**Kokuyo notebooks - Key Features**

1. **Durable spine cloth with a strong adhesive keeps pages securely bound.**
   - Spine cloth made of specially laminated film - resistant to tearing and damage from rubbing or chaffing.
   - Every page remains strongly bound even if a page is torn from the notebook.
   - Safe for children - pin less strong adhesive binding.
   - Spreads flat on opening and allows easy access to double page spread.

2. **Attractive & Functional Cover Design**
   Bright colour designs combined with high functionality generate a special attachment among users.

3. **Ruling Innovations**
   Triangle memory points ▲ located on every page makes drawing precision vertical lines easy.
   This may lead to further innovations based on usage.

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**Note:**

Part B: Multi variate Calculus - A

Rigorous approach

Done as a parallel stream.
The Spectral Theorem (Purely matrix theoretic approach)

Prop: The eigenvalues of a real symmetric matrix are real numbers.

Proof: Let \( A \) be a real symmetric matrix and \( \lambda \) be an (possibly complex) eigenvalue of \( A \).

Let \( \mathbf{v} \) be its eigen vector in \( \mathbb{C}^n \).

\[
A\mathbf{v} = \lambda \mathbf{v}.
\]

Taking complex conjugate,

\[
A^*\overline{\mathbf{v}} = \overline{\lambda} \overline{\mathbf{v}}.
\]

\[
\overline{\mathbf{v}}^* A\mathbf{v} = \overline{\lambda} \mathbf{v} \cdot \overline{\mathbf{v}} = \overline{\lambda} ||\mathbf{v}||^2.
\]

\[
(A\mathbf{v})^* \overline{\mathbf{v}} = \overline{\lambda} ||\mathbf{v}||^2.
\]

\[
\lambda ||\mathbf{v}||^2 = \overline{\lambda} ||\mathbf{v}||^2 \quad \text{or} \quad \lambda = \overline{\lambda}.
\]

Since \( ||\mathbf{v}|| \neq 0 \), \( \lambda = \overline{\lambda} \).

Recall that a complex matrix \( A \) is Hermitian or self adjoint if \( A = A^* \) (the Conjugate transpose of \( A \)).

Ex: Show that the eigenvalues of a Hermitian matrix are real.

Prop: Suppose \( \lambda, \mu \) are distinct eigenvalues of a Hermitian matrix and \( \mathbf{v}, \mathbf{w} \) are the corresponding eigen vectors then

\[
\langle \mathbf{v}, \mathbf{w} \rangle = 0 \quad \text{where} \quad \langle \mathbf{v}, \mathbf{w} \rangle = \sum \mathbf{v}_i \mathbf{w}_i
\]

The corresponding result is true for real symmetric matrices except that in this case \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \).

Proof:

\[
A\mathbf{v} = \lambda \mathbf{v}.
\]

\[
A\mathbf{w} = \mu \mathbf{w}.
\]

\[
\overline{\mathbf{w}}^* A\mathbf{v} = \lambda \langle \mathbf{w}, \mathbf{v} \rangle.
\]

\[
\overline{\mathbf{w}}^* A\mathbf{v} = \lambda \overline{\langle \mathbf{w}, \mathbf{v} \rangle}.
\]
\[ (\mathbf{w}^T \mathbf{v} = \lambda <\mathbf{w}, \mathbf{v}> \]
\[ \mathbf{u}^T \mathbf{v} = \lambda <\mathbf{w}, \mathbf{v}> \]

(Since \( \mathbf{u} = \mathbf{u} \))

\[ <\mathbf{w}, \mathbf{v}> = 0 \text{ Since } \lambda \neq \mu. \]

**Thm (Spectral theorem):**

1. An \( n \times n \) matrix has an orthonormal basis of eigenvectors in \( \mathbb{R}^n \)

   Equivalently \( \exists \mathbf{P} \) orthogonal \( \mathbf{P}^T \mathbf{A} \mathbf{P} = \text{diagonal matrix} \)

2. An \( n \times n \) Hermitian matrix has an orthonormal basis of eigenvectors in \( \mathbb{C}^n \)

   Equivalently \( \exists \mathbf{U} \) unitary \( \mathbf{U}^* \mathbf{A} \mathbf{U} = \text{diagonal matrix} \)

   (Note: \( \mathbf{U} \) unitary simply means \( \mathbf{U}^* \mathbf{U} = \mathbf{I}_n = \mathbf{U} \mathbf{U}^* \))

Each of these statements may be described as:
A real symmetric or Hermitian matrix is
unitarily diagonalizable.

**Proof:** The proofs are similar and we
only do the real case.

Let \( \lambda \) be an eigenvalue of \( \mathbf{A} \) with
eigenvector \( \mathbf{v} \) of unit length.

Choose vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_n \) such that
\( \{ \mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_n \} \) is an orthonormal
basis of \( \mathbb{R}^n \) and put
\[ \mathbf{P} = [\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_n] \] which is an
orthogonal matrix.

\[ \mathbf{AP} = [\lambda \mathbf{v}, \lambda \mathbf{w}_1, \ldots, \lambda \mathbf{w}_n] \]

(Well, to see this apply both \( \mathbf{P} \) and \( \mathbf{P}^T \) and recall
that for any matrix \( \mathbf{C} \), \( \mathbf{C}^\mathbf{P} \) is the \( k\)-th
column of \( \mathbf{C} \).)

\[ \mathbf{AP} = [\lambda \mathbf{v}, \lambda \mathbf{w}_1, \ldots, \lambda \mathbf{w}_n] \]

We now wish to calculate \( \mathbf{P}^T \mathbf{A} \mathbf{P} \).

The \( j \)-th row of \( \mathbf{P}^T \mathbf{A} = \mathbf{P} \)
\( (j > 1) \)
\( (j = 1) \)

\[ \mathbf{v}_1^T \mathbf{A} \mathbf{P} = [\lambda_1, \mathbf{v}_1^T \mathbf{w}_1, \ldots, \mathbf{v}_1^T \mathbf{w}_n] \]

For \( j > 1 \)
\[ \mathbf{w}_j^T \mathbf{A} \mathbf{P} = [0, \mathbf{w}_j^T \mathbf{w}_1, \ldots, \mathbf{w}_j^T \mathbf{w}_n] \]

Thus \( \mathbf{P}^T \mathbf{A} \mathbf{P} = \lambda_1 \mathbf{I}_n \mathbf{P} = \mathbf{P} \mathbf{A} \mathbf{P} \)

where \( \mathbf{A} \) is an \( n \times n \) matrix.

\[ \mathbf{B} = \mathbf{P}^T \mathbf{A} \mathbf{P} \]

Since \( (\mathbf{P}^T \mathbf{A} \mathbf{P})^T = \mathbf{P}^T \mathbf{A} \mathbf{P} \) we see that \( \mathbf{B} \)
and hence \( \mathbf{B} \) is real symmetric.

By induction hypothesis \( \mathbf{B} \) has an orthonormal
basis \( \mathbf{w}_1, \ldots, \mathbf{w}_n \) of eigenvectors in \( \mathbb{R}^n \)
with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and
\( \mathbf{B} \mathbf{w}_j = \lambda_j \mathbf{w}_j \).

Then \( \mathbf{A} \mathbf{w}_j = \mathbf{A} \mathbf{P} \mathbf{P}^T \mathbf{w}_j \)

To compute this (without relying on block
multiplication of matrices). We are
Proving everything from scratch:

\[ u_j^T A v_j = u_i^T A p_i \begin{bmatrix} 0 \\ w_j \end{bmatrix} = \lambda_i u_i^T p_i \begin{bmatrix} 0 \\ w_j \end{bmatrix} = \lambda_i [1, 0, \ldots, 0] \begin{bmatrix} 0 \\ w_j \end{bmatrix} = 0 \]

And for \( k \geq 2 \)

\[ u_k^T A v_j = u_k^T A p_k \begin{bmatrix} 0 \\ w_j \end{bmatrix} = [0, R_k(B)] \begin{bmatrix} 0 \\ w_j \end{bmatrix} \]

Using (\( \ast \)), where \( R_k(B) \) denotes the \( k \)-th row of \( B \)

\[ u_k^T A v_j = \lambda_j R_k(w_j) \quad (k\text{th row of } w_j) \]

\[ p_i^T A v_j = \lambda_j [0] \]

\[ A v_j = \lambda_j v_j \quad (j = 2, 3, \ldots, n) \]

Thus \( v_1, \ldots, v_n \) are eigen vectors of \( A \):

\[ \langle v_j, v_k \rangle = \begin{bmatrix} 0 & w_k \\ w_j & 0 \end{bmatrix}^T \begin{bmatrix} 0 & w_k \\ w_j & 0 \end{bmatrix} = \lambda_j \delta_{jk} \]

If \( j = 1 \) and \( k \geq 2 \)

\[ \langle v_i, v_k \rangle = (p_i \begin{bmatrix} 0 \\ w_k \end{bmatrix})^T \begin{bmatrix} 0 \\ w_k \end{bmatrix} = [0, w_k]^T p_i^T w_k = [0, w_k]^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \]

Thus the vectors \( v_1, \ldots, v_n \) are orthonormal.

The proof is complete.

Theorem (1): If \( A, B \) are commuting real symmetric matrices, they have a common eigen vector in \( \mathbb{R}^n \)

(2) If \( A, B \) are commuting Hermitian matrices, they have a common eigen vector in \( \mathbb{C}^n \)

Proof: Let \( \lambda \) be an eigen value of \( A \) and \( v_1, \ldots, v_p \) be the complete set of eigen vectors of \( A \) corresponding to the eigen value \( \lambda \).

\[ A(Bv_j) = B(Av_j) = \lambda Bv_j \]

\[ Bv_j = \sum_{k=1}^p C_{jk} v_k \]

\[ C_{jk} = u_k^T B v_j \text{ which is a } 1 \times 1 \text{ matrix} \]

\[ = (v_j^T B v_k)^T = v_j^T B v_k = C_{kj} \]

The matrix \( C \) is a \( p \times p \) real symmetric matrix, and \( \lambda \) has eigen value \( \mu \) with eigen vectors \( \begin{bmatrix} \alpha_1 \\ \alpha_{p-1} \\ \alpha_p \end{bmatrix} \). So \( \alpha_1 \ldots, \alpha_p \) are non-zero and \( \alpha_1, \ldots, \alpha_p \) are linearly independent \( \Rightarrow u = \alpha_1 v_1 + \ldots + \alpha_p v_p \neq 0 \).

Claim: \( u \) is the common eigen vector for \( A \) and \( B \)

It is obviously an eigen vector for \( A \).
\[ Bu = \sum_v \alpha_v B \nu_v \]
\[ = \sum_{v, \mu} \alpha_v c_{v \mu} \nu_\mu \]
\[ = \sum_k (\sum_j c_{j \nu} \nu_j) \nu_k \]
\[ = \sum_k \mu \alpha_k \nu_k = \mu u \]

The proof is complete.

Theorem: (1) If \( A, B \) are commuting real symmetric matrices, they have a common orthonormal basis of eigen vectors in \( \mathbb{R}^n \).

Proof: First choose a common eigen vector \( \nu_1 \), for both \( A \) and \( B \): \( \| \nu_1 \|^2 = 1 \), then we can choose an orthonormal basis \( \nu_1, \nu_2, \ldots, \nu_n \) and
\[ P = [ \nu_1, \nu_2, \ldots, \nu_n ] \]
As in the proof of Spectral Theorem
\[ P^T A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & E \end{bmatrix} \]
\[ P^T B P = \begin{bmatrix} \mu_1 & 0 \\ 0 & F \end{bmatrix} \]

Where \( E, F \) are real Symm of size \( (n-1) \times (n-1) \).
\[
(P^T A P, P^T B P) = (P^T B P, P^T A P) \implies EF = FE
\]

So we can induce on the size of \( A \) and \( B \) and thus we may assume inductively that \( E, F \) have a common basis of eigen vectors \( \nu_2, \ldots, \nu_n \) in \( \mathbb{R}^{n-1} \).

But as in the Spectral Theorem if we set
\[ \nu_j = P^T [ 0 \nu_j ] \text{ then also} \]
\[ A \nu_j = \lambda_j \nu_j \quad B \nu_j = \mu_j \nu_j \text{ would hold} \]

\[ ( \lambda_j \text{ is eigenvalue of } E \text{ with eigen-vector } \nu_j \]
\[ \mu_j \text{ is eigenvalue of } F \text{ with eigen-vector } \nu_j \)

thereby providing us with the common basis of eigen vectors.

Remark: Thus commuting real Symm (or Hermitian) matrices can be simultaneously Unitarily Diagonalized.

Normal Matrices: A matrix \( A \) with complex entries is said to be normal if
\[ AA^* = A^* A \]

Thus Hermitian and Unitary matrices are normal.

Prop: Given a \( n \times n \) normal matrix \( A \), \( A \) is
Hermitian matrices \( B \) and \( C \) such that
(i) \( A = B + i C \)
(ii) \( BC = CB \).
Proof: Take \( B = \frac{1}{2} (A + A^*) \)

\[
C = \frac{1}{2i} (A - A^*)
\]

Then \( B \) and \( C \) are Hermitian and

\[ BC = CB \] (since \( AA^* = A^*A \))

If \( A = P + iQ \) where \( P, Q \) are Hermitian

and commuting,

\[ A^* = P - iQ \] and we see that

\[ P = B \] and \( Q = C \) proving uniqueness.

Since \( B, C \) can be simultaneously unitarily diagonalized we get:

Theorem: Every normal matrix has an

orthonormal basis of eigen vectors

we can be unitarily diagonalized.

Conversely, if an \((mxn)\) matrix is

unitarily diagonalizable it is normal.

Proof: Only the last part needs a proof.

Let \( A \) be unitarily diagonalizable

\[
U^*AU = \begin{bmatrix}
\lambda_1 \\
& \lambda_2 \\
&& \ddots \\
&&& \lambda_n
\end{bmatrix}
= D
\]

\[
U^*A^*U = \begin{bmatrix}
\bar{\lambda}_1 \\
& \bar{\lambda}_2 \\
&& \ddots \\
&&& \bar{\lambda}_n
\end{bmatrix}
= \bar{D}
\]

Since \( D \) and \( \bar{D} \) commute,

\[
(U^*AU) (U^*A^*U) = (U^*A^*U) (U^*AU)
\]

\[
\therefore \quad AA^* = A^*A \quad \text{The proof is complete.}
\]
Theory of plane Curves

A parametrized curve in \( \mathbb{R}^n \) is a mapping 
\[ \delta : I \rightarrow \mathbb{R}^n \]
where \( I \) is an open interval in \( \mathbb{R} \), called the parameter interval, and \( \delta \) has derivatives of all orders.

A parametrized curve \( \delta \) is said to be regular if \( \delta'(t) \neq 0 \) for any \( t \in I \).

Note that a curve is not just a point set.

Thus \( \delta_1 : \mathbb{R} \rightarrow \mathbb{R}^2 \) and \( \delta_2 : \mathbb{R} \rightarrow \mathbb{R}^2 \) given by
\[ \delta_1(t) = (\cos t, \sin t) \]
\[ \delta_2(t) = (\cos 2t, \sin 2t) \]
are two distinct curves through they both describe the unit circle \( \{ (x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1 \} \) as a point set. The image \( \{ \delta(t) | t \in I \} \) is called the trace of the curve.

Ex: Show that if \( \delta_1, \delta_2 \) are two curves with the same parameter interval \( I \) then
\[ \frac{d}{dt} \left< \delta_1, \delta_2 \right> = \left< \delta_1', \delta_2' \right> + \left< \delta_1, \delta_2'' \right> \]

We shall use interchangeably the notations
\[ \delta'(t) = \delta(t) = \frac{d}{dt} \delta(t) \]
\[ \| x \| \text{ will always denote the Euclidean norm } \langle x^2 + \ldots + x^n \rangle^{\frac{1}{2}} \]

Def: Given a parametrized curve \( \delta : I \rightarrow \mathbb{R}^n \) the vector \( \delta'(t) \) is the tangent vector to the curve at the point \( \delta(t) \) on the curve.
Def: \( ||\mathbf{\dot{r}}(t)|| \) is called the speed of the curve.

Ex: Trace the curve \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^2 \) given by \( \mathbf{r}(t) = \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \) and determine its speed.

its speed

(2) Show that if \( \mathbf{a}, \mathbf{b} \) are linearly independent unit vectors in the plane then \( \mathbf{r}(t) = \mathbf{a} \cos t + \mathbf{b} \sin t \) describes an ellipse. Determine its major and minor axes.

(3) Show that the curve \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^3 \) \( \mathbf{r}(t) = (a \cos t, a \sin t, bt) \) lies on a cylinder and find the tangent vector at any point on it. Sketch the curve.

(4) Use Exercise (2) to write out a parameterization for the circle of intersection of \( x^2 + y^2 + z^2 = 1 \) and the plane \( x + y + z = 0 \) and do the same for the circle of intersection of \( x^2 + y^2 + z^2 = \frac{1}{3} \).

(5) Consider a quadratic form \( Q(x) \) given by \( x^T A x \) where \( A \) is an \( n \times n \) real symmetric matrix. Discuss conditions under which the quadratic \( x^T A x = 1 \) contains straight lines. If so, how many such lines pass through a given point \( \mathbf{x}_0 \) on the quadric?

Sol. If such lines exist they are given by \( \mathbf{x}_0 + t \mathbf{u} \) and \( \mathbf{x}_0 + s \mathbf{v} \) where \( \mathbf{u}^T A \mathbf{u} = 0 \) and \( \mathbf{v}^T A \mathbf{v} = 0 \).

Thus the directrix of the line is the intersection of a hyperplane and a cone through the origin (look at the cases \( n = 3, n > 3 \)).

(6) Sketch the curve \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^2 \) given by \( \mathbf{r}(t) = (a \cos^2 t, b \sin^2 t) \). Discuss the regularity of the curve.

(7) Show that the curve \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^3 \) given by \( \mathbf{r}(t) = (a_1 + a_2 t + a_3 t^2, b_0 + b_1 t + b_2 t^2, c_0 + c_1 t + c_2 t^2) \) lies on a plane. What kind of curve is it?

(8) A pendulum is suspended at the origin and is set in motion by a force in the direction of a velocity \( \mathbf{u} \). The pendulum rises and reaches its unstable equilibrium as \( t \to \infty \). Find the trajectory of the bob as a parametrized curve.
Arc Length

Let \( \gamma : I \to \mathbb{R}^n \) be a parametrized curve and \( c, d \in I \) with \( c < d \).

The arc length of the piece of the curve \( \gamma \) on \( [c, d] \) is given by

\[
L_{\gamma}[c, d] = \int_{c}^{d} \| \gamma'(t) \| \, dt
\]

denoted by \( L_{\gamma}[c, d] \).

**Exercise:** Explain intuitively why this definition makes sense. arrive at this formula by heuristic considerations and approximating the curve by polygons and obtaining thereby Riemann sums for the integral.

In particular if we fix \( a_0 \in I \) and take an arbitrary point \( t \in I \) ( \( s < a \), not excluded) then

\[
L_{\gamma}[a_0, t] = \int_{a_0}^{t} \| \gamma'(s) \| \, ds
\]

Note that if \( t < a_0 \) then \( L_{\gamma}[a_0, t] \) would be negative and if \( t = a_0 \) it is zero.

The function \( L_{\gamma}[a_0, t] = s(t) \) for short is called the arc length function defined on \( I \).

Thus, \( s(t) \) is the oriented distance measured along the curve from a reference point \( \gamma(a_0) \) chosen on the curve.

Reparametrization. Let \( \phi : J \to \mathbb{R}^n \) be a parametrized curve which is regular and \( \gamma : I \to \mathbb{R}^n \) be a smooth map with a smooth inverse. Then, the curve \( \tilde{\gamma} = \phi \circ \psi \) is called a reparametrization of \( \gamma \).

The reparametrization \( \psi \) is said to be proper or orientation preserving if \( \psi' > 0 \) on \( J \).

**Theorem:** The arc length is invariant under proper reparametrization. Thus if \( \phi \) is as above then

\[
L_{\psi \circ \phi}[c, d] = \int_{c}^{d} \| (\psi \circ \phi)'(s) \| \, ds
\]

**Proof:**

\[
L_{\psi \circ \phi}[c, d] = \int_{c}^{d} \| (\psi \circ \phi)'(s) \| \, ds
= \int_{c}^{d} \| (\psi'(s) \circ \phi'(s)) \| \, ds
= \int_{c}^{d} \| \psi'(s) \| \, ds
= L_{\psi}[\phi(c), \phi(d)]
\]

Proof is complete.

Fixing \( c = a \) and varying \( d \) one can think of using the arc length function to reparametrize a regular curve.

Note that if \( \gamma : I \to \mathbb{R}^n \) is a regular curve then \( \frac{ds}{dt} = \| \gamma'(t) \| > 0 \). Fixing \( t = a_0 \in I \)

We get a function \( \phi : a_t \to \mathbb{R}^n \)

\[
\frac{ds}{dt} = \| \gamma'(t) \|
\]

denoted by the variable \( \beta \), i.e. \( \beta = \phi(t) \).
\[ \phi: [a_0, t] \to [0, L \cdot [a_0, t]] \quad (t > a_0) \]

The reparametrization of \( \gamma \) via arc length is then the parametrized curve

\[ \gamma = \phi \circ \phi^{-1} \]

Now
\[
\frac{d\gamma}{ds} = \gamma'(\phi^{-1}(s)) \frac{d\phi^{-1}(s)}{ds} = \gamma'(s) ; \quad \gamma(s) = \phi(t) \]
\[
\gamma'(t) = \gamma'(s) \quad ||\gamma'(s)||
\]

Thus \( ||\gamma'(s)|| = 1 \) and the arc length parametrization produces a unit speed curve.

In what follows (examples and exercises apart) a Curve will always mean a regular parametrized unit speed curve, Curve with unit speed.

Ex: What happens if we reparametrize a unit speed curve by arc length?

(9) Exercises: Find the arc length function for the Conical helix \( \gamma: I \to \mathbb{R}^3 \) given by \( \gamma(t) = (t\cos t, t\sin t, et) \)

(10) Exercise: On the Sphere \( x^2 + y^2 + z^2 = 1 \) Sketch the meridians and determine all curves on the sphere that cut each meridian at an angle of 45°. Such curves are called loxodromes.
The Osculating Circle and center of Curvature.

Let \( \delta : I \to \mathbb{R}^2 \) be a regular parametrized plane curve parametrized by arc length and let \( s_1, s_2, s_3 \in I \) be three distinct values of the parameter.

Let \( C(s_1, s_2, s_3) \) be the center of the circle through \( \delta(s_1), \delta(s_2) \) and \( \delta(s_3) \). (If it exists.)

The function
\[
\delta(s) = (\delta(s) - C(s_1, s_2, s_3)) \cdot (\delta(s) - C(s_1, s_2, s_3))
\]
vanishes at \( s_1, s_2, s_3 \) so that by Rolle's theorem \( \delta'(s) \) vanishes on \( (s_1, s_2) \) and \( (s_2, s_3) \) any at \( u_1 \) and \( u_2 \). Again \( \delta''(s) \) vanishes on \((u_1, u_2)\); say \( \delta''(s) = 0 \).

Now
\[
\begin{align*}
\delta'(s) &= 2 \delta'(s) \cdot (\delta(s) - C(s_1, s_2, s_3)) \\
\delta''(s) &= 2 + 2 \delta''(s) \cdot (\delta(s) - C(s_1, s_2, s_3))
\end{align*}
\]

Hence \( \delta'(u) \cdot (\delta(u) - C(s_1, s_2, s_3)) = -1 \)
\( \delta'(u) \cdot (\delta(u) - C(s_1, s_2, s_3)) = 0 \) \((i, j, k)\)

Now letting \( s, s_2, s_3 \to s \) a fixed value and assuming that the circle through \( \delta(s_1), \delta(s_2), \delta(s_3) \) has a limiting position \( Z(s, s_2, s_3) \)

\( \Sigma(s) \), or the point \( C(s_1, s_2, s_3) \) must then approach a point \( C_0(s) \).

\( \Sigma(s) \) is called the osculating circle to \( \delta \) at \( \delta(s) \)
and its center \( C_0(s) \) is called the center of curvature.

Then
\[
\begin{align*}
\delta''(s) \cdot (\delta(s) - C_0(s)) &= -1 \quad (4) \\
\delta'(s) \cdot (\delta(s) - C_0(s)) &= 0 \quad (5)
\end{align*}
\]

Note that if \( \delta''(s) = 0 \), \( C_0(s) \) doesn't exist.

Let us now assume \( \delta''(s) \neq 0 \) and proceed to determine \( C_0(s) \).

Since the curve is parametrized by arc length,
\[
|\delta'(s)| = 1 \quad \text{or} \quad |\delta'(s) \cdot \delta''(s)| = 1
\]

Differentiating gives the equation \( \delta'(s) \cdot \delta''(s) = 0 \)
or \( \delta'(s) \) and \( \delta''(s) \) are orthogonal - thereby providing a basis \((\delta, \delta''(s))\).

\[
\delta - C_0(s) = A \delta' + B \delta''(s)
\]

for certain scalars \( A \) and \( B \). Using \((4)\) and \((5)\)

we see that \( B = N \delta' \cdot \delta''(s) \) \( \delta''(s) \) and taking dot product with \( \delta''(s) \) gives

\( A = 0 \).

Whereas dot product with \( \delta''(s) \) gives \[ \|\delta''(s)|^2 B = (\delta - C_0(s) \cdot \delta''(s) = -1 \quad \text{or} \quad B = -1 \]

Thus \( C_0(s) = \delta(s) + \delta''(s) \) \((7)\)

and finally
\[
\delta(s) = \|\delta(s) - C_0(s)||^2 = \|\delta''(s)||
\]

Thus the radius of the osculating circle is \[ \|\delta''(s)||^{-1} \]

Def: The Curvature \( k(s) \) is the reciprocal of the radius of the osculating circle and \( k(s) = \|\delta''(s)||. \)
Exercise: We have tacitly assumed that for the all \( S_1, S_2, S_3 \) sufficiently close to \( S \) and distinct there passes a circle through \( \delta(S_1), \delta(S_2) \) and \( \delta(S_3) \).

Moreover this circle has a limiting position \( \Sigma(S) \) as \( S_1, S_2, S_3 \to S \).

Discuss this. Clearly such a circle cannot exist if \( \delta''(S) = 0 \).

Show that if \( \delta''(S) \neq 0 \) then the osculating circle exists.

Hint: Try a Taylor expansion and neglect higher order terms beyond \( \delta''(S) \).

Well, suppose that \( \delta''(S) \neq 0 \) and \( S_1, S_2, S_3 \) arbitrarily close to \( S \).

\( \delta(S_3) \approx \delta(S_1) \) and \( \delta(S_2) \approx \delta(S_1) \) are colinear (in which case a circle cannot pass through \( \delta(S_1), \delta(S_2), \delta(S_3) \)).

We shall arrive at a contradiction.

Hence we may assume \( S_3 = S_0 \) so we can let \( S_3 \to S_0 \) in the equation

\( (\delta(S_3) - \delta(S_1)) \times (\delta(S_3) - \delta(S_2)) = 0 \).

Second we may assume \( \delta(S_0) = 0 \) and \( \delta''(S_0) = 0 \).

So that by Taylor's theorem

\( \delta(S) = aS + bS^2 + S^3\delta(S) \)

for all \( S \) in a nbhd of \( 0 : n \theta(S) = 1 \).

So \( (aS_1 + bS_2^2 + S^3\delta(S)) \times (aS_2 + bS_2^2 + S^3\delta(S)) \)

\( = 0 \)

\( \implies (a \times b) (S_1 S_2^2 - S_2 S_1) + (a \times \delta(S) S_2^3 - a \times \delta(S) S_2 S_1) S_3 = 0 \)

\( + (b \times \delta(S_1)) S_2^3 S_3 - (b \times \delta(S_2)) S_1^3 S_2 = 0 \)

Dividing by \( S_1 S_2 (S_1 - S_2) \) and letting \( S_1, S_2 \to 0 \)

we get \( a \times b = 0 \).

Note: To deal with the last term after dividing we get

\( S_1 S_2 \left( \frac{(b \times \delta(S_1)) S_2 - (b \times \delta(S_2)) S_1}{S_1 - S_2} \right) \)

Take absolute values and apply the MVT.

But \( a \times \delta = \delta(0) \).

\( a \times b = 0 \implies a = 0 \) or \( b = 0 \) which is a contradiction. Since \( \|S_1 - S_2\| = 0 \) and \( \|S_1 - S_2\| \neq 0 \).

Thus in a sufficiently small neighborhood \( N(S_0) \) of \( S_0 \) such that \( \delta''(S_0) \neq 0 \).

For any three distinct \( S_1, S_2, S_3 \)

\( \delta(S_1), \delta(S_2), \delta(S_3) \) are noncolinear.

Next, to show that \( \Sigma(S) \) exists (3) implies

\( \delta''(v) \cdot (\delta(v) - C(\delta(S_1, S_2, S_3)) = -1 \) for a sequence of values \( v \to S_0 \), and

\( \delta''(v) \cdot (\delta(v) - C(\delta(S_1, S_2, S_3)) = 0 \) for another seq.

\( v \to S_0 \).

\( \delta''(v) \cdot C = \delta''(v) \cdot (\delta(v) - C(\delta(S_1, S_2, S_3))) \)

\( R_H \to 1 + \delta''(v) \cdot \delta(S_1, S_2, S_3) \).

And so \( |C| \leq 1 + \delta''(v) \cdot \delta(S_1, S_2, S_3) \)

\( \delta''(S) \approx \delta''(v) \) has a limit as \( S_1, S_2, S_3 \to S_0 \).

(See figure)

\( \delta''(S) \approx \delta''(v) \)

\( C \implies |C| \leq 1 + \delta''(v) \cdot \delta(S_1, S_2, S_3) \to \infty \) finite.
Likewise $\delta(s)$ has a finite limit as $s_1, s_2, s_3 \to s_0$.

Formula for Curvature for non unit-speed curves:

We now derive the formula for $k(t)$ when the curve is not parametrized by arc length.

Let $s = \phi(t)$ be the arc length function with inverse $t = \psi(s)$.

$\sigma = \sigma(t)$ is the arc length reparametrization of $\delta$.

$$\sigma'(s) = \frac{\sigma'(\psi(s))}{\phi'(t)} = \frac{\psi'(s)}{\|\psi'(s)\|}$$

$$\sigma''(s) = \frac{\sigma''(\psi(s)) - \sigma'(\psi(s)) \sigma'(\psi(s))}{\|\sigma'(s)\|^2}$$

$$= \frac{\|\psi'(s)\|^2 - \sigma'(\psi(s)) \cdot \sigma'(\psi(s))}{\|\sigma'(s)\|^2}$$

$$= \frac{1}{\|\sigma'(s)\|^2} \left\{ \frac{\sigma''(\psi(s))}{\|\sigma'(s)\|^2} - \sigma'(\psi(s)) \cdot \sigma'(\psi(s)) \right\}$$

$$\|\sigma''(s)\|^2 = \frac{\|\sigma'(t) \times \sigma'(t)\|^2}{\|\sigma'(t)\|^3}$$

*Well, $\|\sigma'(t)\| = (\delta'(t), \sigma'(t))^2$.

Diff. w.r.t $s$ gives $\langle \sigma'(t), \sigma''(t) \rangle dt$.

Using Equation (7) we get the center of Curvature

$$C(t) = \sigma(t) + \frac{\sigma''(t) \|\sigma'(t)\|^2 - \sigma'(t) \cdot \sigma''(t)}{\|\sigma'(t)\|^2}$$

$$= \sigma(t) + \frac{\|\sigma'(t)\|^2}{\|\sigma'(t)\|^2}$$

Example: Determine the center of Curvature of the parabola $\delta(t) = (t^2, 2t)$

$$C(t) = \sigma(t) + \frac{\sigma''(t) \|\sigma'(t)\|^2 - \sigma'(t) \cdot \sigma''(t)}{\|\sigma'(t)\|^2}$$

$$= \sigma(t) + \frac{\|\sigma'(t)\|^2}{\|\sigma'(t)\|^2}$$

(13) Example: Determine the center of curvature of the parabola and the ellipse and discuss their regularity properties.

Involutes and Evolutes:

Assume that a string of fixed length which is fastened along the curve $\delta(t)$ is allowed to unwind in such a way that the free portion of the string is a line segment tangent to the curve $\delta(t)$ at the point of contact.

The locus of the free end is called the involute of $\delta$.

To convert this physical description into a mathematical definition we proceed as follows.

Let $\beta(t)$ be the free end of the string and the free portion of the string at time $t$ be the line segment joining $\delta(t)$ and $\beta(t)$.

The tangency condition is that $(\beta(t) - \delta(t))$ is parallel to $\delta(t)$. 

$$k(t) = \|\sigma'(t) \times \sigma''(t)\|^2$$

$$= \frac{\|\sigma'(t) \times \sigma'(t)\|^2}{\|\sigma'(t)\|^3}$$

Using Equation (7) we get the center of Curvature
but makes $\pi$ with $\delta(t)$\footnote{It would make $\pi$ zero if the reference point $\delta(t)$ is taken on the other end resulting in a change in sign in one of the terms in (9) & (10).}

\[
\begin{align*}
|\beta(t) - \delta(t)| &= |\beta'(t) - \delta'(t)| \\
&= \int_{t}^{t_1} \sqrt{(\delta'(s))^2 + (\delta'(s))^2} \, ds = t - t_1, \\
\text{or } \frac{d}{dt} |\beta(t) - \delta(t)| &= |\beta'(t) - \delta'(t)| \quad \text{or} \quad (9)
\end{align*}
\]

\[
\begin{align*}
(\beta(t) - \delta(t)) \cdot (\beta'(t) - \delta'(t)) &= \delta(t) - \delta(t) \\
&= 0. \\
\Rightarrow (\beta(t) - \delta(t)) \cdot \beta'(t) &= 0. \\
\text{Again, } (\beta(t) - \delta(t)) = \lambda \delta'(t) \text{ so that} \\
(\beta'(t) - \delta'(t)) &= 0. \quad (10)
\end{align*}
\]

**Def.** The involute of $\delta(t)$ is a curve $\beta(t)$ such that $\beta(t) - \delta(t)$ is tangential to $\delta(t)$ and $\beta'(t) \cdot \delta(t)$ with this one can actually work backwards but we prove the result formally.

Theorem: $\beta(t)$ is the involute of a unit speed curve $\alpha$ iff for some constant $c$

\[
\beta(t) = \alpha(t) + (c - s) \alpha'(s)
\]

**Proof:** If $\beta(t) = \alpha(t) + (c - s) \alpha'(s)$

then $\beta(t) - \alpha(t)$ is tangent to $\alpha$ and $\beta'(t) = (c - s) \alpha'(s)$ so that $\beta'(t) \cdot \alpha(t) = 0$

Conversely let $\beta(t)$ be an involute of $\alpha$

\[
\beta(t) = \alpha(t) + \lambda(t) \alpha'(t) \text{ for some scalar function } \lambda
\]

Taking dot product with $\alpha$ gives

\[
0 = 1 + \lambda \alpha' \text{ or } \lambda = -1 \text{ and } \lambda(s) = c - s
\]

The proof is complete.

**Remark:** Note that in the definition of an involute, does not make any reference to the parametrization of $\delta$ but is expressed solely in terms of geometric properties that remain invariant under reparametrization.

**Def.** If $\beta(t)$ is an involute of $\alpha(t)$ then $\alpha(t)$ is called an evolute of $\beta(t)$ of a curve

**Theorem:** The evolute of $\beta(t)$ is the locus of its centres of curvature.

**Proof:** To show $\beta(t)$ is an involute of $\alpha(t)$ the condition $\delta(t)$ - $\delta(t)$ is taken of the tangent $\alpha(t)$.

Well, the osculating circle of $\beta$
at the point $\mathbf{B}(t)$ is certainly tangent to
$
\mathbf{B},$ and so
$(\mathbf{B}(t) - \mathbf{C}(t)) \cdot \mathbf{B}(t) = 0 \quad (11)
$
But by form (7) (We assume the given
curve is parameterized by arc length)
$(\mathbf{B}(t) - \mathbf{C}(t)) \cdot \mathbf{B}(t) = -\|\mathbf{B}(t)\|^2$
and $\mathbf{B}(t) \cdot \mathbf{B}(t) = 0$

So that
$(\mathbf{B}(t) - \mathbf{C}(t)) \cdot \mathbf{B}(t) = -\frac{\|\mathbf{B}(t)\|^2}{\|\mathbf{B}(t)\|^2} = -1$

Taking dot product with $\mathbf{B}(t)$ we get:

$1 - \mathbf{C}(t) \cdot \mathbf{B}(t) = -\frac{\|\mathbf{B}(t)\|^2}{\|\mathbf{B}(t)\|^2}$

Now differentiate the equation $\mathbf{B}(t) \cdot \mathbf{B}(t) = 0$
and we see $\mathbf{B}(t) \cdot \mathbf{C}(t) = 0.$ The second
condition has been verified.

But now the condition
$(\mathbf{B}(t) - \mathbf{C}(t)) \parallel \mathbf{C}(t)_{1}$
parallel to $\mathbf{C}(t)$ is
Equivalent to $\mathbf{B}(t) - \mathbf{C}(t)$ is orthogonal to $\mathbf{B}(t)$
which follows from (11). The proof is complete.

From exercise (14) we know the evolute
of an ellipse.

(15) Determine the involute of a circle.

(16) Determine the involute of a parabola

Exercise: Show that for a non-unit speed curve $\mathbf{B}$
the involute is given by
$(\mathbf{B}(t) = \sigma(t) + (C - \int \sigma(t) dt)) \mathbf{B}(t)$

Changing to amounts $c$.

Cyclides (The brachistochrone and tautochrone)

The common cycloid is the curve which is the
base locus of a fixed point on a circle that rolls on
a straight line without slipping (contour line "baseline")
Let $O$ be the instantaneous point of contact of the
Circle with the $x$-axis with $O_0$ = Origin.
and $P$ be the Center of the Circle at time $t$ and
$\mathbf{B}(t)$ be the point on the cycloid along a line
The condition of rolling without slipping
may be translated as
length $O_0 \mathbf{B} = \text{arc (of the circle)}$ for joining
$\mathbf{B}(t)$ and $O_0$

Let $\theta$ be the angle subtended by this arc at
the Center $P_0.$ and is taken as the parameter along the curve.
Then $s = a\theta = \text{length } O_0 \mathbf{B}$ along the $x$-axis.
The abscissa of $\mathbf{B}(t)$ is thus
$a \theta - \text{asine}$

and the ordinate is $a - a \cos \theta.$ Thus
$s: IR \rightarrow IR^2$ in given by
$s(\theta) = (a \theta - \text{asine}, a - a \cos \theta)$
after a complete revolution the point of the
locus is again on the $x$-axis.
At half time, $\theta = \pi$ and the $y$-coordinate
attains a max $= 2a.$
Let us calculate the arc length of one arch of the cycloid:

\[
\begin{align*}
&= \int_{0}^{2\pi} \sqrt{\left( \frac{d\gamma}{d\theta} \right)^2} \, d\theta \\
&= \int_{0}^{2\pi} \sqrt{\left( \frac{a - a \cos \theta}{1 - \cos \theta} \right)^2 + a^2 \sin^2 \theta} \, d\theta \\
&= \int_{0}^{2\pi} a \sqrt{2 - 2 \cos \theta} \, d\theta \\
&= 8a \int_{0}^{\pi} \sin \theta \, d\theta \\
&= 8a.
\end{align*}
\]

Note that the parametrization is not the arc length parametrization.

In fact, \( \frac{ds}{d\theta} = 2a \sin \theta \).

So that \( \theta = 4 \sin^{-1} \left( \frac{x}{\sqrt{8a}} \right) \).

Example: Let us compute the center of curvature of the cycloid.

\[ \dot{\gamma}(\theta) = (a - a \cos \theta, a \sin \theta) \]

\[ \ddot{\gamma}(\theta) = (a \sin \theta, a \cos \theta) \]

\[ \gamma(\theta) \times \ddot{\gamma}(\theta) \mid_{\gamma'(\theta)} = a^2 \left( 1 - \cos \theta \right) \]

\[ \gamma'(\theta) \cdot \ddot{\gamma}(\theta) = a^2 \sin \theta \]

\[ \frac{ds}{d\theta} = 2a \sqrt{1 - \cos \theta} \]

\[ C(\theta) = (a \theta + a \sin \theta, -a + a \cos \theta) \]

\[ R(\theta) = \frac{1}{a} \csc \theta / \left( (1 - \cos \theta)^{3/2} \right) \]

So that the radius of curvature \( \to 0 \) as \( \theta \to 0, 2\pi \).

(The curve is not regular at \( \theta = 0 \) and \( \theta = \pi \) and is maximum at \( \theta = \pi \).

\[ R(\theta) = \frac{1}{a} \sin \theta. \]

Theorem: The evolute of a cycloid is another congruent cycloid obtained by reflection along the base line \( x = \pm a \).

Proof: We shall show that the curves

\[ \delta(\theta) = (a \theta - a \sin \theta, -a + a \cos \theta) \]

\[ C(\theta) = (a \theta + a \sin \theta, -a + a \cos \theta) \]

\[ \theta \in \mathbb{R} \text{ are congruent.} \]
Consider a point \( P \) on \( \gamma \) close to the origin. The center of curvature in a point \( P \) which is a reflection of \( P \) about the point \((R, 0)\) directly on the base line. This suggests the following:

The above isometry \( y \mapsto -y \)

translates \( \delta(t) = (a t - a \pi, -a \sin t, -a - a \cos t) \)

into the curve

\[ \gamma(t) = (a (\pi - t) + a \sin(\pi - t), -a - a \cos(\pi - t)) \]

The parameter reversal

\( \theta \mapsto \pi - \theta \) produces the curve

\[ \delta(\theta) = (a \theta + a \sin \theta, -a - a \cos \theta) \]

This followed by an reflection plus:

isometry

\[ x \mapsto x \]

\[ y \mapsto y + 2a \] preserves \( \gamma(\theta) \).

The proof is complete.

Let us now determine the involute of a cycloid. For convenience we shift the cycloid so that the arches are all tangent to the \( x \)-axis and origin is a point of tangency. This is simply achieved by \( a \theta \) a translation through \( (a \pi, +2a) \)

i.e. subtracting off \( (at - a \sin t, a - a \cos t) \)

the vector \( (a \pi, +2a) \).

To get

\[ \delta(t) = (a t - a \pi, -a \sin t, -a - a \cos t) \]

\( t \in \mathbb{R} \)

We would like to shift the time parameter \( a \theta \) so that at time \( t = 0 \) we are at the origin and at time \( t = \pi \) at the cusp \( C \).

Thus we put \( \theta \mapsto t - \pi \)

\[ \gamma(t) = (at + a \sin t, a - a \cos t) \]

\[ \beta(t) = \delta(t) + (c - s(t)) \delta(t) \]

\( c \in (\mathbb{R}) \)

\[ s(t) = \int_{-a}^{t} \frac{dx}{\sqrt{1 + a^2}} \]

\[ \beta(t) = \delta(t) + (c - 4a \sin^2 t) (\cos t, -\sin t) \]

(Note that \( \sin \theta > 0 \) on \( (0, \pi) \).

Since \( \gamma \) is symmetrical about the line \( x = a \pi \)

So would the involute (and hence it is enough to restrict \( t \in (0, \pi) \).

Exercise: Prove the above claim.

Put \( t = 0 \) on \( \beta(t) = (c, 0) \).

Take \( c = 0 \) so that we start at the involute starts at the origin (which amounts to choosing the length of the string = \( 4a \)).

Then \( \beta(t) = (at - a \sin t, a - a \cos t) \)

We see that the involute is again a cycloid for this specific choice of \( c \).
If we take a different value of \( c \), our curve will be:

\[ \beta'(t) = (a + c \cos t, -a \sin t, \alpha \cos t) + x \delta \]

Investigate this curve. Note that the involute \( \beta(t) \) is dragged down by a unit vector of constant length \( a \).

The direction of drag is along the normal to \( \beta \) by the property of involute. Is \( \beta'(t) \) a cyclïd?

Now \( \beta (0, \pi) = (a, 2a) \) and so the point \( \beta (\pi) \) lies vertically above the cusp \( C \).

At a distance \( 4a = \) length of the string = length of the half arch of the cycloid from origin to \( C \).

The above example will play a crucial role in our discussion of the tautochrone property of the cycloid.

---

**The brachistochrone property of a cycloid:**

*Brachístochrone = Shortest; Chronos = time.*

Let us consider a curve joining two points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) such that \( y_2 < y_1 \). Let a particle slide down along the curve such that the time of descent is least among all choices of curves joining \( A \) and \( B \). To determine the shape of the curve.

This is a problem in calculus of variations.

*Assume \( A \) is the origin and \( B \) is in the fourth quadrant.*

Let us take the curve

\[ \gamma(t) = (x(t), y(t)) \text{ for } 0 \leq t \leq 1 \]

Some function \( f \) of \( x \) on [0, \( x \)]

The instantaneous kinetic energy is then \( \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \)

The work of the particle is 1 and the work done is:

\[ \frac{1}{2} \left( 1 + f'(x)^2 \right) \frac{dy}{dt} \]

which must be equal to the loss in potential energy.

\[ \int_{x_1}^{x_2} \left( 1 + f'(x)^2 \right) dx = \sqrt{y} \text{ or} \]

\[ \frac{\sqrt{y}}{x} \int_{x_1}^{x_2} \sqrt{1 + f'(x)^2} dx \]

**The problem is then to select the curve \( \gamma \) such that:**

\[ I(\gamma) = \int_{x_1}^{x_2} \left( 1 + f'(x)^2 \right)^{1/2} dx \]

is least.

(The Conv. of the integral is taken up at the end)

Let \( f \) be the class of all smooth curves joining \( A \) to \( B \) and we have a map.
Called a "functional" \( I : J \rightarrow \mathbb{R} \) which we seek to minimize.

Let \( m = \inf_{y \in J} I(y) \)

and \( J \) a set of curves in \( J \) s.t.

\[ I(\gamma_n) \rightarrow m \]

but in general \( (\gamma_n) \) may not converge in \( J \) in any reasonable sense. However, in the case at hand one can show the existence of a minimizer \( \gamma = \lim \gamma_n \) by using the so-called Ascoli Arzelà Theorem. However, such an analysis is beyond the scope of this course.

We proceed somewhat heuristically.

In general, the linear functional \( I \) assumes the form

\[ I(\gamma) = \int_{\alpha}^{\beta} F(x, \gamma, \gamma') \, dx \]

Where \( \gamma : [\alpha, \beta] \rightarrow \mathbb{R}^n \) is an admissible curve and \( F : [\alpha, \beta] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is smooth.

If \( \gamma_0 \) is a minimizer then we take a curve \( \gamma + \varepsilon \sigma : [\alpha, \beta] \rightarrow \mathbb{R}^n \) satisfying \( \sigma(\alpha) = \sigma(\beta) = 0 \) (so that the curves \( \gamma_0 + \varepsilon \sigma \) all have the same end points).

Performing a Taylor Series expansion for the function

\[ F(x, y, z) = F(x_0, y_0, z_0) + \partial_1 F(x, y_0, z_0)(y - y_0) + \partial_2 F(x, y_0, z_0)(z - z_0) + \frac{1}{2} (\partial_1^2 F(x, y_0, z_0)(y - y_0)^2 + 2 \partial_2 F(x, y_0, z_0)(y - y_0)(z - z_0) + \partial_2^2 F(x, y_0, z_0)(z - z_0)^2) + R \]

Where \( R \) in the remainder and

\[ \| R \| \leq A (|y - y_0|^2 + |z - z_0|^2)^{3/2} \]

Then \( F = \partial_1 F \) refers to \( \partial_1 F \) and \( \partial_2 F = \partial_2 F \)

But since \( I(\gamma) = \phi(\varepsilon) \) for a minimizer \( \gamma_0 \), \( \phi(0) = 0 \), \( \phi(\varepsilon) \geq \phi(0) \) for all \( \varepsilon \) close to the origin.

\[ \phi(\varepsilon) = \phi(0) + \varepsilon \int \partial_1 F(x, \gamma_0, \gamma_0) \sigma \, dx \]

\[ + \int \partial_3 F(x, \gamma_0, \gamma_0') \sigma' \, dx + \int e^2 G(x) \]

\[ \mathbb{A}(x) = \text{quadratic terms} \]

We perform an integration by parts in the \( \partial_3 F \)

\[ \int_\alpha^\beta \partial_3 F(x, \gamma_0, \gamma_0') \sigma' \, dx = - \int_\alpha^\beta \partial_3 F(x, \gamma_0, \gamma_0') \sigma \, dx \]

\[ = - \int_\alpha^\beta d \left( \partial_3 F(x, \gamma_0, \gamma_0') \right) \sigma \, dx \]

by virtue of the boundary conditions \( \sigma(\alpha) = \sigma(\beta) = 0 \) and we have

\[ \phi(\varepsilon) = \phi(0) + \varepsilon \int \left[ \partial_1 F(x, \gamma_0, \gamma_0) - d \left( \partial_3 F(x, \gamma_0, \gamma_0') \right) \right] \sigma \, dx + \int e^2 G(x) + e^2 R(x) \]

The Next (and \( \phi'(0) = 0 \) now gives

\[ \int_\alpha^\beta \left[ \partial_1 F(x, \gamma_0, \gamma_0) - d \partial_3 F(x, \gamma_0, \gamma_0') \right] \sigma \, dx = 0 \]

Ex: Suppose \( f \) is a continuous function such that

\[ f(x, y, z) = f(x, y, z) \]

for all smooth functions (once diff) and the with
\[ g(x) = g(\beta) = 0, \quad \int f(t)g(t)dt = 0 \]

then \( f(t) \equiv 0 \) on \([\alpha, \beta]\).

Thus, we get the necessary condition for the minimizer \( \ddot{a} \), namely

\[ \dot{y} F(x, y, \dot{y}) - \frac{\partial}{\partial y} \left( \frac{\partial F(x, y, \dot{y})}{\partial \dot{y}} \right) = 0 \]

which is called the Euler-Lagrange equation for the variational problem.

For the case of the brachistochrone

\[ F(x, y, \dot{y}) = (1 + \dot{y}^2)^{1/2}/\sqrt{-y} \]

and a simple calculation gives the Euler-Lagrange eqn.

\[ \frac{\dot{y}^2}{2} \sqrt{1 + \dot{y}^2} (-y)^{-3/2} - \frac{d}{dx} \left( \frac{\dot{y}}{\sqrt{-y} \sqrt{1 + \dot{y}^2}} \right) = 0 \]

\[ \frac{\dot{y}^2}{2} \left( \frac{1}{\sqrt{-y}} \right)^{3/2} - \frac{1}{\sqrt{-y}} \frac{\dot{y}^2}{2} \left( \frac{1 + \dot{y}^2}{\sqrt{-y} \sqrt{1 + \dot{y}^2}} \right)^2 = 0 \]

\[ \frac{1}{\sqrt{-y} \sqrt{1 + \dot{y}^2}} \frac{\dot{y}''}{\sqrt{-y}} + \frac{y''}{\sqrt{-y}} \frac{y'y''}{(1 + \dot{y}^2)^{3/2}} = 0 \]

\[ \frac{1}{\sqrt{1 + \dot{y}^2}} \left( \frac{y''}{\sqrt{-y}} \right)^{3/2} + \frac{y''}{\sqrt{-y}} \left( \frac{1 + \dot{y}^2}{\sqrt{-y} \sqrt{1 + \dot{y}^2}} \right)^{3/2} = 0 \]

\[ 2y'y'' + (1 + \dot{y}^2) = 0 \]

which is on ODE for the desired curve. The initial condition is \( y'(0) = 0 \)
Two points remain to be discussed. 

(i) The meaning of the integral \( \int_0^1 \frac{1}{\sqrt{1+y^2}} \, dx \) near the origin.

(ii) Is the solution obtained a minimum or merely a stationary solution (we have only applied the necessary conditions for local minimum.)

Let us take up the second point and go back to the integral (\(*\)).

\[ \phi(t) = \phi(0) + e^2 \tilde{A}(x, x_0, \theta_0) + R \] (Since the first order terms have vanished)

A direct calculation gives

\[ Q = \frac{1}{2} \int \left( \frac{3}{2} \csc^2 \theta \right) \left( \phi'(\theta) \right)^2 \, d\theta - \frac{1}{2} \left( \sin^2 \theta_2 \right) \left( \phi'(\theta) \right)^2 \, d\theta \]

The quadratic term under the integral has a positive definite \( Q > 0 \)

Now \( \phi(t) > \phi(0) \) for all \( t \) small enough which confirms that \( \phi_0 \) is indeed a minimum.

\[ * \]

\[ \frac{\partial^2 F}{\partial \theta_2^2} = \frac{1}{\sqrt{1+y^2}} \frac{\partial^2}{\partial x^2} = \frac{1}{x_0} \sin^2 \theta_2 \]

\[ \frac{\partial^2 F}{\partial y^2} = \frac{3}{2} \csc^2 \theta \left( 1+x^2 \right) y_0 = \frac{3}{4} \csc^2 \theta_2 \]

\[ \frac{\partial^2 F}{\partial x^2} = \frac{1}{2} \frac{1}{\sqrt{1+y^2}} \left( \frac{3}{2} \csc^2 \theta \right) \]

Since the curve starts at the origin

\[ x(t) = at + bt^2 + \ldots \]

\[ y(t) = ct + dt^2 + \ldots \]

As functions of time.

Also since the particle starts from rest \( x(0) = y(0) = 0 \)

\( a = 0,\ c = 0 \)

The acceleration in the vertical direction \( \neq 0 \)

\( \beta \neq 0 \)

The factor \( dx \) in the integral equals

\[ \frac{2bt + 3ct^2 + \ldots}{\sqrt{2}} \]

Which makes perfect sense.

Exercise: In the above discussion we have assumed that \( F \) is a function from \( \mathbb{R}^2 \rightarrow \mathbb{R} \)

with \( \theta \) depending on \( x \) and the integrand in the objective function \( I \) being \( F(x, \theta(x), \theta'(x)) \)

Discuss what happens if \( \theta \) is \( \theta(x) \) for \( x \) fixed

\( F: I \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} \) with the class \( F \)

Consisting of curves \( \theta: I \rightarrow \mathbb{R}^2 \) with \( \theta(0) = \theta_0 \)

\( \theta(1) = \theta_0 \) fixed and \( \phi = (\phi_1, \ldots, \phi_n) \). What form will the

perturbed curve \( \phi + \epsilon \phi \) assume? How many zero

What will be the nature of the Euler-Lagrange

Equations?

Discuss the isoperimetric problem: Of all closed

Curves \( \gamma \) with a given perimeter the circle

Encloses the least area. Greatest area

Imitate the method of Lagrange multipliers.
The tautochrone property of the cycloid.

A wire has to be constructed such that a bead sliding down the wire reaches the bottommost point in a time $T$ that is independent of the height of release.

Let the bottommost point be the origin and let $(x_0, y_0)$ be the point of release, and $(x, y)$ be any intermediate point. The kinetic energy at $(x, y)$ is $\frac{1}{2} m (\frac{ds}{dt})^2$.

Which must balance the change in potential energy $mg(y_0 - y)$ to get that:

$$\left(\frac{ds}{dt}\right)^2 = 2g(y_0 - y)$$

$$\frac{ds}{dt} = -\sqrt{2g y_0 - y} \quad *$$

Regard $s$ as a function of $y$ and $F(y)$ and

$$F'(y) \frac{dy}{\sqrt{y_0 - y}} = -\sqrt{2g} \frac{dy}{dt}$$

$$\Rightarrow \int_{0}^{y_0} F'(y) dy = -\sqrt{2g} \int_{0}^{T} dt$$

$$\Rightarrow T = \frac{1}{\sqrt{2g}} \int_{0}^{y_0} \frac{F'(y) dy}{\sqrt{y_0 - y}} \quad (***)$$

* $s$ is the arc length measured from the origin.

Thus we have to arrive at integral equation for $F(y)$.

Equation (**) is called an integro-differential equation of convolution type.

We use Laplace transforms to arrive at:

Taking the Laplace transform of (**) we get:

$$\frac{T}{\lambda} = \frac{1}{\sqrt{2g}} \mathcal{L} \left( F' \right) \frac{\sqrt{\pi}}{\sqrt{\lambda}}$$

$$\Rightarrow \frac{T}{\sqrt{\lambda}} = \sqrt{\frac{\pi}{2g}} \mathcal{L} \left( F' \right)$$

$$\Rightarrow \mathcal{L} \left( F' \right) = \sqrt{\frac{2g}{\pi \lambda}} T$$

$$\Rightarrow \mathcal{F}' = \sqrt{\frac{2g}{\pi \lambda}} \frac{T}{\sqrt{\lambda}}$$

$$\Rightarrow \frac{dx}{dy} = \pm \sqrt{\frac{y + a}{y}}$$

(Since $y$ decreases as $x$ increases) in the picture but the situation would be reversed in the following picture.

We take both signs and

Integrating as before we get:

$$2a = a(1 - \cos 2\theta)$$

$$2x = \pm a(2\theta - \sin 2\theta)$$

Which is a cycloid.
Huygen's Isochronus Pendulum:

Recall that the period of a simple pendulum depends upon the amplitude.
C. Huygens constructed an isochronus pendulum whose period is independent of the amplitude of the swing.
For an interesting history of the subject see the work of R. Dugas.

Consider two successive arches $AO$ and $OB'$ of a cycloid and assume that a pendulum is suspended from $O$ and $OC$ is the mean position. If the pendulum is released from any $A$ then the bob traces an inverse of a cycloid $PP'$ and things with amplitude $CO$. But then due to the isochronous property of the cycloid the period would be independent of the amplitude $CO$ thereby providing an isochronous pendulum.
Exercise: For a simple pendulum determine the period as a function of the amplitude.
Space Curves and the Frenet Serret Frame

A space curve is a smooth map \( \gamma : I \to \mathbb{R}^3 \) where \( I \) is an open interval in \( \mathbb{R} \).

As usual, the arc length function is given by
\[
\gamma(s) = \int_0^s |\gamma'(t)| dt
\]
and one can reparametrize to \( \gamma \) by its arc length.

For a curve parametrized by its arc length, \( \gamma(s) \) in the unit tangent vector at the point \( \gamma(s) \).

Now \[
\langle E'(s), E'(s) \rangle = 1
\]
\[
\implies \frac{dE}{ds} \perp E'(s)
\]

Hence \( dE \) is a normal vector to the curve.

For a space curve, unlike the case of a plane curve, there are infinitely many normals to a space curve.

The unit vector \( n(s) = \frac{E'}{\|E'(s)\|} \)
\[
|E'(s)| = |\gamma''(s)|
\]

is called the principal normal to the curve. It exists only at those points \( \gamma(s) \)
for which \( \gamma''(s) \neq 0 \).

It may be noted that when \( \gamma \) is a plane curve \( \gamma''(s) \) is parallel to the

radius vector \( \gamma(s) - \overline{\gamma}(s) \), joining \( \gamma(s) \) and the center of curvature.

Now let us assume \( \gamma''(s) \neq 0 \) and \( s_1, s_2 \) be two values of the parameter close to \( s \).

\[
(\gamma(s_1) - \gamma(s)) \times (\gamma(s_2) - \gamma(s)) \text{ in then a vector } \perp \text{ to the plane } \Pi(s, s_1, s_2)
\]

The three points \( \gamma(s), \gamma(s_1), \gamma(s_2) \).

The circle through \( \gamma(s) \), \( \gamma(s_1), \gamma(s_2) \).
\( \gamma(s), \gamma(s_1), \gamma(s_2) \) also lie on the plane

\[
\overline{\gamma}(s_1) - \gamma(s) = (s_1, s_2) \gamma(s_1) + (s_1, s_2)^2 \gamma(s) + \ldots
\]
\[
\overline{\gamma}(s_2) - \gamma(s) = (s_2, s_2) \gamma(s_2) + (s_2, s_2)^2 \gamma(s) + \ldots
\]
\[
\therefore (\gamma(s_1) - \gamma(s)) \times (\gamma(s_1) - \gamma(s)) = (s_1, s_2, s_2) \gamma(s_1) \times \gamma(s)
\]
\[
= \gamma(s_1) \times \gamma(s)\gamma(s_2) \gamma(s_1)
\]
\[
+ \ldots
\]

Dividing by \( (s_1, s_2)(s_1, s_2) \) and letting \( s_1, s_2 \to s \)
we see that there is a limiting value to the unit normal vector to \( \Pi(s, s_1, s_2) \), namely

\[
\lim_{(s_1, s_2) \to (s, s)} \frac{\gamma(s_1) \times \gamma(s)}{\|\gamma(s_1) \times \gamma(s)\|}
\]

The lines through \( \gamma(s) \) and \( \gamma(s) \)
\( \gamma(s) \) and \( \gamma(s) \)
also lie on \( \Pi(s, s_1, s_2) \) which means that the plane \( \Pi(s, s_1, s_2) \) has a limiting position \( \Pi(s) \)
and orthogonal to \( \gamma(s) \times \gamma(s) \)

\[
\lim_{(s_1, s_2) \to (s, s)} \frac{\gamma(s_1) \times \gamma(s)}{\|\gamma(s_1) \times \gamma(s)\|}
\]
The plane $TT_S$ is called the osculating plane of $C(s)$ at the point $O(s)$.

The plane $TT_S$ is also called the normal plane, and its reciprocal is called the binormal plane.

The normal vector $\mathbf{N}(s)$ is parallel to the osculating plane.

The centre of curvature $K(s)$ is on $TT_S$.
The torsion to the curve.
It measures the rate of change of
The direction of \( b \) i.e. the rate at which \( \delta(s) \) deviates from planar.

Thm: \( \delta(s) \) is a plane curve iff \( b \) is a constant vector.

Proof: If \( \delta \) is a plane curve then \( \delta' \) lies in the plane of \( \delta \)
and so does \( \delta' \).
Thus \( \delta' \times \delta'' \) is constant. viz. the unit-
vector \( b \) that plane.

Conversely if \( \delta' \) is constant then look at

\[
\begin{align*}
f(s) &= (\delta(s)-\delta(0)) \cdot b \\
f'(s) &= \delta'(s) \cdot b = t \cdot b = 0
\end{align*}
\]

or \( f(s) \) is constant. But \( f(s) = 0 \) \( \Rightarrow \)

\( \delta(s)-\delta(0) \cdot b = 0 \quad \forall s \)
\( \Rightarrow \) \( \delta(s) \) lies on the plane passing through \( \delta(0) \) and \( t \) to \( b \).

Theorem: \( \mathbf{n}'(s) = -k(s) \mathbf{t} + \tau \mathbf{b} \)
\( \mathbf{b}'(s) = -\tau(s) \mathbf{n} \)
\( \mathbf{t}'(s) = k(s) \mathbf{n} \)

Proof: Only the first equation needs proof.
\( \mathbf{n} \cdot \mathbf{n} = 0 \) (\( \therefore \mathbf{n} \cdot \mathbf{n} = 1 \))
\( \Rightarrow \mathbf{n}' = \lambda \mathbf{t} + \mu \mathbf{b} \) for some scallars \( \lambda, \mu \).

\( \Rightarrow \lambda = \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{t} + \mathbf{n} \cdot \mathbf{t}' = \mathbf{n} \cdot \mathbf{t}' \)
\( = (\mathbf{n} \cdot \mathbf{t})' = \mathbf{n} \cdot k \mathbf{n} = -k \)

\[ 
\begin{bmatrix}
\mathbf{t}' \\
\mathbf{n}' \\
\mathbf{b}'
\end{bmatrix}
= 
\begin{bmatrix}
0 & k & 0 \\
-k & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{t} \\
\mathbf{n} \\
\mathbf{b}
\end{bmatrix}
\]

The formulas stated in the theorem are called the
Serret-Frenet formulas.

Fundamental theorem for curves in \( \mathbb{R}^3 \):

Let \( k \) and \( \tau \) be two functions on \( (a, b) \rightarrow \mathbb{R} \)
which are continuous and \( k > 0 \).
Then \( \exists! \) unit speed curve \( \delta : (a, b) \rightarrow \mathbb{R}^3 \)
whose curvature and torsion functions are \( k \) and \( \tau \)
respectively.

The uniqueness is in the sense that any two such curves differ by a rigid Euclidean motion.

Proof: Consider the system of ordinary
diff. equations
\[ 
\begin{align*}
\mathbf{v}'_1 &= k \mathbf{v}_2 \\
\mathbf{v}'_2 &= -k \mathbf{v}_1 + \tau \mathbf{v}_3 \\
\mathbf{v}'_3 &= -\tau \mathbf{v}_2
\end{align*}
\]
where \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are vector valued functions.
This is a linear system of 9 ODEs which may be written as a matrix system
\[ 
\begin{bmatrix}
\mathbf{v}'_1 \\
\mathbf{v}'_2 \\
\mathbf{v}'_3
\end{bmatrix}
= \Lambda(s)
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3
\end{bmatrix}
\]
Writing (contrary to usual conventions) the vectors
\( u_1, u_2, u_3 \) as rows.
The system of ODEs has a unique solution for any given initial conditions which we take to be an orthonormal frame:

\[
\begin{bmatrix}
  u_1(s) \\
  u_2(s) \\
  u_3(s)
\end{bmatrix}
\text{is an orthogonal matrix}
\]

With unit determinant.

Denote the skew-symmetric matrix
\[
\begin{bmatrix}
  0 & k & 0 \\
  -k & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]
by \( \Lambda(s) \).

Then consider the solution \( M(s) \) of the system of ODEs
\[
M = \Lambda M
\]
with \( M(0) \) an orthonormal matrix with \( \det = 1 \).

Claim: \( MM^T = I \) \( \forall s \in (a, b) \).

Well,
\[
d\left( MM^T \right) = MM^T + M M^T = \Lambda M M^T + M (\Lambda M)^T = \Lambda M M^T + M M^T \Lambda^T = \Lambda M M^T - M M^T \Lambda
\]
So \( MM^T \) satisfies a linear ODE
\[
\dot{X} = \Lambda X - X \Lambda
\]
with initial cond. \( I \).
But the constant function \( I \) also satisfies the same NVP
\[
MM^T = I \quad \forall s.
\]

Define \( \delta(s) = \int u_i(a) \, da \).

Then \( \| \delta(s) \|_1 = \| u_i \|_1 \) \( \Rightarrow \delta : (a, b) \to \mathbb{R}^3 \) is a unit speed curve.
\[
\ddot{\delta} = \dot{u}_i = k \nu_i \quad \text{(unit tangent vec.)}
\]
\[
\| \ddot{\delta} \|_1 = k \quad \text{so that } -k \text{ is indeed the curvature of } \delta.
\]

The principal normal is \( u_2 \).

Next, \( \nu_3 = \nu_1 \times \nu_2 \), which shows that \( \nu_3 \) is the binormal and \( \nu_3 = -\nu_2 \).
\[
\Rightarrow \nu \text{ is the torsion of } \delta.
\]

Suppose \( \delta_1, \delta_2 \) are two curves with the same \( k \) and \( \tau \) then \( \delta_1 \) is \( \text{orthog} \) to \( \delta_2 \) and \( \text{orthog} \) to \( \delta_0 \).

\[
\Rightarrow (\delta_1 \times \delta_2) = (\delta_1 \times \delta_0)
\]
by skew-symmetric properties.

By Serret-Frenet we have the pair of systems

\[
M_1 = \Lambda M_1, \quad M_2 = \Lambda M_2
\]

But
\[
M_1(0) = \mathcal{P} M_2(0) \quad \text{for some orthogonal P with } \det P = 1
\]

\[
\Rightarrow (M_1 - M_2 \mathcal{P}) = \Lambda M_1 - \Lambda M_2 \mathcal{P} = \Lambda (M_1 - M_2 \mathcal{P})
\]

which shows that \( M_1 \equiv M_2 \).
Def (Helix) A curve \( \gamma: (a, b) \to \mathbb{R}^3 \) of unit speed is called a helix if there exists a constant vector \( \mathbf{u} \) such that
\[
\langle \gamma'(s), \mathbf{u} \rangle \text{ is constant.}
\]
(Or \( \mathbf{u} \perp \gamma'(s) \), \( \mathbf{u} \) is constant, \( \mathbf{u} \) is the axis of the helix.)

\( \hat{\mathbf{u}} = \mathbf{u} / || \mathbf{u} || \)

Unit speed curve \( \gamma \) is called the axis of the helix.

\[ \gamma(t) = (\cos \theta t, \sin \theta t, \theta t) \]

\[ \hat{\gamma}(t) = (-\sin \theta t, \cos \theta t, \theta) \]

Ex: \( \gamma(t) = (\cos \frac{t}{\sqrt{2}}, \sin \frac{t}{\sqrt{2}}, \sqrt{2} t) \)

\( \hat{\gamma}(t) = (-\frac{\sin \frac{t}{\sqrt{2}}}{\sqrt{2}}, \frac{\cos \frac{t}{\sqrt{2}}}{\sqrt{2}}, 1) \)

For this helix, \( \theta = \sqrt{2} \)

\( \mathbf{n} = (-\cos \theta, -\sin \theta, 0) \)

At that point:
\[
\frac{d\mathbf{n}}{ds} = (\sin \theta, -\cos \theta, 0) \quad \text{on} \quad \gamma.
\]

\[ \mathbf{T} = \frac{d\mathbf{n}}{ds} \quad \mathbf{b} = \frac{d\mathbf{n}}{ds} \times \mathbf{T} \quad \mathbf{T} = \frac{1}{\sqrt{2}} \quad \text{Check!}
\]

Show that if \( \gamma \) is a curve in \( \mathbb{R}^3 \) for which the curvature and torsion are both constant, then \( \gamma \) is a cylindrical helix

Hint: Use the fundamental theorem.

* re a helix drawn on a right circular cylinder.

---

Thm: A unit speed curve \( \gamma: (a, b) \to \mathbb{R}^3 \) is a general helix if for some constant \( \mathbf{c} \)
\[
\mathbf{T}(s) = \mathbf{c} \kappa(s) \hat{\mathbf{n}}.
\]

pf: Let \( \gamma \) be a helix with axis \( \mathbf{u} \) another.
\[
\hat{\mathbf{b}} \cdot \mathbf{u} = \kappa \hat{\mathbf{n}} \cdot \mathbf{u} = 0
\]

\[ \hat{\mathbf{T}} \cdot \mathbf{u} = 0 \]

We may certainly assume \( \hat{\mathbf{n}} \cdot \mathbf{u} = 1 \)
\[
\hat{\mathbf{b}} = \lambda \hat{\mathbf{T}} + \mu \hat{\mathbf{b}}
\]

\[ 0 = \lambda \mathbf{T} + \mu \hat{\mathbf{b}} + \kappa \hat{\mathbf{n}} - \mu \mathbf{u} \]

\[ \lambda, \mu \text{ are constants and } \lambda \mathbf{T} - \mu \mathbf{u} = 0 \]

Now \( \lambda^2 + \mu^2 = 1 \):
\[ \mathbf{T} = \lambda \hat{\mathbf{T}} + \mu \hat{\mathbf{b}} \]

Conversely if \( \mathbf{T}(s) = \mathbf{c} \kappa(s) \hat{\mathbf{n}} \) for some constant \( \mathbf{c} \)
Then \( \hat{\mathbf{T}} = \mathbf{c} \kappa \hat{\mathbf{n}} \)
\[
-\frac{d\mathbf{b}}{ds} = \frac{c}{\kappa} \frac{d\hat{\mathbf{T}}}{ds}
\]

\[ \hat{\mathbf{b}} + \mathbf{c} \kappa \hat{\mathbf{n}} = \text{constant} \]

Then \( \mathbf{b} \cdot \mathbf{u} = \text{constant} \)

Showing that the curve is a general helix.

Formula for curvature and torsion for curves not parametrized by arc length.

For a curve \( \gamma: (a, b) \to \mathbb{R}^3 \) not necessarily of unit speed, the curvature and torsion are given by
\[
k(t) = \frac{|| \gamma'(t) \times \gamma''(t) ||}{|| \gamma'(t) ||^2} \quad \tau(t) = \frac{\langle \gamma'(t) \times \gamma''(t), \gamma'''(t) \rangle}{|| \gamma'(t) \times \gamma''(t) ||^2}
\]
Let $\sigma(t) = \sigma(s(t))$ where $\sigma$ is the inverse of the arc length function.

\[ \ddot{s}(t) = (\dot{s}(s))'(s) \sigma(s) = \dot{\sigma}(s) \dot{s}(s) \]

\[ \ddot{\sigma}(t) = (\ddot{s}(s))'(s) = (\dot{s}(s))' \ddot{s}(s) - (\dot{s}(s))' \ddot{s}(s) \]

\[ = k(s) \sigma(s)^2 + \dots \]

\[ \dot{s}(t) \times \ddot{s}(t) = k(\sigma(s))^3 b(s) \]

\[ \| \dot{s} \times \ddot{s} \| = k(\sigma(s))^3 \]

Let $S$ be the arc length function and $\dot{s}$ be the reparametrization of $\sigma$ by its arc length.

Then $\sigma(t) = \sigma(s(t))$

\[ \dot{s}'(t) = \dot{s}(s) \frac{ds}{dt} \]

\[ \ddot{s}(t) = \ddot{s}(s) (\frac{ds}{dt})^2 + \dot{s}(s) \frac{d^2s}{dt^2} \]

\[ \sigma''(t) = \ddot{s}(s) (\frac{ds}{dt})^2 + \dot{s}(s) \frac{d^2s}{dt^2} \]

\[ = (\ddot{s}(s) \times \dot{s}(s)) (\frac{ds}{dt})^2 \]

\[ = (k \hat{n} \times \dot{\sigma}) \frac{ds}{dt} \]

\[ k(\dot{s}) = \frac{\| \dot{s} \times \ddot{s} \|}{\| \dot{s} \| \| \ddot{s} \|} \]

\[ T = \frac{\dot{s}}{\| \dot{s} \|}, \quad b = \frac{\dot{n} \times \ddot{s}}{\| \dot{n} \times \ddot{s} \|} (t \times n) \]

Now $\sigma''(t) = k \left( \frac{d\hat{n}}{dt} \right)^2 + \dot{\sigma} \frac{d^2s}{dt^2}$

\[ \sigma''(t) = \frac{k}{\dot{\sigma}} \frac{d\hat{n}}{dt} \left( \frac{d\hat{n}}{dt} \right)^2 + k \frac{d^2s}{dt^2} \]

Next $\sigma'' = k \left( \frac{d\hat{n}}{dt} \right)^2 + \frac{\dot{\sigma}}{\dot{s}} \frac{d^2s}{dt^2}$

\[ \sigma''(t) \times \sigma'(t) \Rightarrow \sigma''(t) = k^2 T \| \ddot{s} \| \]

\[ \sigma''(t) \times \sigma'(t) = k^2 T \| \ddot{s} \| \]

\[ T = \frac{\sigma'(t) \times \sigma''(t)}{\| \sigma'(t) \times \sigma''(t) \|} \]

(Note: Ex 7-11 on p 93 appears wrong)

Ex Let us calculate the torsion of the twisted cubic $\sigma(t) = (t, t^2, t^3)$

Ans: $3/(9t^4 + 9t^2 + 1)$

In particular, it is not a plane curve (though this can be shown by algebraically).
Surfaces in $\mathbb{R}^3$:

A surface $S$ in $\mathbb{R}^3$ is a subset $S$ with the property that for each point $p \in S$, there is a neighborhood $V_p$ of $p$ in $\mathbb{R}^3$ and a function $x: U \to V_p$ where $U$ is a subset of $\mathbb{R}^2$ such that:

(i) $x$ is differentiable to all orders.
(ii) $x$ is a homeomorphism on its image $x(U)$.
(iii) $x$ is injective, continuous, and $x^{-1}: V_p \to U$ is continuous.
(iv) There exists a continuous function from $V_p \to U$ whose restriction to $V_p \cap S$ is $x^{-1}$.

In other words, the derivative $\frac{\partial x}{\partial x}$ has rank 2 throughout $U$.

Def. The function $x: U \to S$ described above is called a coordinate patch, or a coordinate chart.

Ex: (Plane) Let $\hat{n} \neq 0$ be a unit vector. Let $T_{\hat{n}}$ be the plane containing $\hat{n}$ and having normal vector $\hat{n}$. Assume $\hat{n} \neq 0$.

$x(u,v)$ = $(u, v, g^{-1}(a-u^2 + (9-v)u + v))$

maps $\mathbb{R}^2$ onto $T_{\hat{n}}$. $x(u,v)$ defines a coordinate patch.

(2) Consider $S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 / x_1^2 + x_2^2 + x_3^2 = 1 \}$

Define $G: (0, \pi) \times (0, \pi) \to \mathbb{R}^3$

$G(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$

Then $G$ is a bijection from $(0, \pi) \times (0, \pi)$ onto

$S = \{ (\sin \phi, 0, \cos \phi) / 0 \leq \phi \leq \pi \}$

and defines a coordinate patch.

To show that $G^{-1}$ is continuous, for $(x,y,z) \in S$ observe that $-1 < z < 1$ on $S$ and $\phi = \cos^{-1} z$ is continuous taking values in $(0, \pi)$.

Now let $\text{Arg}: (\text{Re: real axis}) \to (0, \pi)$ be a continuous branch of the argument function.

Then $\frac{x}{z} = \cos \theta$, $\frac{y}{z} = \sin \theta$

and $x \neq 1$.

So $\theta = \text{Arg}(\frac{x + iy}{z})$ and we have $\theta \leq \pi$.

That $G^{-1}$ is continuous.

A closer computation confirms that $G_{\theta} \cdot G_{\phi} \neq 0$. At points along the meridian $(\sin \phi, 0, \cos \phi)$ a different coordinate patch is needed.

(3) Again consider $S = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 / x_1^2 + x_2^2 + x_3^2 = 1 \}$ but this time we take the six coordinate patches each defined on $U = \{ (u,v) \in \mathbb{R}^2 / u^2 + v^2 < 1 \}$.

$G_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$

Images of $G_1$ cover $S$ minus the equator.

To cover the equator we take 4 more patches

$G_2(u,v) = (u, \pm \sqrt{1-u^2-v^2}, v)$

which covers the sphere except the points

Together with $G_1$, $G_2$, ...
The inverse maps are easily written down.

For example

\[(G^{-1})_{3}(x_1,x_2,x_3) = (x_1,x_2)\]

Check that \(DG_{i}^{±}\) has rank 2 everywhere.

(4) Discuss the sphere again but construct two coordinate patches using stereographic projections.

15) Coordinate patch for the hyperboloid of one sheet \(x_1^2 + x_2^2 - x_3^2 = 1\).

\[G(u,v) = (\sqrt{1+v^2}, \cos u, \sqrt{1+v^2} \sin u, v)\]

Defined on \((0, 2\pi) \times 1R\).

Check that \(DG_{i}^{±}\) has rank 2 and that this coordinate patch patch covers the entire surface except for one hyperbola.

Construct a second coordinate patch to cover this.

(6) Hyperboloid again! (Ruled Surface)

Write \(x_1^2 - x_3^2 = 1 - x_2^2\)

\((x_1-x_3)(x_1+x_3) = (1-x_2)(1+x_2)\)

Now consider the line of intersection

\[L_\alpha:\ x_1-x_3 = (1-x_2)x\]

\[x_1+x_3 = (1+x_2)x\]

\(\alpha \in \mathbb{R}\), \(\alpha \neq 0\).

The line lies entirely on the surface.

Likewise the lines \(M_\beta\) given by

\[M_\beta:\ x_1 - x_3 = (1+x_2)\beta\]

\[x_1 + x_3 = (1-x_2)\beta\]

\(\beta \in \mathbb{R} - 10\).

Thus the hyperboloid contains two families \(J_1, J_2\) of lines such that

(a) Any two lines in one family are skew.

(b) Any line of one family meets every line of the other family.

(c) Through each point on the surface there passes exactly one member of each of the family, except for certain exceptional points.

Ex. Verify (a), (b), (c) and determine the exceptional points.

Use (c) to write a coordinate patch for the surface as a function of \(x\) and \(\beta\).

Thus the hyperboloid of one sheet is a doubly ruled surface.

7) Discuss the surface \(z = x^2 - y^2\)

The function \(G: \mathbb{R}^2 \rightarrow \mathbb{R}^3\) given by

\[G(u,v) = (u, v, u^2 - v^2)\]

is a coordinate patch covering the entire surface.

But find another description as a doubly ruled surface.

Hint \((x+y)(x-y) = 1 \cdot z\).

8) Helicoid:
The Cone: Let \( \varnothing(t) \) be a curve not passing through the origin. The image of \( G(s, t) = s \varnothing(t) \) is called a Cone generated by \( \varnothing \). The Cone will not in general be a Surface as it may have self-intersections. Consider \( \varnothing \) described as in the figure. The Cone determined by \( \varnothing \) will in general not be a Surface but if we take a small part of the Curve \( \varnothing \) and notice at the Cone generated by the arc it will be a Surface (with one coordinate patch) and \( G \times E \) is easily seen to be non-zero.

Example: Let \( \varnothing \) be the Curve of intersection of \( y^2 + z^2 = 1 \) and \( x^2 + y^2 = 1 \). Consider the Cone generated by \( \varnothing \) and sketch the Cone.

Use of the inverse function theorem: Suppose \( \phi: \mathbb{R}^3 \to \mathbb{R}^3 \) is a smooth map and \( C \) is a regular value of \( \phi \). (i.e., \( \nabla \phi(c) \neq 0 \)) then show that \( S = \{ x \in \mathbb{R}^3 \mid \phi(x) = c \} \) is a Surface.

Theorem: Suppose \( S \) is a Surface in \( \mathbb{R}^3 \) and \( p \in S \). Let \( \chi: U \to S \) be a coordinate patch. Then, \( \chi^{-1}: S \to U \) is differentiable, bijective and its inverse \( \chi^{-1}: U \to S \) is also differentiable immediately from the following proposition: Let \( \chi: U \to S \) be a coordinate patch. Then \( \chi: \chi(U) \to U \) is a smooth map in the sense that \( \chi \) is a map \( \chi : \mathbb{R}^3 \to \mathbb{R}^3 \) and a smooth map \( \phi: \mathbb{R}^3 \to U \) whose restriction to \( \chi(U) \) is \( \chi^{-1} \).

Proof: Since \( \chi^{-1}: \chi(U) \to U \) is already bijective and continuous, it makes sense to check smoothness locally. So let \( p \in \chi(U) \) and \( q = \chi^{-1}(p) \).

Consider the map \( F: U \times \mathbb{R}^3 \to \mathbb{R}^3 \) (which means the linear vector space)

\[ F(u, v, t) = \chi(u) + t \mathbf{h} \]

Since \( \mathbf{h} = \text{congruent maps to a surface } S \) "passes through \( S \)" by travelling a distance \( c \) along the normal vector \( \mathbf{n} \).

Let us calculate the Jacobian of \( F \) at \( (2, 0) \):

\[ F_u = \mathbf{x}_u + t \mathbf{h} \frac{\partial}{\partial u}; \quad F_v = \mathbf{x}_u + t \mathbf{h} \frac{\partial}{\partial v} \text{ and } F_t = \mathbf{h} \text{ so that} \]
By Continuity, there exists a neighborhood $V$ of $y$ such that $F$ maps $W_1$ into $V$.

Now consider a coordinate patch $(U, x)$ at $p$.

**Then**, $y' = F \circ x$ implies $S_1 \xrightarrow{F} S_2$.

Now $x$ maps a neighborhood of $x'(p)$ into $W_1$ such that the composite function $y' \circ F \circ x$ makes perfect sense and is a map from $N \to V$, $N$, $V$ being open subsets of $\mathbb{R}^2$.

$F$ is said to be differentiable at $p$ if $y' \circ F \circ x$ is differentiable at $x'(p) \in N$.

The notion is independent of the choice of coordinate patches $(U, x)$ and $(V, y)$ as is easily seen by using the theorem stated in the previous section.

**Def:** Surfaces $S_1$ and $S_2$ are said to be **diffeomorphic** if $F$ is a homeomorphism $F: S_1 \to S_2$ such that $F$ and $F^{-1}$ are both differentiable in the sense described above.

**Ex:** (i) The sphere $x^2 + y^2 + z^2 = 1$ is diffeomorphic to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 

\[DF(q, 0) = \begin{bmatrix} x_u & x_v & x \end{bmatrix} \cdot \hat{s} = 11X_u \times X_v \neq 0.\]

Jacobian $F(q, 0) = (x_u X_u + x_v X_v)$.

Let $G = F^{-1}$ which is a smooth map from $W_2$ of $p$.

Thus $x'(p) \mid W_2$ is the restriction to $W_2$ of the smooth map $\pi \circ G$ defined on a neighborhood of $p$.

**Differentiability:**

Let $S_1$, $S_2$ be two surfaces in $\mathbb{R}^3$ and $F: S_1 \to S_2$ be a continuous map.

We now define the notion of smoothness of $F$ by employing local coordinate patches.

Let $p \in S_1$ and $q = F(p)$.

Choose a coordinate patch $(V, y)$ at $q \in S_2$.

\[v_2 \circ y: V \to S \subset \mathbb{R}^3.\]
Show that the relation of differentiability is an equivalence relation.

(ii) The paraboloid \( z = x^2 + y^2 \) and the
plane \( z = x^2 - y^2 \), \( x \geq 1 \)
are diffeomorphic.
Show that they are both diffeomorphic to the
plane

(iii) Explain why the cylinder \( x^2 + y^2 = 1 \) and
the hyperboloid \( x^2 + y^2 - z^2 = 1 \) are
diffeomorphic.

The tangent plane
Let \( p \) be a point on a surface \( S \) and
\((u, v)\) be a coordinate patch containing \( p \).
Let \( x(u_0, v_0) = p \).

Then the curves
\[
\begin{align*}
U &\mapsto x(4u, v) \\
V &\mapsto x(4v, u)
\end{align*}
\]
are curves lying on \( S \) and their tangent
vectors are
\[
X_u(u_0, v_0) \quad \text{and} \quad X_v(u_0, v_0).
\]
Thus \( X_u \times X_v \) at \((u_0, v_0)\) gives the
normal vector \( N \).

It is clear that if \((V, y)\) is another
coordinate patch containing \( p \), then
\[
X_u = x_u \frac{\partial}{\partial u} + y_u \frac{\partial}{\partial v}, \quad X_v = x_v \frac{\partial}{\partial u} + y_v \frac{\partial}{\partial v}
\]
\[
\therefore X_u \times X_v \equiv \left( y_x y_v - y_y x_v \right) (x_x y_v - x_y y_x) \mathbf{J}(y_{\times} x).
\]

Def: A surface \( S \) in \( \mathbb{R}^3 \) is said to be orientable if
there exists a collection of charts \( \{ U, x \} \) such that
\[
S \cap U \quad \text{has positive Jacobian on}
\]
the overlap \( U \cap U'. \)

Let us call
Such a family of charts a coherent atlas.

If a coherent atlas exists (i.e., if the surface is
orientable) then computing the unit normal
using a coherent atlas produces a continuous
normal \( \mathbf{N}(t) \).

\[
\mathbf{G}: p \mapsto \mathbf{N}(p) := N_0 \mathbf{x}^{-1}(p)
\]
\[
S \mapsto S^2 \quad \text{(unit sphere in} \mathbb{R}^3)\]

Called the Gauss map.

Lemma:

Suppose that \( \gamma \) is a curve on \( S \)
and \( N_1, N_2 \) are two continuous normal fields
along \( \gamma \) \([a, b] \rightarrow S\).
Then \( N_1 = N_2 \) a.e. \( \gamma \)
\[
\Rightarrow N_1 = N_2 \quad \text{at} \quad \gamma(t)
\]

\[
\text{Proof:} \quad \text{Clearly } N_1 = \pm N_2 \text{ along } \gamma
\]

So consider the continuous \( \gamma \):
\[
\phi: \gamma \mapsto N_1, N_2 \quad \text{which takes values } \pm 1
\]

\[
\phi(0) = 1
\]
\[
\therefore \phi(1) = 1 \quad \text{which means } N_1 = N_2 \text{ at the}
\]
Terminal point as well.
Theorem: Suppose \((U, x), (V, y)\) are two connected coordinate patches on \(S^2\) such that

(i) \(U \cap V\) is disconnected

(ii) \(T(x^{-1}y) : U \cap V \rightarrow U \cap V\) changes sign.

Then \(S^2\) cannot be orientable.

Proof: Suppose \(S^2\) is orientable, choose a continuous normal field \(N : S^2 \rightarrow S^2\) along \(S^2\) and signs.

Let \(p, q\) be points in \(U \cap V\) at which \(T(x^{-1}y)\) has different signs.

Let \(T(x^{-1}y)\) be a curve in \(U \cap V\) connecting \(p\) and \(q\).

Then, assume \(T(x^{-1}y) > 0\) at \(p\) and \(T(x^{-1}y) < 0\) at \(q\).

Let \(N' = x_u x_v : N'' = y_u y_v \parallel x_u x_v \parallel \parallel y_u y_v \parallel\)

We have \(N' = N'\) at \(p\).

If \(N'' = N''\) at \(q\), then we may choose a normal field \(N = N'' = N\) at \(p\).

Then by the lemma applied to \(S\) we get \(N' = N\) at \(p\).

Applied to \(S\) we get \(N'' = N\) at \(q\).

\(N' = N'' = N\) at \(q\), which is a contradiction.

Use this fact to check that the Möbius band is not orientable.

(Ex: 8.4 on p.114 of the text.)

The Gauss map would play a fundamental role in what follows.

(i) Example: Let us calculate the Gauss map for a point on the cylinder \(x^2 + y^2 = 1\).

A coordinate patch is given by:

\([0, 2\pi] \times \mathbb{R}^2\) with \(x = \cos \theta, y = \sin \theta, z = 0\).

The image of the Gauss map in the equator on \(S^2\) is:

\(N(\theta, z) = (\cos \theta, \sin \theta, 0)\).

(ii) For \(S = S^2\) the Gauss map is the identity map.

(iii) For \(S\) a piece of the plane the Gauss map is constant.

(iv) For the paraboloid \(Z = x^2 + y^2\),

\(N(x, y) = (x, y, \sqrt{1 + 4x^2 + y^2})\).

Show that if \(\phi : \mathbb{R}^3 \rightarrow \mathbb{R}\) is a smooth map and \(\Gamma\) is a regular value of \(\phi\) with \(\phi^{-1}(\Gamma)\) non-empty, then \(\phi^{-1}(\Gamma)\) is an orientable surface.

First show that if \(S\) admits a continuous unit normal field then \(S\) is orientable.
The second fundamental form:

Let \( S \) be an orientable surface and \( N: S \to \mathbb{R}^3 \) be the Gauss map.

Then differentiating the equation,
\[
\langle N, N \rangle = 1
\]
we see that \( \langle N_u, N \rangle = \langle N_v, N \rangle = 0 \) so that \( N_u, N_v \) both belong to the tangent plane \( T_p S \).

Thus we can define a linear transformation \( W: T_p S \to T_p S \) given by
\[
W(x_u) = N_u \\
W(x_v) = N_v
\]

Now \( \langle W(x_u), x_u \rangle = \langle N_u, x_u \rangle \)

But diff. the equations \( \langle N, x_u \rangle = 0 \) w.r.t \( u \)
\( \langle N, x_v \rangle = 0 \) w.r.t \( v \) we get
\( \langle N_u, x_v \rangle = \langle N_v, x_u \rangle \)

Hence \( \langle W(x_u), x_u \rangle = \langle x_u, W(x_v) \rangle \)

From which it follows at once that
\[
\langle W(t_1, t_2) = \langle t_1, W(t_2) \rangle \quad \text{for any pair of tangent vectors} \quad t_1, t_2 \in T_p S.
\]

Def: The map \( W \) is called the Weingarten map.

* Very soon we shall see an invariant description, i.e. one that does not refer to a specific basis for \( T_p S \).

Thm: The Weingarten map is self-adjoint with respect to the standard inner product on \( T_p S \) (inherited from the ambient space \( \mathbb{R}^3 \)).

Def: The quadratic form
\[
\Pi_p: T_p S \to \mathbb{R}, \quad t \mapsto \langle W(t), t \rangle
\]
is called the second fundamental form on the surface \( S \).

By employing Lagrange multipliers we see that
\[
\lambda = \sup_{\|t\|=1} \Pi_p(t) \quad \text{and} \quad \inf_{\|t\|=1} \Pi_p(t) = \mu
\]
are both eigenvalues of \( \Pi_p \) and that they are orthogonal (due to self-adjointness) if \( \lambda \neq \mu \).

If \( \lambda = \mu \) then \( \Pi_p(t) = \lambda \|t\|^2 \quad \text{and so} \quad W(t) = \lambda \text{Id}_{T_p S} \)

We now look at the geometrical interpretation of the second fundamental form.

Recall that if \( \sigma(s) \) is a unit speed curve \( \mathbb{E} \) on \( S \) with tangent vector \( t \) at \( \sigma(s) \) and
\[
\frac{d^2}{ds^2} \mathbb{E} = k(s) \mathbb{N} \quad \text{where} \quad k(s) \text{ is the curvature}
\]

Now, let \( S \) be a surface and \( \mathbb{N} \) be the unit normal vector to \( S \) at \( p \).

Consider the plane \( \mathbb{N} \) passing through \( p \).

Containing \( \mathbb{N} \).
This gives an invariant description of the Weingarten map.

and a tangent vector $\hat{t}$ at $p$.

Choose a unit speed curve $\hat{\gamma} : (c,e) \rightarrow S$

such that $\hat{\gamma}(0) = \hat{p}$.

Then $\hat{\gamma}'(0) = \hat{e}$ and $\hat{\gamma}(s) \in \Lambda$ for all $s$.

Then $\hat{\gamma}$ is a the normal to $\hat{\gamma}(s)$ at $p$.

in the plane $\Lambda_p$

Note that $\hat{\gamma}(s)$ is just the curve of intersection of $\Lambda_p$ and $S$.

Use a coordinate patch $\chi : V \rightarrow S$ and let $t \rightarrow (u(t), v(t))$

be the function $\chi \circ \hat{\gamma}$.

Then $\chi' \circ \hat{\gamma}'(0) = \chi_u \hat{u}(0) + \chi_v \hat{v}(0)$

\[ \hat{\gamma}'(0) = \hat{t} = \chi_u \hat{u}(0) + \chi_v \hat{v}(0) \]

(*) $\quad W(t) = \Lambda_x u(t) + \Lambda_y v(t)$

\[ \frac{d}{d \hat{t}} \frac{d}{d t} (\Lambda x \circ \hat{\gamma}(s)) = \frac{d}{d \hat{t}} \frac{d}{d t} (\Lambda x \circ \hat{\gamma}(0)) \]

Thus the normal $N_x(t)$ is the unit normal to $S$

\[ \frac{d}{d \hat{t}} \frac{d}{d t} (N_x \circ \hat{\gamma}(s)) = k_s(\Lambda x \circ \hat{\gamma}(s)) \hat{\gamma}' \hat{\gamma}' \quad \text{by Gauss' theorem} \]

where $k$ is the curvature (upto sign) of $\hat{\gamma}$ at $p$.

\[ W(t) = k(\hat{e}) \hat{e} + k(t) \hat{t} \]

and

\[ T(t) = k(\hat{e}) \hat{e} \quad \text{Thus the second fundamental form gives the curvature of} \]

Curves that are plane sections of $S$ by the one parameter family of planes through $\hat{\gamma}$ containing $\hat{p}$.

$k_s(p)$ is more appropriately denoted by $k(p, \hat{t})$ which is the curvature at $p$ of $\hat{\gamma}$ in the direction $\hat{t} \in T_p S$.

Thus, the eigen values of the Weingarten map give the maximum and minimum curvature at $p$ of all plane sections (at $p$) through $\hat{\gamma}$ by planes passing through $\hat{p}$.

To calculate the Weingarten map, write

$N_u = \alpha \chi_u + \beta \chi_v$

$N_v = \gamma \chi_u - \delta \chi_v$

The matrix of $W$ w.r.t. the basis $\{ \chi_u, \chi_v \}$ of $T_p S$ is

\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\]

Def: The eigen values of the Weingarten map are called the principal curvatures of $S$ at $p$ and the eigen vectors are the principal curvature directions.

Ex: Determine the principal curvatures and principal curvature directions at a point on the hyperboloid

\[ x^2 + y^2 - z^2 = 1 \]

Def: The Gaussian curvature of $S$ at $p$ is the determinant of the Weingarten map i.e. the product of the principal curvatures.

The trace of the principal curvatures or the trace of the Weingarten map is called the
mean curvature of \( S \) at \( p \).

Let us compute the mean curvature of a surface which is the graph of a function \( f: \mathbb{R} \to \mathbb{R} \).

There is only one coordinate patch

\[ (u, v) \to (u, v, f(u, v)) \]

\[ x_u = (1, 0, f_u) \]
\[ x_v = (0, 1, f_v) \]

\[ N = \left( -f_x, -f_u, 1 \right) / \sqrt{1 + f_x^2 + f_u^2} \]

Let \( \Delta = (1 + f_u^2 + f_v^2)^2 \)

\[ N_u = \frac{1}{\Delta} \left( -f_{uu}, -f_{uv}, 0 \right) + \sigma_1 N \]

for some scalar function \( \sigma_1 \). Likewise

\[ N_v = \frac{1}{\Delta} \left( -f_{uv}, -f_{vv}, 0 \right) + \sigma_2 N \]

Thus writing

\[ N_u = \alpha x_u + \beta x_v \]
\[ N_v = \delta x_u + \epsilon x_v \]

we get the equations

\[ N_u \cdot x_u = -\frac{f_{uu}}{\Delta} = \alpha (1 + f_u^2) + \beta f_u f_v \]

\[ N_u \cdot x_v = -\frac{f_{uv}}{\Delta} = \alpha f_u f_v + \beta (1 + f_v^2) \]

Solving,

\[ \alpha = \frac{1}{\Delta^3} \left( -f_{uu}(1 + f_u^2) + f_{uv} f_{fv} \right) \]

\[ \beta = \frac{1}{\Delta^3} \left( -f_{uv}(1 + f_u^2) + f_u f_v f_{uu} \right) \]

Similarly from \( N_v \cdot x_u \) and \( N_v \cdot x_v \) we get

\[ \sigma_1 = \left( -f_{uv}(1 + f_u^2) + f_{vv} f_{fu} \right) / \Delta^3 \]
\[ \sigma_2 = \left( -f_{vv}(1 + f_v^2) + f_u f_v f_{uu} \right) / \Delta^3 \]

Thus \[ K = \frac{1}{\Delta^6} \left\{\begin{array}{c}
- f_{uv} f_{vv} f_{uu} f_{uv} f_{vv} - f_u f_v \left( f_{vu} + f_{uv} + f_{uu} f_v^2 + f_{vv} f_u^2 \right) \\
- f_{uv} (1 + f_u^2)(1 + f_v^2) - f_{uv} f_{vv} f_{uu} f_{uv} f_{vv} - f_{uv} (1 + f_v^2) f_{uu} f_v f_{vv} f_{vu} + f_{uv} (1 + f_u^2) f_{uu} f_u f_v f_{vv} f_{vu} \end{array}\right\} \]

\[ = \frac{1}{\Delta^6} \left\{\begin{array}{c}
- f_{uv} f_{vv} (1 + f_u^2)(1 + f_v^2) + f_{uu} f_{uv} f_{vv} f_{vu} f_{uv} f_{vv} \\
- f_{uv} (1 + f_u^2)(1 + f_v^2) - f_{uv} f_{vv} f_{uu} f_{uv} f_{vv} f_{vu} + f_{uv} (1 + f_v^2) f_{uu} f_v f_{vv} f_{vu} \end{array}\right\} \]

\[ = \frac{1}{\Delta^6} \left\{\begin{array}{c}
( f_{uu} f_{vv} - f_{uv}^2 )(1 + f_u^2)(1 + f_v^2) \\
- f_{uv}^2 ( f_{uu} f_{vv} - f_{uv}^2 ) + f_{uu} f_{uv} f_{vv} f_{vu} f_{uv} f_{vv} \end{array}\right\} \]

Thus we get

\[ \text{The Gaussian curvature of a graph} \]
\[ (u, v) \to (u, v, f(u, v)) \]

\[ \text{vanishes identically iff satisfaction the Monge- Ampere equation} \]
\[ f_{uu} f_{vv} - f_{uv}^2 = 0. \]
Ex: Verify that for the cone
\[(x, y) \mapsto (x, y, \sqrt{x^2+y^2}) ; \quad 0 < x^2+y^2\]
the Gaussian Curvature Vanishes.

Determine the Gaussian Curvature of the Cylinder, Plane and Unit Sphere.

Show that the Gaussian Curvature of a paraboloid \(z = x^2+y^2\) in positive.

Show that the hyperboloid \(x^2+y^2-z^2=1\) has negative Gaussian Curvature.

The tangent Surface to a Curve:

This is the locus of lines tangent to a given curve.

Let \(\varphi: [a, b] \to \mathbb{R}^3\) be a unit speed curve. The tangent line to the curve at a point \(T(s)\) is given by
\[
\mathbf{t} \mapsto \varphi(s) + \mathbf{t} \varphi'(s)
\]
So the tangent surface is given by the coordinate patch
\[
\gamma: (s, t) \mapsto \varphi(s) + t \varphi'(s)
\]
Now \(\gamma_s = \varphi_s + t \varphi'_s\) and \(\gamma_t = \varphi_t\)
\[
N = \gamma_s \times \gamma_t = \nabla \varphi \times \nabla \varphi \\
\|
\|
\|
\]
\(N_t = 0\) and \(N\) is the Weingarten map in Singular. The Gaussian Curvature is zero.

Determine the surface which is the locus of the one parameter family of lines
\[
y = t(x - b) \\
z = t^2(x - b)
\]
and find the Gaussian curvature of the surface.

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Def: A point \(p \in S\) is said to be an umbilic if the Weingarten map has equal and real eigenvalues at \(p\).

Show that the origin is an umbilic point of the paraboloid
\[
z = x^2+y^2
\]
What about the surface \(z = (x^2+y^2)^2\)?

Determine the umbilical points on the ellipsoid
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]

Suppose that \(S\) is a smooth surface (ie of class \(C^\infty\)) and \(p \in S\) is such that \(T_p \cap T_q \neq 0\).

Then planes parallel to \(T_p\) at a distance \(t \in \mathbb{R}\) from \(p\) meet \(S\) along a circle. Is \(p\) an umbilic point?

So: Let us fix up the origin at \(p\) and take the tangent plane \(T_p\) as the \(x-y\) plane. Then the surface may be parameterized near \(p\) as
\[
(u,v) \mapsto (u,v,f(u,v))
\]
where \(f\) is a function of \(\sqrt{u^2+v^2}\) alone.
\(f(0) = 0\)

Now \(f(t) = at + bt^2 + ct^3 + \ldots\) any.

But the fact that \(f\) is smooth forces \(a = 0\).
\[ f(x, y) = b(u^2 + v^2) + c(u^2 + v^2)^2 + \ldots \]

Now we may assume \( b = 1 \) (by rescaling \( u, v \))

\[ X_u = (1, 0, 2u) + (0, 0, 2b(u^2 + v^2)^2) du + \ldots \]

At \( p \)

\[ x_u = (1, 0, 0) \]
\[ x_v = (0, 1, 0) \]

\[ N_u = \left( (1, 0, 2u) + 2v \cdot \left( \frac{\partial f}{\partial u} \right) \right) \times \left( (0, 1, 2v) + 2u \cdot \left( \frac{\partial f}{\partial v} \right) \right) \]

\( (\gamma \partial) \) terms have vanishing derivatives at \( p \)

\[ N_u = (0, 0, 2) \times (0, 1, 0) \]
\[ N_v = (1, 0, 0) \times (0, 0, 2) \]

\[ \begin{align*}
N_u : x_u &= -2 \\
N_v : x_u &= 0 \\
N_v : x_v &= -2 
\end{align*} \]

So \( W \) is represented by the diagonal matrix

\[ \begin{bmatrix}
-2 & 0 \\
0 & -2
\end{bmatrix} \]

at \( p \) which means \( p \) is an umbilic.

Circular Sections

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

\( a > b > c > 0 \)

The \( xy \) plane slices it along the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

with minor axis along \( (0, b, 0) \)

Where as the \( yz \) plane slices along an ellipse

with major axis along \( (0, b, 0) \)

By rotating the planes through the \( y \)-axis

we get a family of ellipses with one axis along the \( y \)-axis, and continually decreasing from a major axis in one case to a minor axis in another.

For one of these planes the two axes must become equal.

Let \( \lambda x + \mu z = 0 \) be these planes

through the \( y \)-axis. It cuts the ellipsoid along an ellipse with axis \( (0, b, 0) \). The other axis must be orthogonal to this and so we look for it in the form

\[ \alpha x^2 + \beta z^2 = 1 \]

\( (\lambda > 0) \)

Then the ellipse is

\[ \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \]

The ellipse as axes \( b \) and \( \sqrt{\lambda + \mu} \left( \frac{a^2}{a^2 + c^2} \right)^{1/2} \)

So this will be a Circular Section if

\[ \lambda + \mu = b^2 \left( \frac{a^2}{a^2 + c^2} \right) \]

\[ \lambda \left( 1 - \frac{b^2}{c^2} \right) = c^2 \left( \frac{\mu}{a^2} + \frac{b^2}{c^2} \right) \]

Taking \( \lambda = 1 \) (WLOG), \( \mu = \pm \frac{b^2 - c^2}{a^2 - b^2} \)

So we have

The family of planes \( x = \pm \frac{b^2 - c^2}{a^2 - b^2} \)

Where Slice the ellipsoid along Circles which diminish in points where these planes touch the ellipsoid.
Let the point of contact be \((x, y, z)\).

Then we see comparing \((\star)\) with

\[
\frac{y_1}{a^2} + \frac{y_2}{b^2} + \frac{y_3}{c^2} = 1
\]

we get

\[
y_1 = 0; \quad k = \frac{a^2}{x_1}, \quad a^2 \over x_1, c^2 = \pm \sqrt{b^2 - c^2}
\]

and

\[
x_1^2 + \frac{z_1^2}{c^2} = 1
\]

which give the four umbilical points

\[
(x_1, 0, z_1);
\]

\[
x_1^2 = \frac{a^4 \left(a^2 - b^2\right)}{(a^2 - c^2) \left(a^2 + c^2 - b^2\right)}
\]

\[
z_1^2 = \frac{c^4 \left(b^2 - c^2\right)}{(a^2 - c^2) \left(a^2 + c^2 - b^2\right)}
\]

Further Exercises:

1. Consider \(N\) as a map from \(S\) to \(\mathbb{R}^3\)

so that \(N : U \to \mathbb{R}^3\) and \(D(N)\) is a linear map.

Let \(\mathbf{p}\) be a point on \(S\) and \(\mathbf{e}\) be a
tangent vector to \(S\) at \(\mathbf{p}\) and \(\mathbf{t}\) be a
tangent vector to \(S\) at \(\mathbf{p}\) and \(\mathbf{t} : \mathbb{R} \to S\) be a
curve such that

\[
\mathbf{t}(0) = \mathbf{p}; \quad \mathbf{t}(\theta) = \mathbf{e}
\]

Show that

\[
W(\mathbf{e}) = \frac{d}{ds} \left( N \mathbf{t}(s) \right)
\]

\[
= D(N) \mathbf{t}(s)
\]

\[
= (D\mathbf{n}) \mathbf{e} \quad \text{regarding} \quad D\mathbf{n} \text{as a}
\]

2. Suppose \(\sigma : \mathbb{R} \to S\) is a line of curvature on \(S\) such that

\(\sigma(s)\) is a principal curvature direction

at \(\sigma(s)\).

Show that

\[
\frac{d}{dt} \left( N \sigma(s) \right) = \lambda \sigma(s)
\]

\(\lambda\) is an eigen value of \(W\) i.e. the associated

principal curvature.

This formula is called Rodrigues formula.

3. Suppose \(S_1, S_2\) are two surfaces intersecting along

a regular curve which is a line of curvature on \(S_1\).

Then the angle of intersection of \(S_1, S_2\) is

constant along \(\sigma\) iff \(\sigma\) is also a line of curvature

of \(S_2\).

4. Determine the lines of curvature on the

hyperboloid

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

5. For a surface of revolution show that the lines

of curvature are parallels and meridians.

6. Determine explicitly the entries of \(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\)

of the

Weingarten map \(\mathbf{W}\) in the basis \(\{\mathbf{x}, \mathbf{y}\}\)

and write the result purely in terms of

\(\mathbf{x}, \mathbf{y}, \mathbf{x}_n, \mathbf{x}, \mathbf{x}_n\) and \(\mathbf{x}_n\) abbreviating

\(\mathbf{x}_n \mathbf{x}_n = \mathbf{N}\) if need be.

That is to say formulas instead of

\(\mathbf{Nu}, \mathbf{Nv}, \mathbf{Nu}, \mathbf{Nv}\)
These equations are called the Weingarten equations.

(7) Determine the second fundamental form of
Enneper's surface
\[ X(u, v) = \left( \frac{u - u^3}{3}, \frac{v - v^3}{3}, u^2 - v^2 \right) \]
and prove that its mean curvature is zero.

(8) Find the mean curvature of the graph
\((u, v) \to (u, v, f(u, v))\) and determine the
PDE that \( f \) would have to satisfy in order that
its mean curvature = 0.

(9) Let \( \Lambda \) be a plane that cuts \( T_p S \) at angle \( \theta \)
and \( \Sigma \) be the intersection of \( \Lambda \) with \( S \).
Find the curvature of \( \Sigma \)
(Moebius's theorem)

Appendix: Eigenvalues of a real symmetric matrix
Suppose \( V \) is a finite dimensional inner product space and \( A: V \to V \) is self-adjoint.
Then
\[ \lambda_1 = \sup_{u_1 \neq 0} \langle Au, u \rangle \text{ is the largest eigenvalue of } A \]
\[ \lambda_2 = \sup_{u_1, u_2 \neq 0} \langle Au_1, u_2 \rangle \text{ is the next largest eigenvalue of } A \text{ where } u_1 \text{ is the } \]
\text{unit vector such that } \lambda_1 = \langle Au_1, u_1 \rangle = \sup_{u \neq 0} \langle Au, u \rangle

\[ \inf \langle Au, v \rangle \text{ is the smallest eigenvalue of } A. \]

Proof: Fix a unit vector \( u, v \) in the basis.
\[ Q(x) = \langle Ax, x \rangle \text{ and let } \lambda \text{ be the Lagrange multiplier: } \]
\[ \sup \langle Ax, x \rangle \text{ is attained at any } u, \text{ on } \]

The Unit sphere
\[ L(x, y) = Q(x) - \lambda \left( 1x^2 + 1 \right) \]
Thus \[ \partial L / \partial x = 0, \quad \partial L / \partial y = 0 \text{ at } u, \]
Thus \( \lambda \) is an eigenvalue of \( A \) where \( u \) is the
Corresponding eigenvector. Call this \( \lambda_1 \)
\[ \lambda_1 = \langle Au, u \rangle = \begin{pmatrix} \lambda_1 \end{pmatrix} \]
1.e The eigenvalue \( \lambda = \lambda_1 

Now let \( \lambda_2 = \sup_{u_1, u_2 \neq 0} \langle Au, u \rangle = \begin{pmatrix} \lambda_2 \end{pmatrix} \)
\( \left( \begin{array}{c} u_1 \cr u_2 \end{array} \right) \neq 0 \text{ and } \langle u_1, u_2 \rangle = 0 \)
We now set up the Lagrangian
\[ L_2(x, u, v, \mu) = Q(x) - \lambda \left( 1x^2 + 1 \right) - \mu \langle u, v \rangle \]
and at \( u_2 \)
\[ \partial L_2 / \partial \mu = 0, \quad \partial L_2 / \partial \lambda = 0, \quad \partial L_2 / \partial u = 0 \]
Thus \( \lambda_2 = \sup_{u_2} \langle Au_2, u_2 \rangle - \mu_0 \langle u_2, u_2 \rangle = 0 \).
Taking dot product with \( u_2 \) we get
\[ \langle Au_2, u_2 \rangle = 0 \]
Taking dot product with \( v \) gives
\[ \lambda_2 = \langle Au_2, v \rangle - \mu \langle u_2, v \rangle \]
\[ = 2 \langle u_2, Au_2 \rangle = 2 \lambda_1 \langle u_2, v \rangle = 0 \]
So that
\[ Au_2 = u_2, \]
\[ \langle u_2, u_2 \rangle = 1, \]
\[ \langle u_2, v_1 \rangle = 0. \]
\[ \therefore u_2 \text{ is an eigen vector of } A \text{ with} \]
eigenvalue \[ \lambda_2 \]
eigenvalue \[ \lambda_2 \]
Call \[ u_2 = \lambda_2 . \]

Proceed further and optimize \( Q(x) \) subject to the Constraints \[ \| x \| = 1 \]
and \[ \langle x, u_1 \rangle = 0, \]
and we get the next eigenvalue.

If \( u_1, \ldots, u_{k-1} \) have been found such that
\[ \langle u_i, u_j \rangle = d_{ij}, \]
\[ A u_i = \lambda_i u_i, \]
then by
\[ \lambda_2 = \max_{\| x \| = 1} \langle Au, u \rangle \]
\[ \langle u, u_k \rangle = 0 \quad (k = 1, 2, \ldots, j-1) \]
Let
\[ L(x, \lambda, u_1, \ldots, u_j) = L \]
\[ = Q(x) - \lambda (1x_1^2 - 1) - \sum_{k=1}^{j-1} \langle x, u_k \rangle \]
\[ \frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0 \]
gives
\[ Ax = 2x - \sum_{j=1}^{k-1} y_j u_j = 0 \quad \text{for each } j \]
\[ \langle x, u_j \rangle = 0 \]

Dotting with \( u_1, \ldots, u_{k-1} \) in succession gives
\[ y_j = 0 \quad \text{for each } j \]
whereby
\[ Ax = 2x \]
so that the optimization problem

\[ \text{Sup } \langle Ax, x \rangle \text{ has an extremum at an eigen vector } u_k \text{ orthogonal} \]
\[ \langle x, u_j \rangle = 0, \quad j = 1, 2, \ldots, k-1 \]
\[ \text{and the value of the supremum } = \lambda_k, \text{ the next eigenvalue in the list.} \]
The first fundamental form:

Let \( S \) be a surface in \( \mathbb{R}^3 \) at a point \( p \in S \), the tangent plane \( T_p S \) inherits an inner product from its ambient space, usually called the first derived by \( T_p \).

Thus we have the smoothly varying positive definite symmetric inner product \( p \mapsto \mathbf{I}_p \) known as the first fundamental form.

If \( (u,v) \) is a coordinate patch with \( X(u,v) = p \) then the tangent to the coordinate curves

\[
\begin{align*}
    u & \mapsto X(u,0) \\
    v & \mapsto X(0,v)
\end{align*}
\]

generates the vector space \( T_p S \) and \( T_p \) is uniquely determined by

\[
\begin{align*}
    \mathbf{I}_p (X_u, X_u) &= X_u(u,0) \cdot X_u(u,0) \\
    \mathbf{I}_p (X_u, X_v) &= X_u(u,0) \cdot X_v(u,0) \\
    \mathbf{I}_p (X_v, X_v) &= X_v(0,v) \cdot X_v(0,v)
\end{align*}
\]

These are the coefficients of the bilinear form \( \mathbf{I}_p \).

The traditional notation for these coefficients is

\[
\begin{align*}
    g_{11}(u,v) &= X_u \cdot X_u \\
    g_{12}(u,v) &= g_{21}(u,v) = X_u \cdot X_v \\
    g_{22}(u,v) &= X_v \cdot X_v
\end{align*}
\]

\((g_{ij})\) are known as the components of the metric tensor.

Recall that if \( \gamma \) is a curve lying on \( S \) then the arc length of \( \gamma \) from base point \( a \) to \( b \) is given by

\[
\ell = \int_a^b \| \dot{\gamma}(t) \| \, dt
\]

But if \( \sigma = \gamma^{-1} \circ \gamma \) a curve in \( U \)

\[
\sigma'(t) = X_u \frac{du}{dt} + X_v \frac{dv}{dt}
\]

\[
\langle \sigma'(t), \sigma'(t) \rangle = g_{11}(\frac{du}{dt})^2 + 2g_{12} \frac{du}{dt} \frac{dv}{dt} + g_{22}(\frac{dv}{dt})^2
\]

Thus the differential of arc length is expressible in terms of the components of the metric tensor (which is why it is called the metric tensor).

Transformation law of \( \sigma_{ij} \):

Let \( (v,w) \) be another coordinate patch and \( p \in X(u,v) \).

and for \( (u,v) \in X^{-1}(v,w) \)

\[
\begin{align*}
    X(u,v) &= Y(\tilde{u}, \tilde{v}) \quad \text{for some } (\tilde{u}, \tilde{v}) \in \tilde{Y}(\tilde{x}(u,v))
\end{align*}
\]

and by implicit function theorem, \( \tilde{u}, \tilde{v} \) are smooth functions of \((u,v)\):

\[
\begin{align*}
    X_u &= Y_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + Y_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u} \\
    X_v &= Y_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + Y_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}
\end{align*}
\]
\[ G_{ij}(u,v) = \tilde{G}_{ij} \frac{\partial u}{\partial u} \frac{\partial u}{\partial v} + \tilde{G}_{ij} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{ij} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{ij} \frac{\partial u}{\partial u} \frac{\partial u}{\partial v} + \tilde{G}_{ij} \frac{\partial v}{\partial v} \frac{\partial v}{\partial v} \]

\[ G_{12}(u,v) = \tilde{G}_{12} \frac{\partial u}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{12} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{12} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} \]

\[ G_{22}(u,v) = \tilde{G}_{22} \frac{\partial u}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{22} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} + \tilde{G}_{22} \frac{\partial v}{\partial u} \frac{\partial v}{\partial v} \]

Changing notation slightly \((U, V)\) with coord. \((u, v)\) etc.

Let us denote \( g_{ij} = x_u \cdot x_v \cdot x_u \cdot x_v \)

\[ g_{ij} = \sum_{i', j'} g_{i'j'} \frac{\partial x_{j'}}{\partial x_i} \frac{\partial x_{j'}}{\partial x_i} \]

That is \( g_{ij} \) are the components of a Covariant tensor of rank 2.

**Area of a piece of a Surface:**

Let \( S \) be a Surface and \( A \) be a piece of \( S \) contained in the image of a coordinate patch \((x, U)\). The area of \( A \) is by definition

\[
\int \int_{x^{-1}(A)} || x_u \times x_v || \, du \, dv
\]

The motivation for this is of course...
\[ \left\| x_1 x_2 \right\| \, dx_1 \, dx_2 = \left\| y_1 y_2 \right\| \, \frac{\partial (u,v)}{\partial (x_1,x_2)} \, du \, dv \]

\[ \int y_1 y_2 \, dx_1 \, dx_2 \] by the change of variables formula.

How does one compute (or first define) area of a piece of \( S \) when this piece is not contained in the image of a single coordinate patch? Some issue for the arc length of a curve in terms of \( du, dv \).

We shall merely sketch the answer to this question. First of all we shall only be concerned with \( A \subseteq \mathbb{R}^3 \) such that \( A \) is compact and boundary of \( A \) is a finite union of piecewise smooth arcs (otherwise the Jordan Content may not make sense and one may be forced to employ Lebesgue theory).

Now use the fact that every open cover has a \( \mathcal{B} \) Lebesgue number and so \( \mathbb{R}^3 \) can be chopped into a family of non-overlapping cubes each of whose sides is less than \( \frac{1}{\mathcal{B}} \) of the Lebesgue number.

Only finitely many of these cubes meet \( S \) and each of these gives a piece that lies entirely in one of the coordinate patches.

Now add up the contributions from each of these cubes.

Exercises:

Determine the components of the first fundamental form of:

(i) The paraboloid \( (u,v) \mapsto (u,v, u^2 + v^2) \)

(ii) The hyperbolic paraboloid \( (u,v) \mapsto (u,v, u^2 - v^2) \)

(iii) The surface of revolution obtained by rotating \( f = f(x) \) about the \( x \)-axis

(iv) In the hyperboloid of one sheet \( x^2 + y^2 - z^2 = 1 \) and the hyperboloid \( z = \sqrt{x^2 + y^2} \)

find the area of the portion of the surface contained within a quadilateral determined by four rulings (two of each family).

The integral may not be computable explicitly.

(v) Find the area of a piece of the tangent surface to a curve \( \gamma \) and also a piece of a cone determined by \( \gamma \) (not passing through the origin).

(vi) Consider the curve of intersection of \( \gamma^2 + z^2 = 1 \) and \( x^2 + z^2 = 1 \) and \( S \) be the cone determined by this curve. Find the area of this cone.
Tangential Components of the acceleration vector on the Covariant derivative.

Now suppose that \( \delta: I \rightarrow S \) is a curve on \( S \) then \( \delta'(t) \) lies on the tangent space to \( S \) at \( \delta(t) \) but the acceleration vector \( \delta''(t) \) need not be tangential to \( S \) (even if \( \delta \) is unit speed). For instance, look at the "arc of a circle" on a Sphere \( x^2 + y^2 + z^2 = 1 \), namely, the intersection of the Sphere with the plane \( z = c \), \( c \neq 0 \). The acceleration vector points towards the centre of this circle \( (0, 0, c) \) ≠ origin and so is not tangent to the Sphere.

Let us determine the Components of the acceleration vector along the tangent plane, known as the Covariant derivative of the Velocity vector field.

Before taking this up it is useful to look at the Case of Coordinate Curves on a Coordinate patch \((u, v)\) on the Surface.

Let us consider the Coordinate Curves \( u \rightarrow X(u, v) \) and \( v \rightarrow X(u, v) \) through a point \( X(u_0, v_0) \) on \( S \).

Differentiating the relations \( g_{11} = X_u \cdot X_u \), \( g_{12} = X_u \cdot X_v \), \( g_{22} = X_v \cdot X_v \) with respect to \( u \) and \( v \) we obtain a System of Six Equations which determine

\[
\begin{align*}
X_u \cdot X_{uu} &= \frac{1}{2} \frac{\partial g_{11}}{\partial u} \\
X_u \cdot X_{uv} &= \frac{1}{2} \frac{\partial g_{12}}{\partial u} \\
X_u \cdot X_{vu} &= \frac{1}{2} \frac{\partial g_{12}}{\partial v} \\
X_u \cdot X_{vv} &= \frac{1}{2} \frac{\partial g_{22}}{\partial u} \\
X_v \cdot X_{uu} &= \frac{1}{2} \frac{\partial g_{11}}{\partial v} \\
X_v \cdot X_{uv} &= \frac{1}{2} \frac{\partial g_{12}}{\partial v} \\
\end{align*}
\]

We are now ready to calculate the projections of \( X_{uu}, X_{uv} \) and \( X_{vu} \) on \( T_p S \).

Well, \( X_{uu} = \lambda g_{11} + \mu g_{12} + \nu g_{22} \)

Taking dot product \( u \cdot v \rightarrow X_u \cdot X_v \) we get

\[
\begin{align*}
X_u \cdot X_u &= \lambda g_{11} + \mu g_{12} + \nu g_{22} \\
X_u \cdot X_v &= \lambda g_{12} + \mu g_{22} \\
\end{align*}
\]

Which can be immediately solved using Gramm's rule:

\[
\begin{bmatrix}
\lambda \\
\mu \\
\nu
\end{bmatrix} = \frac{1}{2} \left[ \begin{array}{ccc}
\partial u g_{11} & \partial v g_{12} & \partial v g_{12} \\
\partial u g_{12} & \partial v g_{22} & \partial v g_{22} \\
\partial u g_{22} & \partial v g_{12} & \partial v g_{12}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\partial u g_{11} \\
\partial v g_{12} \\
\partial v g_{12}
\end{array} \right]
\]

\[
\begin{align*}
\lambda &= \frac{1}{2} \left[ \begin{array}{ccc}
\partial u g_{11} & \partial v g_{12} & \partial v g_{12} \\
\partial u g_{12} & \partial v g_{22} & \partial v g_{22} \\
\partial u g_{22} & \partial v g_{12} & \partial v g_{12}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\partial u g_{11} \\
\partial v g_{12} \\
\partial v g_{12}
\end{array} \right] \\
\mu &= \frac{1}{2} \left[ \begin{array}{ccc}
\partial u g_{11} & \partial v g_{12} & \partial v g_{12} \\
\partial u g_{12} & \partial v g_{22} & \partial v g_{22} \\
\partial u g_{22} & \partial v g_{12} & \partial v g_{12}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\partial u g_{12} \\
\partial v g_{22} \\
\partial v g_{22}
\end{array} \right] \\
\nu &= \frac{1}{2} \left[ \begin{array}{ccc}
\partial u g_{11} & \partial v g_{12} & \partial v g_{12} \\
\partial u g_{12} & \partial v g_{22} & \partial v g_{22} \\
\partial u g_{22} & \partial v g_{12} & \partial v g_{12}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\partial u g_{22} \\
\partial v g_{12} \\
\partial v g_{12}
\end{array} \right]
\end{align*}
\]
\[ \text{where} \quad \left[ g_{ij} \right]^{-1} = [ g^{ij} ] \]

\[ \therefore \lambda = \frac{1}{2} \left\{ g^{12} \left( \frac{\partial g_{12}}{\partial u} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{11}}{\partial v} \right) \right\} \]

\[ m = \frac{1}{2} \left\{ g^{21} \frac{\partial g_{11}}{\partial u} + \right. \]

\[ \left. g^{22} \left( \frac{\partial g_{12}}{\partial u} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{11}}{\partial v} \right) \right\} \]

\[ \lambda \text{ and } m \text{ are denoted by } \Gamma^1 \text{ and } \Gamma^2 \text{ respectively.} \]

\[ \therefore \Gamma^1_{uv} = \Gamma^2_{u1} X_u + \Gamma^2_{v1} X_v + \lambda N \]

Similarly we can compute the coefficients of \( X_u \), \( X_v \) \( \text{in the projection of } X_{uv} \) \( \text{on } T_p S \)

\[ X_{uv} = \Gamma^1_{v2} X_u + \Gamma^2_{v2} X_v + \alpha N \]

\[ X_{uv} = \frac{1}{2} \left( \Gamma^1_{i2} X_u + \Gamma^2_{i2} X_v + \beta N \right) \]

The coefficients \( \Gamma^{ij}_{kl} \) are known as Christoffel Symbols. The equality of mixed partials is expressed as

\[ \Gamma^{ij}_{kl} = \Gamma^{ij}_{lk} \]

\[ \Gamma^{ik}_{lj} = \frac{1}{2} \sum_{k=1}^{2} g^{ik}\left( \frac{\partial g_{lj}}{\partial u} + \frac{\partial g_{uj}}{\partial v} - \frac{\partial g_{lj}}{\partial v} \right) \]

Exercise: Verify this. We have done it for \( \Gamma^1_{11}, \Gamma^2_{11}, \Gamma^1_{22}, \Gamma^2_{22} \).

Need to do it for the other four.

Suppose that \( \gamma : I \to S \) is a curve parametric and \( \gamma = x^1(t), x^2(t) \)

\[ \gamma' = (x^1, x^2) = x^1 \dot{u} + x^2 \dot{v} \]

\[ \gamma'' = x^1 \ddot{u} x^2 \dot{v}^2 + x^1 \dot{u} \dot{v} + x^2 \dot{u}^2 \dot{v} \]

\[ = \left( \Gamma^1_{11} x^1 + \Gamma^1_{12} x^2 \right) u'' + \left( \Gamma^2_{11} x^1 + \Gamma^2_{12} x^2 \right) v'' \]

\[ + \left( \Gamma^1_{22} x^1 + \Gamma^2_{22} x^2 \right) u' v' \]

\[ + x^1 \ddot{u} + x^2 \ddot{v} + \alpha N \]

for some scalar function \( \alpha \).

* Def: A curve on \( S \) whose acceleration has no tangential component is said to be a geodesic.

If now \( \gamma \) is a geodesic then

\[ \Gamma^1_{11} u'^2 + \Gamma^1_{12} u' v' + 2u' \dot{u} \Gamma^1_{12} + v'' = 0 \quad (\#) \]

\[ \Gamma^2_{11} u'' + \Gamma^2_{12} u'' v' + 2u' \dot{v} \Gamma^2_{12} + v'' = 0 \quad (\#) \]

which is a system of coupled second order ODEs for \( u, v \).

\[ x^1(u(t), v(t)) \text{ is then the geodesic on } S. \]

* Reads: \( \sum_{k=1}^{2} \Gamma^{ik}_{lj} u_k' u_l' + u_k'' = 0 \quad (k = 1, 2) \)

Let us determine the geodesics on the unit sphere.

\[ x(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \]

\[ x_\theta = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0) \]

\[ x_\phi = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi) \]

\[ g_{12} = x_\theta \cdot x_\phi = 0 \quad ; \quad g_{11} = x_\theta \cdot x_\theta = \sin^2 \phi \]

\[ g_{22} = 1 \]
Thus \( g^{12} = 0; \quad g^{11} = \csc^2 \phi; \quad g^{22} = 1 \)

The Christoffel symbols are:

\[
\Gamma^1_{12} = \cot \phi; \quad \Gamma^2_{12} = 0
\]

\[
\Gamma^1_{11} = 0; \quad \Gamma^2_{11} = -\sin \phi \cos \phi
\]

\[
\Gamma^2_{22} = 0; \quad \Gamma^2_{22} = 0
\]

\[
\therefore \quad \phi'' - 2 \sin \phi \cos \phi \theta^2 = 0
\]

\[
\theta'' + 2 \cot \phi \theta' \phi' = 0.
\]

The other case is \( \theta' = \infty \)

Note that the spherical polar coordinates are well adapted to finding geodesics through the equator meeting it orthogonally but we proceed generally. The second equation integrates to

\[
\theta' \sin^2 \phi = \alpha \quad \text{(const of integration)}
\]

\( \alpha > 0, \quad \theta' = 0 \)

The first equation now gives

\[
\phi'' - \sin \phi \cos \phi \cdot \frac{\alpha^2}{\sin^2 \phi} = 0
\]

\[
\frac{d}{dt} \left( \frac{1}{2} \phi'^2 + \frac{\alpha^2}{2} \sin^{-2} \phi \right) = 0
\]

\[
\phi'^2 + \alpha^2 \csc^2 \phi = \beta
\]

By rescaling time variable we may assume \( \beta = 1 \)

\[
\cos \phi = \sqrt{1 - \alpha^2} \sin (t + \theta_0)
\]

This amounts to changing \( \alpha \) by \( \alpha^2 \) throughout.

\[
\alpha' = \frac{\alpha}{\cos^2 (t + \theta_0) + \alpha^2 \sin^2 (t + \theta_0)}
\]

Let us assume that the geodesic crosses the equator at \( t = 0 \) (so \( \phi = \frac{\pi}{2} \) when \( t = 0 \))

\[
\cos \phi = \sqrt{1 - \alpha^2} \sin t
\]

\[
\theta' = \frac{\alpha}{\cos^2 t + \alpha^2 \sin^2 t}
\]

\[
\theta' = \frac{\alpha}{\cos^2 (t + \theta_0)}
\]

\( \theta' = \frac{\alpha}{\cos^2 (t + \theta_0)} \)

\( \sin \phi \cos \theta \sin \phi \pi = \alpha \theta' \phi' (\cos \phi \sin \phi \pi + \sin \phi \pi) \)

\( \sin \pi \phi \pi + \sin \pi \phi \pi = \alpha \phi ' (\cos \pi \phi \pi - \cos \pi \phi \pi) \)

\( \tan (\pi + \pi) = \alpha \tan \pi \quad \lambda \) is one more cons. of integrations.

Taking \( \lambda = 0 \),

\[
\tan \theta = \alpha \tan \pi
\]

\[
\cos \phi = \sqrt{1 - \alpha^2 \tan^2 \pi}
\]

\[
\sin \phi = \sqrt{\cos^2 \pi + \alpha^2 \sin^2 \pi} = \cos \theta \sqrt{1 + \alpha^2 \tan^2 \pi}
\]

\[
\cos \theta = \frac{1}{\sqrt{1 + \alpha^2 \tan^2 \pi}}
\]

This curve lies on the plane \( \sqrt{1 - \alpha^2} y - \alpha z = 0 \)

If \( \lambda = 0 \) is not assumed we would get a plane not passing through \( x \)-axis.

Not \( \lambda = 0 \) just means that the geodesic was at \( (1, 0, 0) \) when \( t = 0 \).
Note that a geodesic may not be reparametrizable for its acceleration of the reparametrization. Therefore, it would produce a tangential component.

Thm. (i) The geodesics have the property that the arc length function is proportional to $t$.

(i) The rescaling of time parameter keeps the geodesic equations invariant.
(ii) is immediate.

(i) Let $\delta = X(u(t), v(t))$ be a geodesic $\delta'' = \sigma N$ for some scalar function $\sigma$.

For all $t$, $\|\delta''\|^2 = \sigma < \delta', \delta''> = 0$

Thus, $\|\delta''\|^2$ is constant in time, i.e., $\delta(t) = ct$.

Thus, the geodesics may be assumed to be unit speed curves.

The initial value problems of systems of ODEs

Consider a system of second order ODEs:

$$x_i'' = F_i(t, x, x')$$

where $F_i$ are smooth functions defined on an open set in $\mathbb{R}^{2n}$. For simplicity, we assume that $F_i$ are defined on a box

$$\Omega = \{(t_0, x, t_0, x') \in \mathbb{R}^{2n} : \|x - x_0\| < \delta, \|x' - x_0'\| < \delta\}$$

The initial value problem consists of a system together with a set of initial conditions:

$$x_i(t_0) = \alpha_i, \quad x_i'(t_0) = \beta_i$$

for $i = 1, 2, ..., n$.

Thereby, providing a point $(t_0, \alpha, \beta) \in \Omega$. Assume that $\Omega$ is chosen such that $Q = x_0, \beta = y_0$.

Define $y_i = x_i'$ for $i = 1, 2, ..., n$ and consider the system of first order ODEs:

$$\begin{align*}
x_i'' &= \dot{y}_i \quad (i = 1, 2, ..., n) \\
y_i'' &= F_i(t, x, y)
\end{align*}$$

With initial conditions:

$$x_i(t_0) = \alpha_i, \quad y_i(t_0) = \beta_i$$

To each solution of this system, there corresponds a unique solution of $\Omega$ and vice versa. Hence it suffices to discuss the system with the given initial conditions $x_i(t_0) = \alpha_i, y_i(t_0) = \beta_i$.

Let $z = (x_1, y_1, ..., x_n, y_n)$ and

$$\Omega = \{ (t_0, x_0) \in \mathbb{R}^{2n} : \|x - x_0\| < \delta, \|y - y_0\| < \delta \}$$

For $\alpha, \beta \in \mathbb{R}^{2n}$, then $\|x - x_0\| < \delta$ and $\|y - y_0\| < \delta$.

That is to say, $G$ satisfies a Lipschitz condition on a Strichartz domain in $\mathbb{R}^2$.
Let \( M = \text{Snp} \{ \| G(t, z) \| / |t-b| < a \}
\]\( \{ |z-z_0| < b \} \)
\( \text{and} \quad A = \text{Min} \{ a, b \} \}\) \( \{ |z-z_0| < b \} \)
\( \{ |t-b| < a \} \)

Claim: For any point \((t_0, z)\) with \( |z-z_0| < b/3 \) \( |t-t_0| < b/3 \)

1. Solution \( \phi(t) \) of the IVP defined on the interval \([t_0-h, t_0+h]\) with \( i.e. t \).

The solution depends continuously on \( \phi \).

Proof: (Existence) Solving the IVP is equivalent to solving the integral equation

\[
\phi(t) = \phi(z) + \int_{t_0}^{t} G(s, \phi(s)) ds
\]

Define the successive iterates

\[
\phi_0(t) = \phi(z) \quad \text{and} \quad \phi_n(t) = \phi(z) + \int_{t_0}^{t} G(s, \phi_{n-1}(s)) ds.
\]

First of all, we must ensure that the iterates are all defined on the common interval \([t_0-h, t_0+h]\).

In other words, \((s, z_n(s)) \in I_n \) so that \( G(s, z_n(s)) \) makes sense for each \( n \) and \( s \in [t_0-h, t_0+h] \).

Well, this is certainly true when \( n=0 \):

\[
\| z_n(t) - \phi(z) \| \leq \int_{t_0}^{t} \| G(s, z_{n-1}(s)) \| ds
\]

\( \leq M |t-t_0| \leq \frac{b}{3} \leq \frac{b}{2} \)

Now

\[
\| z_n(t) - z_0(t) \| \leq \| z_n(t) - z_0(t) \| + \| z_0(t) - \phi(z) \| \leq \frac{b}{3} + \frac{b}{2} = \frac{2}{3} b < b
\]

So, since \((s, z_n(s)) \in I_n \) if \((s, z_{n-1}(s)) \in I_n \) for all \( s \in [t_0-h, t_0+h] \). The claim is proved by induction.

Now we show \((z_{n}(t)) \) converges uniformly on \([t_0-h, t_0+h]\). Well,

\[
\| z_n(t) - \phi(z) \| \leq \int_{t_0}^{t} \| G(s, \phi(s)) \| ds
\]

\( \leq M |t-t_0| \leq \frac{b}{3} \)

Assume inductively that

\[
\| z_n(t) - z_{n-1}(t) \| \leq \frac{b}{3 n}
\]

Then

\[
\| z_n(t) - z_{n-1}(t) \| \leq \frac{b}{3 n}
\]

\( \leq \int_{t_0}^{t} \| G(s, \phi(s)) - G(s, z_{n-1}(s)) \| ds
\]

\( \leq \frac{b}{3 n} \int_{t_0}^{t} 1 ds
\]

\( \leq \frac{b}{3 n} (t-t_0) \)

Thus

\[
\sum_{n=0}^{\infty} \| z_{n+1}(t) - z_n(t) \| \text{ converges uniformly on } [t_0-h, t_0+h].
\]

The partial sums of the series converge uniformly.

So \((z_n(t)) \) converges uniformly on \([t_0-h, t_0+h]\) to \( \phi(t) \).

Letting \( n \to \infty \) in \((\ast)\) we see that

\[
\phi(t) = \phi(z) + \int_{t_0}^{t} G(s, \phi(s)) ds \quad \text{the integral in the RHS is continuous, } \phi(t) \text{ is } C^1 \quad \text{and to}
\]

\( \phi(t) \text{ is } C^1 \) and so
by the fundamental thm of calculus
\[ \frac{\partial y}{\partial t} = f(t, y(t)) \]
Uniqueness and Continuity:
Changing the initial condition may change domain, but we have ensured that for all \( y \) such that \( ||y - z_0|| \leq b/a \)
The solutions are defined on \([a-b, b+a]\)
To prove Continuity w.r.t. \( y \) as well as uniqueness
we need the following

Lemma (Gronwall): Suppose \( u, v : [a,b] \to \mathbb{R} \)
are non-negative functions such that
for constants \( C, k \)
nonneg \[ u(t) \leq C + k \int_a^t u(s)v(s)\,ds \]
then \( u(t) \leq C e^{kt} \int_a^t v(s)\,ds \)

Proof: Assume first \( C > 0 \).
Put \( U(t) = C + k \int_a^t u(s)v(s)\,ds \geq C > 0 \)
Also by hypothesis \( u(t) \leq U(t) \)

Next, let \( C = 0 \).
Then obviously \( u(t) \leq 1 + k \int_a^t v(s)u(s)\,ds \)

by applying the previous case with \( C = \infty \)
we infer
\[ 0 \leq u(t) \leq \frac{1}{k} \int_0^t \exp(k) v(s)\,ds \]
Letting \( \rightarrow \infty \) we get \( u(t) \to 0 = C \exp(k) \int_0^t v(s)\,ds \)

Now suppose \( z(t) \) and \( \tilde{z}(t) \) are two solutions
of the IVP then
\[ z(t) = \xi + \int_0^t f(s, z(s))\,ds \]
\[ \tilde{z}(t) = \xi + \int_0^t f(s, \tilde{z}(s))\,ds \]
\[ ||z(t) - \tilde{z}(s)|| \leq L \int_0^t ||z(s) - \tilde{z}(s)||\,ds \]

Applying Gronwall's Lemma with \( C = 0, \ k = L \)
\( \nu = 1 \) and \( \nu = ||z(s) - \tilde{z}(s)|| \) we conclude
\[ u(t) \to 0 \text{ or } \xi(t) \leq \tilde{\xi}(s) \]

Continuity w.r.t. \( \xi \)
Again if \( z(t, \xi) \) denotes the solution of the system
of ODEs with initial value \( \xi \)
and \( \tilde{z}(t, \tilde{\xi}) \) is the solution of the IVP with
initial value \( \tilde{\xi} \) then
\[ z(t) = \xi + \int_0^t f(s, z(s))\,ds \]
\[ \tilde{z}(t) = \tilde{\xi} + \int_0^t f(s, \tilde{z}(s))\,ds \]
\[ ||z(t) - \tilde{z}(s)|| \leq L \int_0^t ||z(s) - \tilde{z}(s)||\,ds \]

Apply Gronwall with \( C = 0 \)
\( \xi = \tilde{\xi} \) and \( k = L \)

by}\[ ||z(t) - \tilde{z}(s)|| \leq ||\tilde{\xi} - \tilde{\xi}|| + L \int_0^t ||z(s) - \tilde{z}(s)||\,ds \]

\( = ||\tilde{\xi} - \tilde{\xi}|| \exp(L(t - s)) \)
Proving Continuity \( \mathcal{C} \) in \( \mathbb{R}^n \).

It is true but harder to prove that the solution \( \frac{dx(t, \xi)}{dt} = \mathbf{F}(t, x(t, \xi)) \) is differentiable with \( \xi \). However, the following is important:

If we define the ODE
\[
\frac{d\xi}{dt} = F(t, \xi) \quad \text{w.r.t. } \xi
\]
then
\[
\frac{d}{dt} \left( \frac{\partial \xi}{\partial \xi_j} \right) = \sum_{k=1}^{n} \frac{\partial F_k}{\partial \xi_j} \frac{d\xi_k}{dt}
\]

Thus, each of the \( \xi \)-functions \( \frac{d\xi_j}{dt} \), \( j = 1, 2, \ldots \), satisfies the same linear ODE
\[
\frac{d\xi_j}{dt} = A(t) \xi_j
\]
where
\[
A(t) = \frac{\partial F(t, \xi(t))}{\partial \xi}
\]
To obtain initial conditions observe that
\[
\frac{\partial \xi_j}{\partial \xi_j} \bigg|_{t=0} = \frac{\partial}{\partial \xi_j} \left( \frac{\partial \xi_j}{\partial \xi_j} \bigg|_{t=0} \right) = \frac{\partial \xi_j}{\partial \xi_j} = 1
\]

In particular
\[
\frac{\partial (\xi_1, \ldots, \xi_n)}{\partial (\xi_1, \ldots, \xi_n)} \bigg|_{t=0} = 1
\]

By the Abel–Liouville theorem,
\[
\frac{\partial (\xi_1, \ldots, \xi_n)}{\partial (\xi_1, \ldots, \xi_n)} = \exp \int_0^t \text{Tr}(A(s)) \, ds
\]
and
\[
\frac{\partial (\xi_1, \ldots, \xi_n)}{\partial (\xi_1, \ldots, \xi_n)} = \exp \int_0^t \text{Tr}(A(s)) \, ds
\]

We shall not use this result though.

Returning now to geodesics,

Thm: Let \( S \) be a surface in \( \mathbb{R}^3 \) and \( \mathbf{p} \in S \). For each \( \mathbf{p} \in T_p S \), \( \exists \) a unique geodesic \( \gamma : I \rightarrow S \) such that
\[
\gamma(0) = \mathbf{p}
\]
\[
\gamma'(0) = \mathbf{v}
\]

We shall choose a coordinate patch \( (U, \mathbf{x}) \) containing \( \gamma \) such that
\[
\mathbf{x} \circ \gamma : (x_1(0), x_2(0), x_3(0)) = \mathbf{p} \quad \mathbf{v} = (v_1(0), v_2(0), v_3(0))
\]

Then \( \dot{x}_1(0), \dot{x}_2(0) \) are expressible in terms of \( w \) and we have the initial conditions
\[
x_1(0) = x_1(0), \quad \dot{x}_1(0) = \ddot{x}_1(0)
\]
\[
x_2(0) = x_2(0), \quad \dot{x}_2(0) = \ddot{x}_2(0)
\]

With these we can uniquely solve the pair of ODEs
\[
u'' + \Gamma_{12}^1 \nu^1 + \Gamma_{12}^2 \nu^2 + 2 \Gamma_{12}^3 \nu^1 \nu^2 = 0
\]
\[
u'' + \Gamma_{22}^1 \nu^1 + \Gamma_{22}^2 \nu^2 + 2 \Gamma_{22}^3 \nu^1 \nu^2 = 0
\]
and \( \mathbf{x} \circ \gamma : (x_1(0), x_2(0), x_3(0)) \) is the desired geodesic.

Remark: The geodesic through \( \mathbf{p} \) in the direction \( \mathbf{v} \) is defined only on an interval \( I \). In general

I may not be whole of \( \mathbb{R}^3 \)

The maximal interval on which \( \gamma \) exists is called the life span of the geodesic. The finite life span is due to the non-linearity of the system of ODEs.

Thm: If \( S \) is a compact surface then...
The geodesics on $S$ have have $\mathbb{R}$ as their interval of existence.

We shall not prove this theorem here.

Geodesics are locally length minimizing curves.

Note that if $p$, $q$ are two points on a surface $S$ then may not be a geodesic on $S$ connecting $p$ and $q$.

Ex: $S = \mathbb{R}^2 - (0)$ and $p = (1,1)$,

$q = (-1, -1)$

There is no geodesic connecting $(1,1)$ and $(-1,-1)$.

Second, if $\exists$ a geodesic connecting $p$ and $q$, it need not be unique.

Ex: Consider a sphere and two points $m$, $n$.

There are exactly two arcs of great circles joining them one of which is the line of shortest distance.

We prove the converse.

Thm: If $S$ is a surface and $(u, v)$ is a coordinate patch and for $p, q \in x(u)$,

$\gamma : [a, b] \rightarrow S$ is a curve minimizing the length of least joining $p$ and $q$ then $\gamma$ is a geodesic.

Proof: $\| \gamma'(t) \|^2 = \sum g_{i,j}(u,v) u^i w^j$

We change notations and write $u = u_1$, $v = u_2$.
\[
\int_a^b \sum_{i,j} \left( \sum_k \frac{\partial g_{ij}}{\partial u_k} \phi_i u_j u_k \right) \, dt + 2 \int_a^b \sum_{i,j} g_{ij} u_j u'_i \, dt = 0
\]

We now integrate by parts.

\[
\sum_{i,j} \int_a^b \left( \sum_k \frac{\partial g_{ij}}{\partial u_k} \phi_i u_j u'_k \right) \, dt - 2 \int_a^b \sum_{i,j} g_{ij} u_j u'_i \phi_i \, dt = 0
\]

Replacing \( k \) by \( l \).

\[
\sum_{i,j} \int_a^b \left( \sum_l \frac{\partial g_{ij}}{\partial u_l} \phi_i u_j u'_l \right) \, dt - 2 \int_a^b \sum_{i,j} g_{ij} u_j u'_i \phi_i \, dt = 0
\]

In the second integral replace the dummy index \( i \) by \( l \) and use the fundamental theorem of the calculus of variations.

Since \( \phi_l \) are arbitrary we see that for each \( l \),

\[
\sum_{i,j} \frac{\partial g_{ij}}{\partial u_l} u_j u'_i - \sum_i u_i' \phi_l = 0
\]

\[
- \sum_k \sum_k \frac{\partial g_{ij}}{\partial u_k} u_j u'_k = 0
\]

\[
\sum g_{kl} u_l u'' + \frac{1}{2} \sum_{i,j,k} \left( \frac{\partial g_{kl}}{\partial u_i} + \frac{\partial g_{kl}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k} \right) u_i u'_j u'_k = 0
\]

\[
\sum g_{kl} u_l u'' + \frac{1}{2} \sum_{i,j,k} \left( \frac{\partial g_{kl}}{\partial u_i} + \frac{\partial g_{kl}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k} \right) u_i u'_j u'_k = 0
\]

Thus \( \gamma \) is a geodesic.
The exponential map:

Let $S$ be a surface and $p, q \in S$. By the fundamental existence-uniqueness theorem on ODEs, given any $u_{0} \in S$ and $v_{0} 

Let us now consider the set of all geodesics with initial conditions $u_{0}, v_{0}$.

The geodesics would be the set of all geodesics $u, v$ with $u(0) = a$, $v(0) = b$.

Then we have the geodesic equations:

$\frac{d^{2}u}{dt^{2}} + \frac{\gamma_{12}}{\gamma_{11}} \frac{du}{dt} \frac{dv}{dt} = 0$

$\frac{d^{2}v}{dt^{2}} + \frac{\gamma_{21}}{\gamma_{22}} \frac{du}{dt} \frac{dv}{dt} = 0$

The geodesic equations:

$\frac{d^{2}u}{dt^{2}} = 0$

$\frac{d^{2}v}{dt^{2}} = 0$

The geodesic equations:

$\frac{d^{2}u}{dt^{2}} = 0$

$\frac{d^{2}v}{dt^{2}} = 0$

$g_{11} = \sqrt{1 + \left(\frac{dv}{dt}\right)^{2}}$

$g_{12} = \frac{0}{\sqrt{1 + \left(\frac{dv}{dt}\right)^{2}}} = 0$

The geodesic equations:

$\frac{d^{2}u}{dt^{2}} = 0$

$\frac{d^{2}v}{dt^{2}} = 0$

The geodesic equations:

$\frac{d^{2}u}{dt^{2}} = 0$

$\frac{d^{2}v}{dt^{2}} = 0$

The geodesic equations:

$\frac{d^{2}u}{dt^{2}} = 0$

$\frac{d^{2}v}{dt^{2}} = 0$
If $c=0$ then $\theta$ is constant and $w$ are

length rays in the $xy$-plane. The geodesics are

the meridians in this case.

In any surface of revolution the meridians

are geodesics.

Ex: Assume then $c \neq 0$. With $u = \gamma \cos \theta$

\[ u' = \gamma' \cos \theta - \gamma \sin \theta \theta' \]

\[ u'' = \gamma' \sin \theta + \gamma \cos \theta \theta'' \]

\[ \theta'' = \text{constant} \Rightarrow \theta' = c \]

Now

\[ uu'' + vv'' + \frac{4 \gamma^2 (u'^2 + v'^2)}{\sqrt{1 + \gamma^2}} = 0 \]

\[ \begin{aligned}
& (uu' + vv')' - (u'^2 + v'^2) + 4 \gamma^2 (u'^2 + v'^2) = 0 \\
& \frac{1}{2} (\gamma^2)'' + (\gamma^2 + \gamma^2 \theta'^2) \left\{ \frac{\gamma'}{\sqrt{1 + \gamma^2}} - 1 \right\} = 0 \\
& \frac{1}{2} (\gamma^2)'' + (\gamma^2 + \gamma^2 \theta'^2) \left\{ \frac{\gamma'}{\sqrt{1 + \gamma^2}} - 1 \right\} = 0 \\
& \gamma'' + \frac{4 \gamma^2 \theta'^2}{\sqrt{1 + \gamma^2}} - \gamma \theta'' = 0 \\
& \gamma'' + \frac{4 \gamma^2 \theta'^2}{\sqrt{1 + \gamma^2}} + \frac{4 \gamma'^2 \theta'^2}{\sqrt{1 + \gamma^2}} = 0 \\
& \gamma'' + \frac{4 \gamma^2 \theta'^2}{\sqrt{1 + \gamma^2}} + \frac{4 \gamma'^2 \theta'^2}{\sqrt{1 + \gamma^2}} = 0 \\
\end{aligned} \]

This can be integrated and one can show that

the geodesics (other than meridians) wind around
densely.

The exponential map. (Left incomplete.)

Let $p \in S$ be a fixed point. By a simple

Compactness argument we see that $\exists h > 0$ such that

for every $w$ with $\|w\| = 1$.

There is a solution $\Phi(t, p, w)$ of the geodesic

equation with initial conditions $(p, w)$ defined

on $(-2h, 2h)$. That is the only as we vary $w$
on the unit circle in $T_p S$, the geodesics have all
a common domain $(-2h, 2h)$.

Thus if $t, w \in T_p S$ and $\|w\| \leq h$

\[ x \circ \Phi(\|w\|, p, \frac{w}{\|w\|}) \text{ makes sense and is a}
\]

point of $S$.

Def. $E(w) = x \circ \Phi(\|w\|, p, \frac{w}{\|w\|})$

$E: B \to S$ where $B$ is the ball of

radius $h$ in $T_p S$ is called the exponential

map.

It is useful to have a slight reformulation

Let $w \in T_p S$ and $\|w\| \leq h$

Define $\psi(t) = \Phi(\frac{t}{\|w\|}, p, w)$

Then $\psi$ satisfies the DE for geodesics with

initial conditions $\psi(0, p, \frac{w}{\|w\|})$ same as the

initial conditions of $\Phi(t, p, \frac{w}{\|w\|})$

Thus, by uniqueness.
* Since $\Phi(\frac{t}{\|v\|}, p, w) = \Phi(t, p, \frac{w}{\|v\|})$ is defined for $|t| < \|v\|$

Thus $\Psi'_{|v|} = \Phi(1, p, w)$

Thus the exponential map $E(w)$ is defined to be $\Psi(1) = \Phi(1, p, w)$

We now want to compute the derivative of $E(w)$ w.r.t $w$

Fixing a basis $x_u, x_v$ for $T_p S$ and writing $w = g_x x_u + h_x a x_v$

$E(w)$ is equal to $x(u(t, a), v(t, a_2))$ where

$u(t, a_1, a_2, a_3, a_4)$ and $v(t, a_1, a_2, a_3, a_4)$ are the solutions of the pair of ODEs

$u'^{\nu} + \sum_{\nu} \sum_{i,j} u_i u_j = 0 \quad (k = 1, 2)$

With initial conditions $u_1(0) = u_1(0) = a_1$

We shall now show that $E$ is a diffeomorphism of a neighborhood of $0$ on $S$ or on $p$ on $S$

Thus $E$ sets up a coordinate chart on $S$ which is

The spherical polar coordinates on the

A sphere is an example. The meridians are geodesics

and the parallels parameterise the family of

geodesics.
\[ \frac{\partial E}{\partial a_x} \times \frac{\partial F}{\partial a_y} = \left( \frac{\partial E}{\partial a_x} \times \frac{\partial F}{\partial a_y} \right) \left( \frac{\partial u}{\partial a_x} \frac{\partial v}{\partial a_y} - \frac{\partial u}{\partial a_y} \frac{\partial v}{\partial a_x} \right) \]

As it suffices to show that
\[ \frac{\partial u}{\partial a_y} \frac{\partial v}{\partial a_x} - \frac{\partial u}{\partial a_x} \frac{\partial v}{\partial a_y} \neq 0 \quad \text{(at time } t = 1) \]

will not be completed.

---

The Compatibility Conditions of Gauss and Codazzi-Mainardi: Theorem Egregium.

We recall from the Chapter on Covariant Differentiation, for a surface patch \( X: U \rightarrow S \subset \mathbb{R}^3 \):

\[ X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_u + \lambda_1 N \]
\[ X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{22}^2 X_u + \lambda_2 N \]
\[ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_u + \lambda_3 N \]

We now apply the conditions \( \frac{\partial X_{uv}}{\partial u} = \frac{\partial X_{vu}}{\partial v} \)

and \( \frac{\partial X_{uv}}{\partial v} = \frac{\partial X_{tu}}{\partial u} \) and derive from it a set of four equations known as the Gauss' equations. Among other things it would yield a formula for the Gaussian curvature in terms of the components of the metric tensor and their partial derivatives.

\[ \frac{\partial X_{uv}}{\partial v} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + \frac{\partial \Gamma_{11}^1}{\partial v} X_u + \frac{\partial \Gamma_{11}^2}{\partial v} X_v + \frac{\partial \lambda_1}{\partial v} N + \lambda_1 N_u \]

\[ = \Gamma_{11}^1 \left( \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + \lambda_2 N \right) + \]
\[ \Gamma_{11}^2 \left( \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + \lambda_3 N \right) + \frac{\partial \Gamma_{11}^1}{\partial v} X_u + \frac{\partial \Gamma_{11}^2}{\partial v} X_v + \frac{\partial \lambda_1}{\partial v} N + \lambda_1 N_u \]

Recall that \( N_u = \alpha X_u + \beta X_v \)
\[ N_v = \delta X_u + \varepsilon X_v \]

and \( \lambda_1 = X_{uu} \cdot N \)

Now, \( X_u \cdot N = 0 \Rightarrow X_{uu} \cdot N + X_u \cdot N_u = 0 \)

\[ \therefore \lambda_1 = -X_u \cdot N_u = -\alpha g_{11} - \beta g_{12} \]
\[ \frac{\partial x_{uv}}{\partial u} = \frac{\partial x_{uv}}{\partial u} + \frac{\partial x_{uv}}{\partial v} \]

Then, we have:

\[ x_{uv} = \frac{\partial x_{uv}}{\partial u} + \frac{\partial x_{uv}}{\partial v} \]

Again, \( x_{uv} \).  \( N \) gives

\[ x_{uv} \cdot N = \frac{\partial x_{uv}}{\partial v} \]

It is convenient to use the second expression

\[ x_{uv} \cdot N = -x_{uv} \cdot N = -x_{uv} \cdot (\alpha x_{uv} + \beta x_{uv}) \]

Thus, \( \alpha_{12} \) gives

\[ \frac{\partial x_{uv}}{\partial u} = \frac{\partial x_{uv}}{\partial v} \]

Comparing the coefficients of \( x_u \) and \( x_v \) gives the pair of equations:

\[ \Gamma_{v_{12}} + \Gamma_{v_{22}} = \Gamma_{v_{12}} + \Gamma_{v_{22}} \]

Using the above expressions, we get the pair of equations:

\[ \Gamma_{v_{12}} + \Gamma_{v_{22}} = \Gamma_{v_{12}} + \Gamma_{v_{22}} \]

(1)
Equations (1)-(4) are known as Gauss' eqns.