{k-1}}$ of $P_{k-1}^+(E)$, must have a Whittaker model and is $\nu^{-1/2}$-distinguished by $P_{k-1}(F)$.*

Proof. Since ${}_k\Delta$ is distinguished by $P_k(F)$, the largest quotient ${}_k\Delta_{N_k(F)}$ of ${}_k\Delta$ on which $N_k(F) = F^{k-1}$ operates trivially is non-zero, and is distinguished by $\mathrm{GL}_{k-1}(F)$. Clearly ${}_k\Delta_{N_k(F)}$ is a smooth representation for

$$\mathrm{GL}_{k-1}(F) \rtimes (N_k(E)/N_k(F)) = \mathrm{GL}_{k-1}(F) \rtimes F^{k-1}.$$

Thus we are in the context of Clifford theory which applies to any smooth representation of a group in the presence of an abelian normal subgroup, cf. [5, §5.1 C], for a similar analysis, and [6, §3] for developing the Clifford theory in greater generality.

Note that for $k \geq 2$, the action of $\mathrm{GL}_{k-1}(F)$ on the set of non-trivial characters of $N_k(E)/N_k(F) = F^{k-1}$ is transitive.

It follows from [6, Proposition 1] that the representation ${}_k\Delta_{N_k(F)}$ of

$$\mathrm{GL}_{k-1}(F) \rtimes (N_k(E)/N_k(F)) = \mathrm{GL}_{k-1}(F) \rtimes F^{k-1}$$

has a filtration with two subquotients, which are (with un-normalized induction):

(1) $\mathrm{ind}_{P_{k-1}(F)\rtimes F^{k-1}}^{\mathrm{GL}_{k-1}(F)\rtimes F^{k-1}}({}_k\Delta_{N_k(E),\psi_{k-1}})$,
(2) ${}_k\Delta_{N_k(E)}$.

Since we know that ${}_k\Delta_{N_k(F)}$ is distinguished by $\mathrm{GL}_{k-1}(F)$, at least one of the representations above is distinguished by $\mathrm{GL}_{k-1}(F)$. In case (1), by Mackey theory, ${}_k\Delta_{N_k(E),\psi_{k-1}}$, a smooth representation of $P^+_{k-1}(E)$, is $\nu^{-1/2}$-distinguished by $P_{k-1}(F)$, whereas in case (2), ${}_k\Delta_{N_k(E)}$, is distinguished by $\mathrm{GL}_{k-1}(F)$. By the hypotheses of the lemma, ${}_k\Delta_{N_k(E)}$, is not distinguished by $\mathrm{GL}_{k-1}(F)$, leaving us with only option (1).

This completes the proof of the lemma. □

Proof of Proposition 4.3. For the proof of the proposition, we will apply the previous lemma to the representation ${}_k\Delta = \pi^{(n-k)}|_{P^+_k(E)}$ where $\pi^{(n-k)}$ is the $(n-k)$-th derivative of Bernstein-Zelevinsky, which is a representation of $\mathrm{GL}_k(E)$, starting with $k = n$, and ${}_n\Delta = \pi|_{P^+_n(E)}$. It follows from Bernstein-Zelevinsky that ${}_k\Delta_{N_k(E)} = \nu^{1/2}\pi^{(n-k+1)}$, a smooth representation of $\mathrm{GL}^+_{k-1}(E)$. Further, $\nu^{1/2}{}_k\Delta_{N_k(E),\psi_{k-1}} = {}_{k-1}\Delta$, a smooth representation of $P^+_{k-1}(E)$. This implies that the way we have defined ${}_k\Delta$, decreasing induction hypothesis holds if we can ensure that the condition, "${}_k\Delta_{N_k(E)} = \nu^{1/2}\pi^{(n-k+1)}$, a smooth representation of $\mathrm{GL}^+_{k-1}(E)$, is not distinguished by $\mathrm{GL}_{k-1}(F)$", is satisfied. This is where we will use the temperedness hypothesis.

Recall that a tempered representation π of $\mathrm{GL}_n(E)$ is of the form $\pi = \pi_1 \times \cdots \times \pi_r$ where π_i are irreducible unitary discrete series representations of $\mathrm{GL}_{n_i}(E)$. It is known that any unitary discrete series representation π_i is the unique irreducible quotient representation of $\rho_i\nu^{-(n_i-1)/2} \times \cdots \times \rho_i\nu^{(n_i-1)/2}$ for a unitary supercuspidal representation ρ_i of some $\mathrm{GL}_{m_i}(E)$ for $m_i|n_i$, and that $\pi_i^{(k)} = 0$ if k is not a multiple of m_i, and for $k = m_i r$, $\pi_i^{(m_i r)}$ is the unique irreducible quotient of $\rho_i\nu^{-(n_i-1)/2+r} \times \cdots \times \rho_i\nu^{(n_i-1)/2}$.

The Leibnitz rule for derivatives allows one to calculate the derivative of $\pi = \pi_1 \times \cdots \times \pi_r$, and from the recipe of the derivatives of a discrete series recalled above, we find that any non-zero positive derivative $\pi_i^{(k)}$ has a central character $\omega(\pi_i^{(k)})$ whose absolute value $|\omega(\pi_i^{(k)})|$ is a positive power of ν unless $k = 0$ or $k = n_i$. Since a distinguished representation Λ of $\mathrm{GL}_n(E)$ must have $\Lambda^\sigma \cong \Lambda^\vee$, in particular, $|\omega(\Lambda)| = 1$. This implies that $\nu^{1/2}\pi^{(k)}$ cannot be $\mathrm{GL}_{n-k}(F)$ distinguished, unless it is a representation of $\mathrm{GL}_0(E) = 1$. □

Remark 4. We believe that Propositions 4.2 and 4.3 remain valid for finite fields, but have not been able to find a proof, except as mentioned earlier in the case of cuspidal representations where the proof given here for p-adic fields remains valid, and the case of $\mathrm{SL}_2(E)$ independently proved in Lemma 4.1.

Remark 5. The proof of Proposition 4.3 given here is based on an idea contained in [2] that although the restriction to mirabolic of a representation of $\mathrm{GL}_n(E)$ has two subquotients, the non-generic component cannot carry $P_n(F)$-invariant linear forms because of the presence of the modulus character. Since the modulus character for finite fields is trivial, we are not able to rule this possibility out for finite fields. Note that [2] uses a lemma, [2, Lemma 2.4], according to which (using un-normalized induction unlike [2, Lemma 2.4]),

$$\mathrm{Hom}_{P_n(F)}[\mathrm{ind}_{P_k(E)}^{P_n(E)}(\pi \times \psi_{n-k}), \mathbb{C}] \cong \mathrm{Hom}_{\mathrm{GL}_k(F)}[\pi, \mathbb{C}];$$

here $P_n(E)$ is the mirabolic subgroup of $\mathrm{GL}_n(E)$, $P_k(E)$ is the subgroup of $\mathrm{GL}_n(E)$ contained in the $(k, n-k)$-parabolic and containing its unipotent radical with Levi replaced by $\mathrm{GL}_k(E) \times \mathrm{U}_{n-k}(E)$ where $\mathrm{U}_{n-k}(E)$ is the upper triangular unipotent subgroup of $\mathrm{GL}_{n-k}(E)$, and ψ_{n-k} is its generic character. For the proof of this lemma, [2] refers to the main lemma of Flicker's paper [9], whose proof is rather long winded. Our proof here does not need [2, Lemma 2.4], but rather gives a proof of it.

5. Fibers of the base change map from $\mathrm{SU}(n)$ to $\mathrm{SL}_n(E)$

In this section we consider Langlands parameters for the groups $\mathrm{SU}(n)$ and $\mathrm{SL}_n(E)$. Our aim here is to compute the number of parameters of $\mathrm{SU}(n)$ that lift to a given parameter of $\mathrm{SL}_n(E)$.

A Langlands parameter of $\mathrm{SL}_n(E)$

$$\phi : W'_E \to \mathrm{PGL}_n(\mathbb{C})$$

gives rise to an element of $H^1(W'_E, \mathrm{PGL}_n(\mathbb{C}))$, where the Weil-Deligne group W'_E of E acts trivially on $\mathrm{PGL}_n(\mathbb{C})$. It is well-known that such a parameter ϕ lifts to a Langlands parameter $\widetilde{\phi}$ of $\mathrm{GL}_n(E)$

$$\widetilde{\phi} : W'_E \to \mathrm{GL}_n(\mathbb{C}),$$

which can be thought of as an element of $H^1(W'_E, \mathrm{GL}_n(\mathbb{C}))$ with the W'_E-action on $\mathrm{GL}_n(\mathbb{C})$ being trivial. Indeed, the above observation follows from a theorem of Tate according to which $H^2(W'_E, \mathbb{C}^\times) = 0$ for the trivial action of W'_E on $\mathbb{C}^\times$. We note that though Tate's theorem is usually stated in terms of the absolute Galois group $\mathrm{Gal}(\bar{E}/E)$ instead of the Weil-Deligne group W'_E, i.e., $H^2(\mathrm{Gal}(\bar{E}/E), \mathbb{C}^\times) = 0$ with $\mathrm{Gal}(\bar{E}/E)$ acting trivially on $\mathbb{C}^\times$ (cf. [18, Theorem 4]); the W'_E-version, $H^2(W'_E, \mathbb{C}^\times) = 0$, and its relation to lifting of continuous projective representations is known too, cf. [17, Theorem 1, Theorem 8]. We will continue to call the vanishing of $H^2(W'_E, \mathbb{C}^\times)$ as Tate's theorem.

That a Langlands parameter for $\mathrm{SL}_n(E)$ lifts to a Langlands parameter for $\mathrm{GL}_n(E)$ is related to the fact that an irreducible admissible representation π of $\mathrm{SL}_n(E)$ occurs in the restriction of an irreducible admissible representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$.

As in the case of $(\mathrm{GL}(n), \mathrm{SL}(n))$, an irreducible representation of $\mathrm{SU}(n)$ occurs in the restriction of an irreducible admissible representation of $\mathrm{U}(n)$. We will check below that a Langlands parameter for $\mathrm{SU}(n)$ lifts to a Langlands parameter for $\mathrm{U}(n)$.

Since the Langlands dual group of U(n) is

$$^L\mathrm{U}(n) = \mathrm{GL}_n(\mathbb{C}) \rtimes W'_F,$$

where W'_F acts by projection to Gal(E/F), and via

$$\sigma(g) = J^t g^{-1} J^{-1},$$

where J is the anti-diagonal matrix with alternating $1, -1$. We will denote the group $\mathrm{GL}_n(\mathbb{C})$ with this action of W'_F by $\mathrm{GL}_n(\mathbb{C})[\tau]$; similarly for $\mathrm{PGL}_n(\mathbb{C})$. Thus a Langlands parameter for U(n) gives rise to an element of $H^1(W'_F, \mathrm{GL}_n(\mathbb{C})[\tau])$, where W'_F acts on $\mathrm{GL}_n(\mathbb{C})$ as above. Similarly, a Langlands parameter for SU(n) gives rise to an element of $H^1(W'_F, \mathrm{PGL}_n(\mathbb{C})[\tau])$.

Thus the fact that a Langlands parameter for SU(n) lifts to a Langlands parameter for U(n) follows from the following lemma.

Lemma 5.1. *Let W'_F operate on $\mathbb{C}^\times$ by $z \mapsto z^{-1}$ via the quotient $W'_F \to W'_F/W'_E \cong \mathbb{Z}/2$. Denote the corresponding representation of W'_F by $\mathbb{C}^\times[\tau]$. Then,*

$$H^2(W'_F, \mathbb{C}^\times[\tau]) = 0.$$

Proof. Consider the restriction-corestriction sequence

$$H^2(W'_F, \mathbb{C}^\times[\tau]) \to H^2(W'_E, \mathbb{C}^\times) \to H^2(W'_F, \mathbb{C}^\times[\tau]).$$

Since the composite map is multiplication by 2, and since $H^2(W'_E, \mathbb{C}^\times) = 0$ by Tate's theorem, it follows that

$$2H^2(W'_F, \mathbb{C}^\times[\tau]) = 0.$$

Using the exact sequence,

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{C}^\times[\tau] \xrightarrow{[2]} \mathbb{C}^\times[\tau] \longrightarrow 1,$$

since $2H^2(W'_F, \mathbb{C}^\times[\tau]) = 0$, it follows that we have an exact sequence

$$H^1(W'_F, \mathbb{C}^\times[\tau]) \to H^1(W'_F, \mathbb{C}^\times[\tau]) \to H^2(W'_F, \mathbb{Z}/2) \to H^2(W'_F, \mathbb{C}^\times[\tau]) \to 0.$$

Now,

$$H^2(W'_F, \mathbb{Z}/2) = \mathbb{Z}/2,$$

since this is the 2-torsion in the Brauer group. Therefore, to prove that

$$H^2(W'_F, \mathbb{C}^\times[\tau]) = 0,$$

it suffices to prove that

$$2H^1(W'_F, \mathbb{C}^\times[\tau]) \neq H^1(W'_F, \mathbb{C}^\times[\tau]).$$

A cocycle in $H^1(W'_F, \mathbb{C}^\times[\tau])$ upon restriction to W'_E gives rise to a character of $E^\times$ which is trivial on elements of $F^\times$ which arise as norms from $E^\times$. It can be seen that a character $\chi : E^\times/\mathbb{N}mE^\times \to \mathbb{C}^\times$ extends to a cocycle on W'_F with values in $\mathbb{C}^\times[\tau]$ if and only if χ is trivial on $F^\times$, and then the cocycle is unique up to coboundary. Thus,

$$H^1(W'_F, \mathbb{C}^\times[\tau]) = \mathrm{Hom}(E^\times/F^\times, \mathbb{C}^\times) = \mathrm{Hom}(U(1), \mathbb{C}^\times),$$

where the second equality is the result of the identification $\chi \to \chi'$ via $\chi'(x/x^\sigma) = \chi(x)$. (Since $\mathbb{C}^\times[\tau]$ is the L-group of $U(1)$, $H^1(W'_F, \mathbb{C}^\times[\tau]) \cong \mathrm{Hom}(U(1), \mathbb{C}^\times)$ is the usual Langlands correspondence for tori.) Clearly, a character χ' of $U(1)$ has a square root if and only if $\chi'(-1) = 1$, and therefore

$$H^1(W_F', \mathbb{C}^\times[\tau])/2H^1(W_F', \mathbb{C}^\times[\tau]) = \mathbb{Z}/2,$$

proving the lemma. □

We are interested in computing the number of Langlands parameters of $\mathrm{SU}(n)$ that lift to a given Langlands parameter of $\mathrm{SL}_n(E)$. Thus, we need to analyse the fiber of the restriction map

$$H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \xrightarrow{P\Phi} H^1(W_E', \mathrm{PGL}_n(\mathbb{C})).$$

For this, observe that the above map fits into the following commutative diagram:

$$\begin{tikzcd} H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \arrow[r, "P\Phi"] & H^1(W_E', \mathrm{PGL}_n(\mathbb{C})) \\ H^1(W_F', \mathrm{GL}_n(\mathbb{C})[\tau]) \arrow[r, "\Phi"] \arrow[u, "P_F"] & H^1(W_E', \mathrm{GL}_n(\mathbb{C})) \arrow[u, "P_E"] \end{tikzcd}$$

where Φ is the restriction map which corresponds to lifting a Langlands parameter of $\mathrm{U}(n)$ to a Langlands parameter of $\mathrm{GL}_n(E)$, and the maps P_F and P_E are the natural maps on cohomology induced from the homomorphism $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. Note that we have proved in the preceding paragraphs that both the maps P_F and P_E are surjective; surjectivity of P_E follows from Tate's theorem and surjectivity of P_F is a consequence of Lemma 5.1.

The map Φ which takes a $\mathrm{U}(n)$-parameter to a $\mathrm{GL}_n(E)$-parameter is well understood: its image consists precisely of conjugate self-dual Langlands parameters of W_E' of parity $+1$ if n is odd, and parity -1 if n is even. We will need to make use of another well-known fact about the map Φ for which we refer to [16, Proposition 7] for a proof.

Lemma 5.2. *The restriction map*

$$H^1(W_F', \mathrm{GL}_n(\mathbb{C})[\tau]) \xrightarrow{\Phi} H^1(W_E', \mathrm{GL}_n(\mathbb{C}))$$

is injective.

We will have many occasions to use the following lemma, cf. [19, Proposition 42].

Lemma 5.3. *Suppose G is a group with an action of W_F', and Z is a central subgroup of G left invariant by the action of W_F'. Then elements ϕ_1, ϕ_2 of $H^1(W_F', G)$ which lie over the same element of $H^1(W_F', G/Z)$ are translates of each other by an element of $H^1(W_F', Z)$, i.e., $\phi_2 = \phi_1 \cdot c$ for some $c \in H^1(W_F', Z)$.*

The following proposition is a simple consequence of the previous two lemmas using the definitions of strong and weak equivalence introduced at the end of §2.

Proposition 5.4. *Let $\rho \in H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau])$. Let $\widetilde{\rho} \in H^1(W_F', \mathrm{GL}_n(\mathbb{C})[\tau])$ be such that $P_F(\widetilde{\rho}) = \rho$. Then the cardinality of the set*

$$\{\mu \in H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \mid P\Phi(\mu) = P\Phi(\rho)\}$$

equals $q(\Phi(\widetilde{\rho}))$, which is the number of strong equivalence classes in the weak equivalence class of $\Phi(\widetilde{\rho})$ (among conjugate self-dual representations of a given parity).

Proof. By Lemma 5.3, parameters for $\mathrm{SL}_n(E)$ can be identified to parameters for $\mathrm{GL}_n(E)$ up to twisting by characters $\chi : E^\times \to \mathbb{C}^\times$. Similarly, by Lemma 5.3, parameters for $\mathrm{SU}_n(F)$ can be identified to parameters for $\mathrm{U}_n(F)$ up to twisting by characters $\chi : E^\times/F^\times \to \mathbb{C}^\times$ (because $H^1(W_F', \mathbb{C}^\times[\tau]) \cong \mathrm{Hom}(U(1), \mathbb{C}^\times) = \mathrm{Hom}(E^\times/F^\times, \mathbb{C}^\times)$). By Lemma 5.2, parameters for $\mathrm{U}_n(F)$ embed into parameters for $\mathrm{GL}_n(E)$ by the base change map Φ. Thus the cardinality of the fiber of the base change map

$$H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \xrightarrow{P\Phi} H^1(W_E', \mathrm{PGL}_n(\mathbb{C}))$$

is the number of strong equivalence classes in the weak equivalence class of $\Phi(\widetilde{\rho})$ among conjugate self-dual representations of a given parity ($= (-1)^{n-1}$). $\square$

We next restate Theorem 2.3 taking into account Lemma 5.2 according to which parameters for $\mathrm{U}_n(F)$ embed into parameters for $\mathrm{GL}_n(E)$ by the base change map Φ.

Theorem 5.5. *An irreducible admissible generic representation $\widetilde{\pi}$ of $\mathrm{GL}_n(E)$ is distinguished by $\mathrm{GL}_n(F)$ if n is odd, respectively $\omega_{E/F}$-distinguished if n is even if and only if its Langlands parameter $\widetilde{\rho}_{\widetilde{\pi}}$ is in the image of*

$$\Phi : H^1(W_F', \mathrm{GL}_n(\mathbb{C})[\tau]) \to H^1(W_E', \mathrm{GL}_n(\mathbb{C})),$$

and moreover,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}[\widetilde{\pi}, \omega_{E/F}^{n-1}] = |\Phi^{-1}(\widetilde{\rho}_{\widetilde{\pi}})|. \tag{3}$$

The main theorem of this paper is the SL(n)-analogue of Theorem 5.5.

Theorem 5.6. *An irreducible admissible generic representation π of $\mathrm{SL}_n(E)$ is distinguished by $\mathrm{SL}_n(F)$ if and only if*

(1) its Langlands parameter ρ_π is in the image of the base change map:

$$P\Phi : H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \to H^1(W_E', \mathrm{PGL}_n(\mathbb{C})),$$

(2) π has a Whittaker model for a non-degenerate character of $N(E)/N(F)$.

Further, if $\mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] \neq 0$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}] = |P\Phi^{-1}(\rho_\pi)|. \tag{4}$$

Proof. Choose $\widetilde{\pi}$ as in Proposition 3.7 and $\widetilde{\rho}$ as in Proposition 5.4 so that $\Phi(\widetilde{\rho}) = \widetilde{\rho}_{\widetilde{\pi}}$. Such a choice does exist by the first part of Theorem 5.5. Thus, the assertion (1) about the Langlands parameter ρ_π follows from the commutativity of the diagram:

$$\begin{tikzcd} H^1(W_F', \mathrm{PGL}_n(\mathbb{C})[\tau]) \arrow[r, "P\Phi"] & H^1(W_E', \mathrm{PGL}_n(\mathbb{C})) \\ H^1(W_F', \mathrm{GL}_n(\mathbb{C})[\tau]) \arrow[u, "P_F"] \arrow[r, "\Phi"] & H^1(W_E', \mathrm{GL}_n(\mathbb{C})). \arrow[u, "P_E"] \end{tikzcd}$$

The assertion (2) about Whittaker models is part of the conclusion of §4.

For the assertion on $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_n(F)}[\pi, \mathbb{C}]$, observe that the left hand side of (4) is $q(\widetilde{\pi})$ by Proposition 3.7, whereas the right hand side of (4) is $q(\widetilde{\rho}_{\widetilde{\pi}})$ by Proposition 5.4. Since $q(\widetilde{\pi}) = q(\widetilde{\rho}_{\widetilde{\pi}})$, this proves the theorem. $\square$

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