Dynamical Systems

Chapter IX : The Kepler Problem

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§9.1 Short historical Sketch

Theories of planetary motion may be traced back to remote antiquity. The earth was believed to be stationary around which all celestial bodies revolve. Some of the planets (the outer planets) occasionally exhibit retrograde motions.

Aristarchus seems to have maintained the heliocentric view but no definite written account survives. The ancient works culminated with Ptolemy's Almagest (~ 150 A.D) which, placing the earth at the centre provides a complex, but mathematically correct, description of the motion of celestial bodies as superpositions of circular motions in the form of epicycles. Indeed, it is through the Almagest that one learns of the significant contributions of Hipparchus (~ 135 BC) and Apollonius of Perga. (~ 230 BC)

Ptolemy's work has been successfully used for predicting the positions of the sun, moon and planets. The almagest superseded all earlier works (see Chapter 3 of A.V. Helden's book) and unquestionably remained the most ~~ath~~ authoritative account for about 13 centuries.

The Almagest makes extensive use of Euclid's elements, but nevertheless, the modern reader would find the intricate clockwork of epicycles a tough maze to Comprehend (see pp 33-34 of Heiden)

~~Tycho~~ Copernicus improved Ptolemy's theory by shifting the origin to the Sun but epicycles could not be eliminated completely.

Tycho Brahe refuted the Copernican theory but he was too good an astronomer to reject the simplifications in the Copernican theory. Tycho maintained that the five planets (Mercury, Venus, Mars, Jupiter & Saturn) revolve around the sun but this system in turns revolves around the earth.

A major leap forward was taken by Johannes Kepler through an unorthodox approach to the subject. After a prolonged struggle of over 25 years he arrived at his three famous laws governing the planetary motions:

(a) The planets revolve in elliptical orbits with the Sun at one focus.

(b) The radius vector joining the Sun and Earth sweeps out equal areas in equal intervals of time.

(c) The square of the period is proportional to the cube of the semi major axis.

The first two laws were published in 1609 in Astronomia Nova Physica Coelestiis, tradita Commentariis de Motibus

Stellae Martis. The last appears in his Harmonices Mundi, Libri V published in 1619 (?).

Interestingly Conic Sections have beeen studied ~~and~~ extensively by Apollonius of Perga who authored an eight-volume treatise on the subject. But it occured to neither Apollonius or his immediate contemporaries that these may ~~a~~ be applied to planetary theory. See Einstein's remarks in S. Chandrashekar's essay on Science and Scientific attitudes.

The works of Apollonius, Hipparchus and Ptolemy marks a turning point in the History of Astronomy in transforming vague doctorines and speculations to a science founded upon observations and mathematical principles. This empirical approach to Astronomy ends with Kepler. Another epoch begins with Newton's principia. Newton demonstrates that the Kepler's laws were consequences of his universal law of gravitation thereby giving a cause for the observed behaviour of planetary motion rather than ~~a~~ merely providing a mathematical description of planetary motion (as was done hitherto). Thus dynamical principles were introduced into Astronomy.

See Arnold's book [ ] for interesting details on Hooke's contributions, and S. Chandrashekar [ ]

Goals: The Kepler problem involves the study of the motion of two point masses moving under their mutual gravitational attraction.

Often, one of the bodies (the primary) is much more massive and may be regarded as being fixed and the force may be considered as being directed towards the primary.

The Kepler problem serves as a paradigm for completely integrable Hamiltonian system. The theory of completely integrable Hamiltonian systems implies the existence of a coordinate system (the action-angle variables) in which the motion is reduced to a coupled system of harmonic oscillators. The phase space foliates into "invariant tori". We shall derive these "ab-initio" without recourse to the general theory of Hamiltonian systems. The coordinate transformation through which the system may be recast as a system of linear oscillators was discovered by Levi Civita and Bohlin (see Arnold [ ] and Boccaletti et al [ ])

A thumbnail historical sketch can be found in the article by D. Saari. Besides D. Saari's work, the articles by Needham [ ] and Pourciau [ ] may be consulted.

§9.2 The Classical Integrals of Motion:

Consider a rectangular system of coordinates in which $\vec{r}_1$, $\vec{r}_2$ are the position vectors of the two point masses. By Newton's second law of motion

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \frac{G\, m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{\|\vec{r}_1 - \vec{r}_2\|^3} \qquad (9.1)_1\ \cancel{(9/2)_1}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \frac{G\, m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{\|\bar{r}_1 - \bar{r}_2\|^3} \qquad (9.1)_2\ \cancel{(9/2)_2}$$

Denoting by $\vec{v}_i$ the velocity vector $\frac{d\vec{r}_i}{dt}$ $(i=1,2)$ we may write (9.1) as a system of first order ODES

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 \qquad (9.2)_1$$

$$\frac{d\vec{r}_2}{dt} = \vec{v}_2 \qquad (9.2)_2$$

$$m_1 \frac{d\vec{v}_1}{dt} = \frac{G m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{\|\vec{r}_1 - \vec{r}_2\|^3} \qquad (9.2)_3$$

$$m_2 \frac{d\vec{v}_2}{dt} = \frac{-G m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{\|\vec{r}_1 - r_2\|^3} \qquad (9.2)_4.$$

On adding the last two equations, we get

$$\frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = 0$$

Thus $\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2$ give 3 first integrals of motion:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{p} \qquad (9.3)$$

Multiplying $(9.2)_1$ by m_1, ~~(9/2)~~ $(9.2)_2$ by m_2 and adding, $\frac{d}{dt}(m_1\vec{r}_1 + m_2\vec{r}_2) = \vec{p}$ (const)

Thus, $m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{p}t + \vec{c}$ (9.4)

and $m_1\bar{r}_1 + m_2\bar{r}_2 - \bar{p}t$ are three more (time Dependent) first integrals.

Theorem 9.1 The Conservation of linear momentum $\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2$ furnishes a set of three first integrals (9.3). The Conservation of $m_1\vec{r}_1 + m_2\vec{r}_2 - \vec{p}t$ furnishes three more. These are obviously functionally independent. (9.4) implies that the centre of mass $\vec{\mu}$ given by

$$\vec{\mu} = (m_1\vec{r}_1 + m_2\vec{r}_2) / (m_1 + m_2) \quad (9.5)$$

moves uniformly in a straight line.

Let us now choose a coordinate system whose origin is at the centre of mass $\vec{\mu}$. Thm 9.1 implies that this coordinate system is inertial with respect to the one originally chosen.

Define $\vec{R}_1 = \vec{r}_1 - \left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right) = \frac{-m_2}{m_1+m_2}(\bar{r}_2 - \bar{r}_1)$

$$\vec{R}_2 = \vec{r}_2 - \left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\right) = \frac{m_1}{m_1+m_2}(\vec{r}_2 - \vec{r}_1)$$

(9.6)

$$\frac{d\vec{R}_1}{dt} = \frac{d\vec{r}_1}{dt} - \vec{p} \; ; \quad \frac{d^2R_1}{dt^2} = \frac{d^2\vec{r}_1}{dt^2}$$

$$m_1 \frac{d^2\vec{R}_1}{dt^2} = \frac{-G m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{\|\vec{r}_1 - \vec{r}_2\|^3}$$

$$\therefore \frac{d^2\vec{R}_1}{dt^2} = -G(m_1+m_2)\vec{R}_1 \Big/ \left\|\left(\frac{m_1+m_2}{m_2}\right)\vec{R}_1\right\|^3$$

$$\therefore \frac{d^2\vec{R}_1}{dt^2} = -\frac{G m_2^3}{(m_1+m_2)^2} \frac{\vec{R}_1}{\|\vec{R}_1\|^3} \qquad (9.7)_1$$

Likewise $\frac{d^2\vec{R}_2}{dt^2} = -\frac{G m_1^3}{(m_1+m_2)^2} \cdot \frac{\vec{R}_2}{\|\vec{R}_2\|^3} \qquad (9.7)_2.$

Note that if $\vec{R}_1(t)$ solves $(9.7)_1$,

put $\gamma_1 = G m_1^3/(m_1+m_2)^2$; $\gamma_2 = G m_2^3/(m_1+m_2)^3$

$\ddot{R}_1 = \gamma_2 R_1/\|R_1\|^3$. Replace t by t/λ^a

$\ddot{R}_1(t/\lambda^a) = \gamma_2 \lambda^{-2a} R_1/\|R_1\|^3$. Mult by λ

and we get $\ddot{R}_2 = \gamma_2 \lambda^{1-2a} R_1(t/\lambda^a)/\|R_1(t/\lambda^a)\|^3$

where $R_2 = \lambda R_1(t/\lambda^a)$

a to be determined.

$\therefore \ddot{R}_2 = \gamma_2 \lambda^{1-2a} \lambda^{-1} R_2(t)/\|\lambda^{-1} R_2(t)\|^3$

$\therefore \ddot{R}_2 = \gamma_2 \lambda^{3-2a} R_2/\|R_2\|^3$

We now choose λ, a such that

$\gamma_2 \lambda^{3-2a} = \gamma_1$

i.e. $\lambda^{3-2a} = \gamma_1/\gamma_2 = (m_1/m_2)^3$

We may choose $\lambda = m_1/m_2$ and $a = 0$.

Thm 9.2: (i) if $\vec{R}_1(t)$ is a solution of (9.7), then $\vec{R}_2(t) = (m_1/m_2)\vec{R}_1(t)$ solves $(9.7)_2$. That is to say $\vec{R}_1(t)$ and $\vec{R}_2(t)$ trace out homothetic

(Similar) Curves in the moving coordinate System.

We shall see later that the paths traced out are homothetic ellipses.

(ii) The System of differential equations for the two body problem (9.2) is equivalent to the decoupled System (9.7) of One body problems in which the motion is due to a central force directed towards the Centre of mass $\vec{\mu}$.

(iii) Either of the two bodies describes a Conic about the Centre of mass $\vec{\mu}$ as a focus.

<u>Remark:</u> In view of Theorem 9.2 it suffices to study the System of ODES of the form

$$\frac{d^2\vec{R}}{dt^2} = -\gamma\, f(\vec{R})\,\vec{R} \qquad (9.8)$$

where $f(\vec{R})$ is a Scalar valued function depending only on $\|\vec{R}\|$.

The Study of (9.8) will be postponed. We continue to derive additional first integrals for the System (9.2)

We work with (9.7) which is equivalent to (9.2) via the transformation (9.6)

$$\frac{d}{dt}\left(\vec{R_1} \times \frac{d\vec{R_1}}{dt}\right) = \vec{R_1} \times \frac{d^2\vec{R_1}}{dt^2}$$

$$= \vec{R_1} \times \left(-\frac{G\, m_2^3}{(m_1+m_2)^2}\right)\frac{\vec{R_1}}{\|R_1\|^3} = 0$$

so $\vec{R_1} \times \frac{d\vec{R_1}}{dt}$ ~~(sgnd)~~ and $\vec{R_2} \times \frac{dR_2}{dt}$

are six additional first integrals.

However, thanks to the homothetic relation between R_2 and R_1 the latter three are proportional to the former. We therefore have 3 additional first integrals.

$$\vec{L} = \vec{R_1} \times \dot{R_1} \qquad (9.9)$$

Theorem 9.3: The conservation of angular momentum provides three additional first integrals.

$$\text{Note that } \nabla(\|\vec{R}\|^{-1}) = -\frac{\vec{R}}{\|\vec{R}\|^3} \qquad (9.10)$$

$$\text{and so } \frac{\vec{R}}{\|\vec{R}\|^3} \cdot \frac{d\vec{R}}{dt} = \frac{d}{dt}\left(-\|\vec{R}(t)\|^{-1}\right) \qquad (9.11)$$

Multiply (9.7), scalarly by $\frac{d\vec{R_1}}{dt}$ and we get

$$\frac{d}{dt}\left(\frac{1}{2}\, \dot{R_1}\cdot\dot{R_1}\right) = \frac{G m_2^3}{(m_1+m_2)^2}\frac{d}{dt}\left(\|\vec{R_1}(t)\|^{-1}\right)$$

$$\therefore \frac{d}{dt}\left\{\frac{1}{2}\dot{R_1}\cdot\dot{R_1} - \frac{G m_2^3}{(m_1+m_2)^2}\|\vec{R_1}(t)\|^{-1}\right\} = 0$$

We therefore one more first integral:

Theorem 9.4: The conservation of energy provides yet another first integral

$$E := \frac{1}{2}\left\|\frac{d}{dt}\vec{R}_1\right\|^2 - \frac{G m_2^3}{(m_1+m_2)^2}\frac{1}{\|\vec{R}_1(t)\|} \qquad (9.12)$$

The 10 first integrals

Three linear momentum integrals

Three integrals of centre of mass

Three angular momentum integrals

and One energy integral

are known as the classical integrals.

For the two body problem, the phase space is 12 dimensional.

The vector field (9.2) has zero divergence and so the flow is volume preserving. This gives the constant as an integral invariant. The Jacobi-last integral theorem provides an additional first integral. We shall derive this ab-initio without invoking the general principles laid down in Chapter V.

§9.3: Derivation of Kepler's Laws of Planetary Motion

We begin with equation (9.8). Denoting R by either $\vec{R_1}$ or $\vec{R_2}$

$$\frac{d}{dt}(R \times \dot{R}) = 0 \qquad (9.13)$$

$$\therefore \quad R \times \dot{R} = \vec{c} \qquad (9.14)$$

Case(i): $\vec{c} = 0$. In this case $\dot{R} = \lambda R$ for some scalar valued function.

$$\dot{R} e^{\Lambda(t)} - \lambda R e^{\Lambda(t)} = 0 \qquad (9.15)$$

$$\therefore \frac{d}{dt}\left(R e^{\Lambda(t)}\right) = 0 \quad \text{provided} \quad \dot{\Lambda} = -\lambda$$

We deduce that $R(t) = R(0) e^{-\Lambda(t)}$ and the motion takes place along a straight line (in the moving coord system given by (9.6). But this comment will not be made explicit hereafter. All of the analysis in this section refers to the moving coord).

Exercise: Suppose that a particle starts from rest and its motion is governed by (9.8) where $\gamma > 0$ and $f(r)$ is of the form $g(r)/r$ where $g > 0$. Prove that the particle collides with the centre of force in finite time.

Thus the flow is not complete. That is not all solutions are defined on $\mathbb{R}$.

Substituting $\dot{R} = \lambda R$ into the ODE (9.8)

$$\dot{\lambda} + \lambda^2 = -\frac{g(R)}{R} \qquad (9.15)$$

$\dot{\lambda} + \lambda^2 < 0$ so λ is monotone decreasing

$\lambda(0) = 0$ so λ is negative

$\therefore \frac{d\lambda}{\lambda^2} < -dt$ integrating from ϵ to t

(denoting $\lambda(\epsilon)$ by λ_ϵ)

$$\frac{1}{\lambda_\epsilon} - \frac{1}{\lambda} < -(t-\epsilon)$$

Note that $\lambda\lambda_\epsilon > 0$

$$\therefore \lambda - \lambda_\epsilon < -\lambda\lambda_\epsilon(t-\epsilon)$$

$$\lambda(1 + \lambda_\epsilon(t-\epsilon)) < \lambda_\epsilon$$

$$\therefore \lambda < \lambda_\epsilon / (1 + \lambda_\epsilon(t-\epsilon)) \quad \text{for } t \geq \epsilon$$

$$~~\exp~~ \int_\epsilon^t \lambda(s)ds < \log(1 + \lambda_\epsilon(t-\epsilon))$$

$$\therefore \exp 2\int_\epsilon^t \lambda(s)ds < (1 + \lambda_\epsilon(t-\epsilon))^2$$

~~From (9.14)~~, Now, $\dot{R} = \lambda R$ gives on $R(t)$ mult scalarly by R

$$\frac{d}{dt} R^2 = 2R^2\lambda$$

$$\therefore R^2(t) = R^2(\epsilon) \exp \int_\epsilon^t 2\lambda(s)ds$$

$$\therefore (R(t)\cdot R(t)) \leq \|R(\epsilon)\|^2 (1 + \lambda_\epsilon(t-\epsilon))^2$$

as time evolves, $(1 + \lambda_\epsilon(t-\epsilon)) \to 0$

(note that $\lambda_0 = 0$ so $1 + \lambda_\epsilon(t-\epsilon) = 1$ to begin with and $\lambda_\epsilon < 0$).

$\therefore R(t) \to 0$ in finite time.

(See H. Pollard p 3.)

Assume now $\vec{c} \neq 0$, equation (9.14) shows that

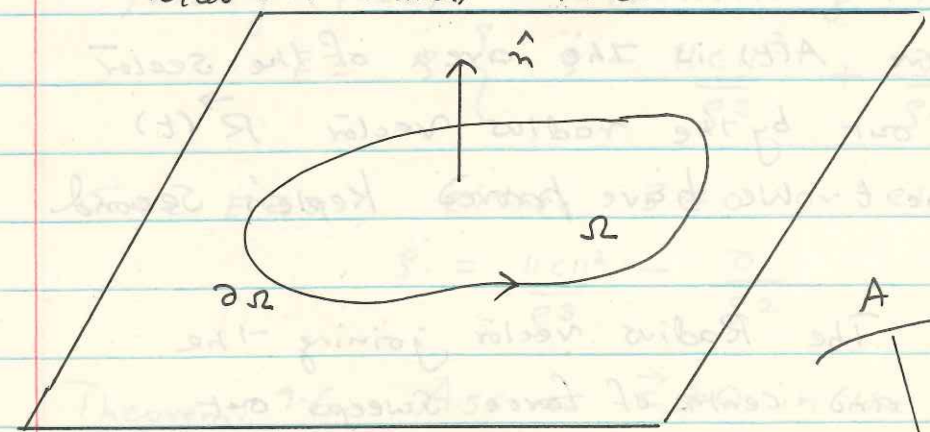
$$R(t)\cdot\vec{c} \equiv 0 \qquad (9.17)$$

Thus, the motion takes place in the plane given by the equation $\vec{x}\cdot\vec{c} = 0$

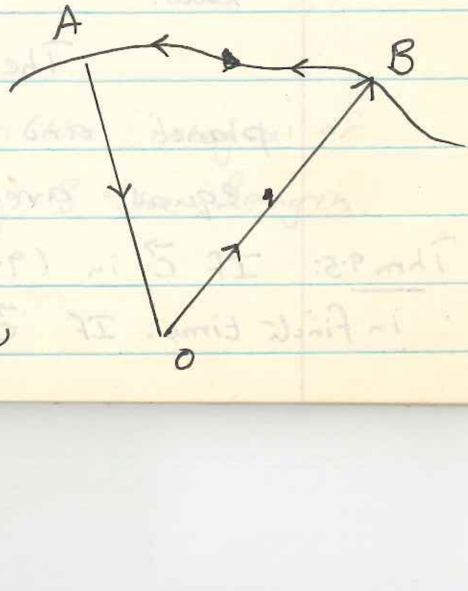
Let Ω be a region in this plane bounded by the smooth curve $\partial\Omega$. On applying Stokes' theorem to the vector field $\vec{c}\times\vec{r}$ we get

$$\oint_{\partial\Omega} (\vec{c}\times\vec{r})\cdot d\vec{r} = 2\vec{c}\cdot\hat{n}\ \text{area}(\Omega)$$
$$= 2\|\vec{c}\|\ \text{area}(\Omega)$$

Where $\hat{n} = \vec{c}/\|\vec{c}\|$ and $\partial\Omega$ is oriented consistently with $\hat{n}$ in accordance with the right hand rule. Remark: An analogue of this holds for conical surfaces as well.



$$\hat{n} = \frac{\vec{c}}{\|\vec{c}\|}$$



Consider Ω in the form of a sector with vertex at the origin and arc AB to be a part of the traj $\vec{R}(t)$

The arrows have been depicted so that at B,

$$R \times \dot{R} = \vec{c} \text{ points above the plane}$$

along the segments OA and OB,

$$R \times \dot{R} = 0$$

Along arc BA, $(\vec{c} \times \vec{R}) \cdot d\vec{R}$

$$= (\vec{c} \times \vec{R}) \cdot \frac{d\vec{R}}{dt}$$

$$= (\vec{c} \times \vec{R}) \cdot \dot{R} = \vec{c} \cdot (\vec{R} \times \dot{R}) = \|\vec{c}\|^2$$

$\therefore \int_{\partial\Omega} (\vec{c} \times \vec{r}) \cdot d\vec{r} = \|c\|^2 \Delta t$ where Δt is the time of transit from B to A.

and area (sector) $= \frac{1}{2} \|\vec{c}\| \Delta t$

$$\therefore \frac{dA}{dt} = \frac{1}{2} \|\vec{c}\| \qquad (9.18)$$

where $A(t)$ is the area of the sector swept out by the radius vector $\vec{R}(t)$ in time t. We have proved Kepler's second law:

The Radius vector joining the planet and centre of force sweeps out equal areas in equal intervals of time.

Thm 9.5: If $\vec{c}$ in (9.14) the particle moves in a straight line and reaches the singularity (origi in finite time. If $\vec{c} \neq 0$ the motion takes place in the plane

$\vec{X} \cdot \vec{c} = 0$ and if $A(t)$ is the area swept out by the vector $R(t)$ ~~then~~ from 0 to t, $\dot{A} = \frac{1}{2} \|\vec{c}\|$. Thus areal velocity is const. Radius vector sweeps out equal areas in equal intervals of time.

To derive Kepler's first law we need the analysis of energy integral:

$$E = \frac{1}{2} \|\dot{R}_1\|^2 - \gamma \frac{1}{\|R_1\|} \qquad (9.19)$$

where $\gamma = +G\, m_2^3 / (m_1 + m_2)^2$; m_2 = Mass of Sun

let us denote $\|R_1\|$ by ρ

Exercise: $\|\dot{R}_1\|^2 = \dot{\rho}^2 + \rho^{-2} \|c\|^2$ and (9.20)

$$E = \frac{1}{2}(\dot{\rho}^2 + \rho^{-2}\|c\|^2) - \gamma \rho^{-1} \qquad (9.21)$$

$$\dot{E} = \dot{\rho} \left\{ \ddot{\rho} - \frac{\|c\|^2}{\rho^3} \right\} + \frac{\dot{\rho}\gamma}{\rho^2} \qquad (9.22)$$

Sol: $\rho = \|R_1\|$; $\dot{\rho} = (R_1 \cdot \dot{R}_1) / \|R_1\| = \rho^{-1}(R_1 \cdot \dot{R}_1)$

$$\therefore \dot{\rho}^2 = \rho^{-2} (\|R_1\|^2 \|\dot{R}_1\|^2 - (R_1 \times \dot{R}_1)^2)$$

$$\dot{\rho}^2 = \rho^{-2} (\rho^2 \|\dot{R}_1\|^2 - \|c\|^2)$$

$$\therefore \dot{\rho}^2 = \|\dot{R}_1\|^2 - \|c\|^2 \rho^{-2}$$

proving (9.20). Differentiating (9.21) we get

$$\dot{E} = \dot{\rho} \left\{ \ddot{\rho} - \frac{\|c\|^2}{\rho^3} + \frac{\gamma}{\rho^2} \right\}$$

Since $\dot{E} \equiv 0$, either $\dot{\rho} \equiv 0$ or else

$$\ddot{\rho} = \frac{\|c\|^2}{\rho^3} - \frac{\gamma}{\rho^2}$$

<u>Theorem 9.6:</u> Assume $\vec{c} \neq 0$ in the integral of angular momentum. Denoting by ρ the length $\|R_1(t)\|$,

The motion is either a Circle $(\dot{\rho} \equiv 0)$

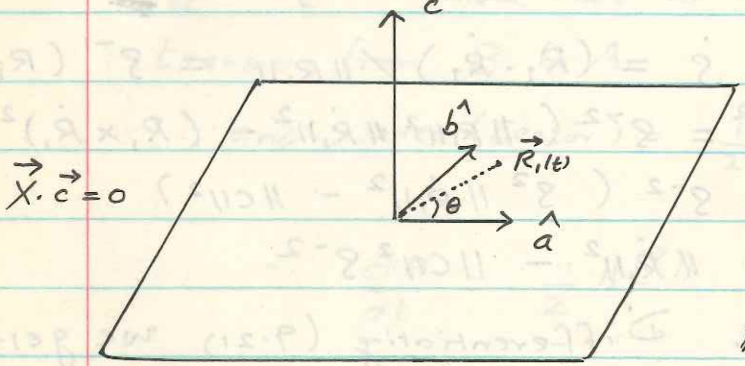
Or ρ evolves according to

$$\ddot{\rho} = \frac{\|c\|^2}{\rho^3} - \frac{\gamma}{\rho^2} \qquad (9.23)$$

we prove now:

The Substitution $\rho = \frac{1}{u}$ transforms (9.23) to the forced "harmonic oscillator" with respect the angular variation.

Let $\hat{a}$ be a unit vector in the plane $X \cdot \vec{c} = 0$ and $\hat{b} = \frac{\vec{c}}{\|\vec{c}\|} \times \hat{a}$



Let θ be the angle between $\vec{R}_1(t)$ and $\hat{a}$ so that

$\cancel{\vec{R}_1(t) = (\hat{a}\cos\theta + \hat{b}\sin\theta)\|R_1\|}$

$$\vec{R}_1(t) = \rho(\hat{a}\cos\theta + \hat{b}\sin\theta)$$

Equation (9.14) implies

$$\vec{c} = (\hat{a} \times \hat{b})\, \rho^2 \dot{\theta}$$

$$\therefore \quad \|\vec{c}\| = \rho^2 \frac{d\theta}{dt} \qquad (9.24)$$

Thm 9.7: The Conservation of angular momentum implies $\rho^2 \frac{d\theta}{dt} = \|\vec{c}\|$ provided $\vec{c} \neq 0$

Exercise: Prove that the following are equivalent, where $R(t)$ is the position vector of a moving point.

(i) $\vec{R}(t)$ is parallel to the acceleration vector

(ii) $\vec{R}(t) \times v(t)$ is const ; $v(t) =$ velocity vector

(iii) $\vec{R}(t)$ is planar and rate of change of sectorial area swept by $R(t)$ is constant.

(B. Pourciau)

The Binet's Equation:

We now transform the ODE (9.23) to a forced harmonic oscillator using (9.24)

Theorem 9.8: The substitution $u = 1/\rho$ transforms (9.23) to the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma}{\|C\|^2} \qquad (9.25)$$

pf: $\dot{u} = -\rho^{-2}\dot{\rho}$

$$\therefore \frac{du}{d\theta} = \frac{du}{dt} \Big/ \frac{d\theta}{dt} = \frac{-\rho^{-2}\dot{\rho}}{\|C\|\rho^{-2}}$$

$$\therefore \frac{d^2u}{d\theta^2} = -\frac{1}{\|C\|}\frac{d}{d\theta}(\dot{\rho}) = \frac{-1}{\|C\|}\frac{(\ddot{\rho})}{\dot{\theta}}$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{-\rho^{-2}\ddot{\rho}}{\|C\|^2}$$ Now using (9.23) we get

$$\frac{d^2u}{d\theta^2} = \frac{\gamma}{\|C\|^2} - \frac{1}{\rho} \qquad \therefore \frac{d^2u}{d\theta^2} + u = \frac{\gamma}{\|C\|^2}$$

Equation (9.25) is called Binet's Eqn in [ ]

Exercise: Use Binet's equation to show that circular motion occurs if and only if

$$E\,\|C\|^2 + \frac{\gamma^2}{2} = 0 \qquad (9.26)$$

$\dot{\rho} \equiv 0 \Rightarrow$ second factor vanishes

(Warning: Prove that the two cases in (9.22) are not mutually exclusive

Solution: $u = \gamma/\|C\|^2$, the constant solution corresponds to circular motion. This gives

$$\rho = \|C\|^2/\gamma, \quad \dot{\rho} = 0.$$

Using the energy equation (9.21) we get

$$E = \frac{\|C\|^2}{2\rho^2} - \frac{\gamma}{\rho} = \frac{\gamma^2\|C\|^2}{2\|C\|^4} - \frac{\gamma}{\|C\|^2/\gamma}$$

$$\therefore E = -\frac{\gamma^2}{2\|C\|^2} \quad \text{which is } (9.26)$$

Conversely, if $E = -\gamma^2/2\|C\|^2$, we get from the energy integral,

$$\frac{\dot{\rho}^2}{2} + \frac{1}{2}\left(\frac{\|C\|^2}{\rho^2} - \frac{2\gamma}{\rho} + \frac{\gamma^2}{\|C\|^2}\right) = 0$$

which forces $\dot{\rho} \equiv 0$ and $\rho \equiv \|C\|^2/\gamma$

Showing that the motion is circular.

Note: One must also prove that $\dot{\rho} = 0 \Rightarrow \ddot{\rho} = \frac{\|C\|^2}{\rho^3} - \gamma/\rho^2$

Exercise: Polar equation of the Conic:

We consider a polar coordinate system with the origin at the pole and initial line $\perp$ to the directrix.

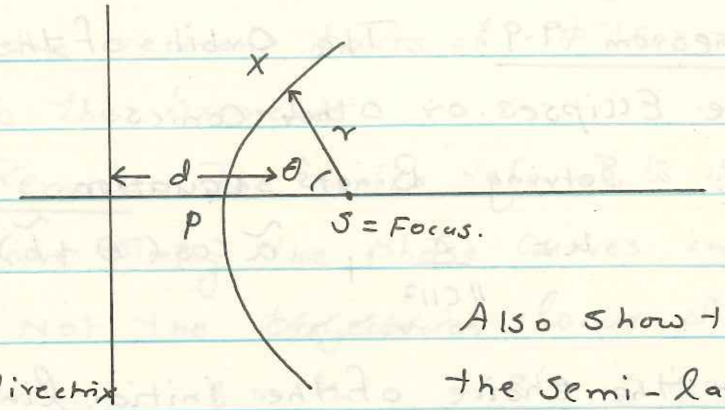
P is a point closest to the directrix and X is a general point on the conic

(i) The polar equation of the conic is

$$r = (de) \,/\, (1 + e\cos\theta) \qquad (9.27)$$

$r =$ distance (XS), $\theta = \measuredangle PSX$; $e =$ eccentricity

$d =$ dist of S from directrix, $S =$ focus.



Also show that de equals the semi-latus rectum.

(ii) Vector equation of a Conic: The unit vector along the ray emenating from $S \perp$ directrix and of length e, is called the eccentric vector and denoted by $\vec{e}$.

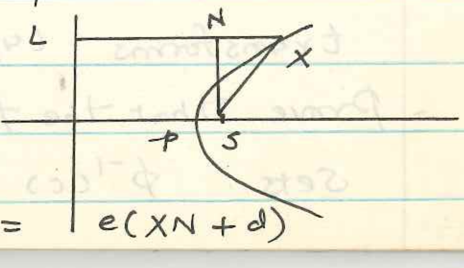
Show that equation (9.27) can be rewritten as

$$\vec{r} \cdot \left(\frac{\vec{r}}{\|\vec{r}\|} + \vec{e} \right) = l = \text{semi-latus rectum.} \qquad (9.28)$$

(Vector equation of the Conic)

<u>proof</u>: (ii) follows from (9.27) for,

$r + re\cos\theta = \vec{r} \cdot \left(\frac{\vec{r}}{\|\vec{r}\|} + \vec{e} \right)$ and de is the semi latus rectum as can be seen by applying definition to the end point of the semi-latus rectum.



Draw $XL \perp$ directrix

$SN \perp XL$; $\quad SX = e\,XL$

$= e(XN + NL) = e(XN + d)$

but $XN = -XS\cos\theta = -r\cos\theta$. So

$$r = e(d - r\cos\theta)$$

$$\therefore r(1 + e\cos\theta) = de \quad \text{which is } (9.27)$$

Theorem 9.9: The Orbit of the planets are Ellipses or other Conics.

pf: Solving Binet's equation

$$u = \frac{\gamma}{\|c\|^2} + \tilde{a}\cos(\theta + \tilde{b})$$

but the choice of the initial line on the plane $X \cdot \vec{c} = 0$ is as yet arbitrary. Rotating the initial line, we may assume $\tilde{b} = 0$ and we get ($\because \rho = u^{-1}$)

$$\rho = \frac{\|c\|^2/\gamma}{1 + a_0\cos\theta} \qquad (9.29)$$

Which is a conic with eccentricity $a_0 = e$ (in futu
and semi latus rectum $\|c\|^2/\gamma$.

Exercise: (i) Write Binet's equation as a first-order system for (u, v). Determine a first-integral ϕ and sketch the level curves $\phi^{-1}(c)$

The relation $u = 1/\rho$ and $v = u'$ transforms (u, v) to $(\rho, \rho') = (U, V)$ say.

Prove that the transformation maps the level sets $\phi^{-1}(c)$ into Conics.

Show that the form of the conic changes when c passes through a critical value. Study this change, identify the conics for various values of c. In the case of a hyperbola describe the relevant parts of $\phi^{-1}(c)$ that get mapped to the branches.

Remark: The conics referred to in this exercise are simply the phase curves in the (ς, ς') plane. - Not the ~~trajectories~~ locus of $\vec{R}(t)$

Sol: $v = u'$; $\varsigma = 1/u$ $\therefore \varsigma' = -\frac{1}{u^2}u' = -v/u^2$

$$\therefore (\varsigma, \varsigma') = \left(\tfrac{1}{u}, -v/u^2\right) = (U, V)$$

the level sets $\phi^{-1}(c)$ are circles

$$(u-a)^2 + v^2 = b^2; \quad a = \gamma/\|c\|^2$$

$$\therefore (1-aU)^2 + V^2 = b^2U^2$$

$$\therefore \left(U - a/(a^2-b^2)\right)^2 + V^2/(a^2-b^2) = b^2/(a^2-b^2)^2$$

When $b = a$, the value of ϕ is critical

for $b < a$, the locus in (U,V) plane is an ellipse

for $b = a$, the locus is a parabola

note that points $u = 0$ are singularities of the transformation $(u,v) \mapsto (U,V)$

for $b > a$ the circles cut the axis $u = 0$ at two points. The two arcs are mapped to the two branches of a hyperbola.

Exercise: Use the energy integral (9.21) and theorem 9.7 to obtain directly an ODE for $\left(\frac{d\rho}{d\theta}\right)^2$ and use this to show that for $\vec{c} \neq 0$
the orbit is an ellipse if the energy $E<0$ and a hyperbola if $E>0$.

(Chicone, p 214, Ex 3.9)

Exercise: Show that the integral of energy has only one critical point and the critical value is a global minimum. This corresponds to circular orbits.

Exercise: Substituting (9.29) into the energy equation, determine the eccentricity in terms of the Energy E, square of the angular momentum $\|\bar{c}\|^2$ and the constants G, m_1, m_2. Deduce that a depends only on E*.

Ans: $$e^2 = 1 + \frac{2E}{\gamma^2}\|c\|^2 \qquad (9.30)$$

(See H. Pollard, p 8) (and so $a = -\gamma/2E$)

which is in perfect agreement with our earlier result namely $e=0$ iff $E = -\frac{\gamma^2}{2\|c\|^2}$

Warning: to get rid of $\dot{\theta}$, use Thm 9.7

So: $$\dot{\rho} = \frac{(-a_0 \sin\theta)\, l\, \dot{\theta}}{(1+a_0\cos\theta)^2} = \frac{l(-a_0\sin\theta)}{(1+a_0\cos\theta)^2}\frac{\|c\|}{\rho^2}$$

$$= -a_0 \sin\theta\, \|c\|/l = -a_0\sin\theta\, \frac{\gamma}{\|c\|}$$

*: This is important in connection with regularization [ ].

$$\dot{\rho}^2 + \|c\|^2/\rho^2 = \gamma^2/\|c\|^2 \left(2a_0\cos\theta + a_0^2+1\right)$$
$$-2\gamma/\rho = -2\gamma^2/\|c\|^2 \left(1+a_0\cos\theta\right)$$
$$\therefore\ 2E = \left(\gamma^2/\|c\|^2\right)\left(a_0^2-1\right)$$

So using the notation $a_0 = e$ we get

$$e^2 = 1 + 2E\|\bar{c}\|^2/\gamma.$$

Ans: $2\tan^{-1}\dfrac{\|c\|\sqrt{2E}}{GM}$.

Ex: For a hyperbolic traj, determine ∡ between asymptotes:

Theorem 9-10: (Kepler's 3rd law)

if T is the period of the elliptical orbit and a the semi-major axis

T^2 is "proportional" to a^3.

Specifically $\quad 4\pi^2 a^3 = \gamma T^2 \qquad (9.31)$

proof: By the law of areas $\dfrac{dA}{dt} = \dfrac{\|c\|}{2}$

integrating over a complete period,

$$\text{Area (Ellipse)} = \|c\|\,T/2$$
$$\therefore\ 4\pi^2a^2b^2 = \|c\|^2T^2$$

But $\|c\|^2/\gamma$ = semi-latus rectum $= b^2/a$

$\therefore\ 4\pi^2a^3 = T^2\gamma$. The proof is complete.

Exercise: If V is the orbital speed, prove $V^2 = \gamma\left\{\frac{2}{\rho} - \frac{1}{a}\right\}$.*

Exercise: (J.M.A. Danby, Chapter 6, p136, problem 6)

The perihelion distance of a parabolic comet is q (measured in Astronomical Units) is less than one.

Assuming the earth's orbit to be circular and that the comet moves in the ecliptic

Show that the time t (measured in sidereal years)

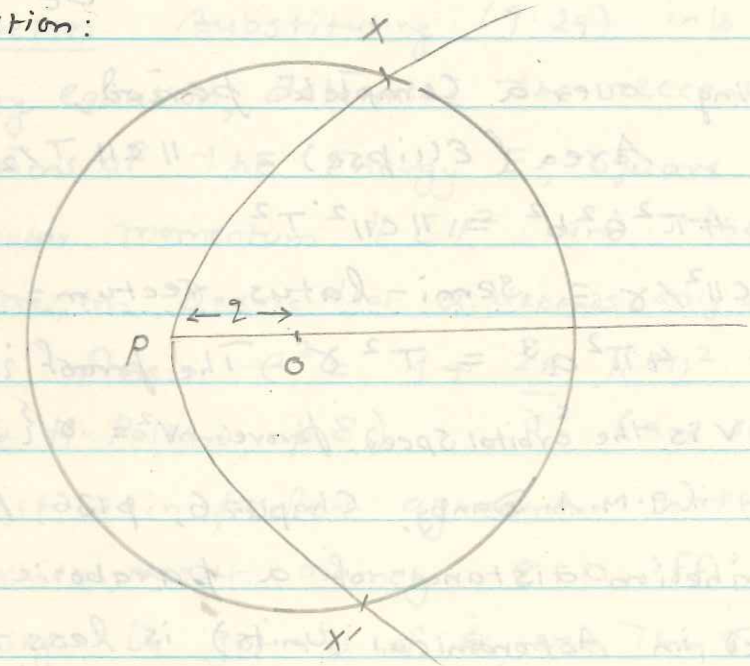
* For circular orbits $a=\rho$; For comets $a \to +\infty$ and so $1/a$ can be neglected.

is given of the

spent by the comet within the earth's orbit is $t = \frac{1}{3\pi}(1+2q)\sqrt{2-2q}$

Hint: Energy $E = 0$ giving an ODE for $\frac{d\rho}{dt}$
$t = 2\int_{\rho=q}^{\rho=1} dt$. Finally, to determine the constants use the relations
$\gamma = GM$; (M = Mass of Sun)
$\|C\|^2/\gamma$ = semi-latus rectum (which must be computed in terms of q) and γ to be computed using Kepler's third law using the earth's orbit.

Solution:



Let X, X' be the points where the orbit of the comet meets the earth's orbit and P the perihelion.

The semi latus rectum of the parabola $= 2q$

$$\therefore 2q = \|C\|^2 / \gamma$$

$$\therefore \|C\|^2 = 2q\gamma$$

$E = 0$ gives (using (9.21)) the ODE

$$\dot{\rho}^2 + \frac{\|C\|^2}{\rho^2} = \frac{2\gamma}{\rho}$$

$$\therefore \dot{\rho}^2 = 2\gamma(\rho - q)/\rho^2$$

at the points X and P, values of ρ are 1 and q

$$\therefore t = 2\int_{\rho=q}^{\rho=1} dt = \frac{2}{\sqrt{2\gamma}} \int_q^1 \frac{\rho\, d\rho}{\sqrt{\rho - q}}$$

$$\therefore \sqrt{\gamma}\, t = \frac{2}{3}(1+2q)\sqrt{2-2q}$$

Using Kepler's 3rd law equation (9.31) with $T = 1$ sidereal year and $a = 1$ AU

$$\sqrt{\gamma} = 2\pi$$

$$\therefore t = \frac{1}{3\pi}(1+2q)\sqrt{2-2q}.$$

Remarks: (i) Kepler's 3rd law also makes sense for ~~even~~ parabolic and hyperbolic orbits if interpreted correctly. See pp 20-21 and p24 of the book by F.T. Geyling and H.R. Westerman

(ii) Although we have proved that the orbits of planets are ellipses, this is with respect to an inertial coordinate system moving uniformly in space

with its centre of gravity at the origin.
So in absolute space the trajectories are actually helices.
See interesting remarks on the applications to Binary Stars in Danby [ ] p 126.

(iii) Note that Kepler's Laws as originally stated are only approximate. The centre of mass is at the focus and not the sun. However, for the sun-planet system, the CM is close to the sun's centre compared to the sun-planet distance and so to a good approximation it is true to that the sun is at the focus. See Newton's comments on pp 377-378 of [ ]
(iv) The statement T^2 is "proportional" to a^3 in Kepler's 3rd law (thm 9.10) is not quite precise since the constant of "proportionality" depends on m_1. However, if $m_1 << m_2$ as is the case for planets, $\frac{m_2}{m_1+m_2} \sim 1$ and γ is almost independent of m_1.
In this sense, the proportionality factor is the same for all planets w.r.t sun

Additional Exercises:

(1) Show that the greatest value of $\frac{d\varrho}{dt}$ in an elliptic orbit occurs at the ends of the latus rectum. Find this value for a planet whose orbit has semi-major axis a and eccentricity e. (Danby, p14) #8)

(2) Compute the time averages: ~~of~~

(i) $\langle a^3/\varrho^3 \rangle$ (ii) $\langle a^4/\varrho^4 \rangle$ (iii) $\langle a^5/\varrho^5 \rangle$

where $\langle f \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(s)\,ds$

Ans: (i) $(1-e^2)^{-3/2}$ (ii) $(1+\frac{e^2}{2})(1-e^2)^{-5/2}$

(iii) $(1+\frac{3}{2}e^2)(1-e^2)^{-7/2}$

Sol: More generally we prove

$$\int_0^T \frac{a^j}{\varrho^j}\,dt = T(1-e^2)^{3/2-j} \sum_{k=0}^{j-2} \binom{j-2}{k} \frac{e^k}{2^k} \binom{k}{k/2} \epsilon_k$$

$$\epsilon_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

T is a complete period. The result follows at once

$$\int_0^T \frac{a^j}{\varrho^j}\,dt = \int_0^{2\pi} \frac{a^j}{\varrho^j} \frac{1}{\dot\theta}\,d\theta = \int_0^{2\pi} \frac{a^j}{\|c\|\,\varrho^{j-2}}\,d\theta$$

$$= \int_0^{2\pi} \frac{a^j \gamma^{j-2}}{\|c\|^{2j-3}} (1+e\cos\theta)^{j-2}\,d\theta$$

$$= \frac{2\pi\, a^j \gamma^{j-2}}{\|c\|^{2j-3}} \sum_{k=0}^{j-2} \binom{j-2}{k} \frac{e^k}{2^k} \binom{k}{k/2} \epsilon_k$$

Now, using Kepler's 3rd law $\frac{2\pi a^2}{\|c\|} = T(1-e^2)^{-1/2}$

and $\left(\gamma / \|c\|^2 \right)^{j-2} = (1-e^2)^{2-j}$.

(3) A planet describes a circular orbit but receives a small impulse in the direction of motion. If $\tilde{R}(t)$ is the new position and $R(t)$ the position vector had there been no impact prove that the loci of $R(t)$ and $\tilde{R}(t)$ remain close (as sets) although $\| R(t) - \tilde{R}(t) \|$ may become large.

Thus, we have Orbital stability but instability in time ("Secular instability")

Solution: Assume that $\dot{R}(t_0)$ is perturbed to $\dot{R}(t_0)(1+\delta)$ and $\dot{R}(t_0) \perp R(t_0)$ since the motion is circular.

The Angular momentum vector $\vec{c}$ changes to $\vec{c}(1+\delta)$ (without a change in direction)

The semi-latus rectum of the perturbed orbit (which need not be a circle) changes to $\|c\|^2 (1+\delta)^2 / \gamma$.

Equation (9.30) shows that the eccentricity changes by a small amount and so in any case the perturbed orbit is an ellipse with axes B, A say

$$B^2/A = \|c\|^2 (1+\delta)/\gamma > \|c\|^2/\gamma = a$$

$\therefore A = \frac{A^2}{A} \geqslant \frac{B^2}{A} > a$. So the Major axis has increased by an amount proportional to δ. By Kepler's 3rd law the period T is perturbed by an amount proportional to δ.

Let T_2 be the perturbed period and T_1 the original period $\vec{R_2}(t)$ the perturbed position vector, $\vec{R_1}(t)$ the position vector in the absence of perturbation.

At time $t = t_1$ the two vectors $\bar{R_i}(t)$, $i = 1, 2$ are in conjunction. After time ~~$t_1 + k(T_2 - T_1)$~~ $t_1 + kT_1$

$\vec{R_1}(t) = \vec{R}(t_1)$ where as $\vec{R_2}(t)$ lags behind

~~Choose k large enough so that $t_1 + k(T_2 - T_1)$ is close to T_2~~

~~which is possible since $T_2 - T_1$ is small~~

~~and then $\vec{R_2}(t_1 + k(T_2 - T_1))$ is close to~~

~~opposition whereas $\vec{R_1}$~~

Choose k so large that $k(T_1 - T_2) \sim -T_2/2$

Which is possible since $|T_1 - T_2|$ is small

and $\vec{R_2}(t_1 + kT_1)$

$$= \vec{R_2}(t_1 + kT_2 + k(T_1 - T_2))$$

$$= \vec{R_2}(k(T_1 - T_2)) \sim \vec{R_2}(-T_2/2)$$

So $R_2(t_1 + kT_1)$ and $R_1(t_1 + kT_1)$ are in Opposition.

4. Orbit of Ganymede is almost circular with mean distance 0.007156 AU. Orbital period of Ganymede is 7.155 days. Determine the ratio of masses of Jupiter to the mass of the Sun.

Sol: a = Semi-Maj axis of ~~a~~ Ganymede and T is ~~is~~ the period. The Corresp quantities for Earth are unity

$$4\pi^2 a^3 = G M_J T^2$$ Since Mass of Ganymede is much smaller to that of Jupiter. For the earth

$$4\pi^2 = G M_S$$

M_S, M_J are the masses of Sun, Jupiter respectively

$$\therefore a^3 = \frac{M_J}{M_S} T^2$$

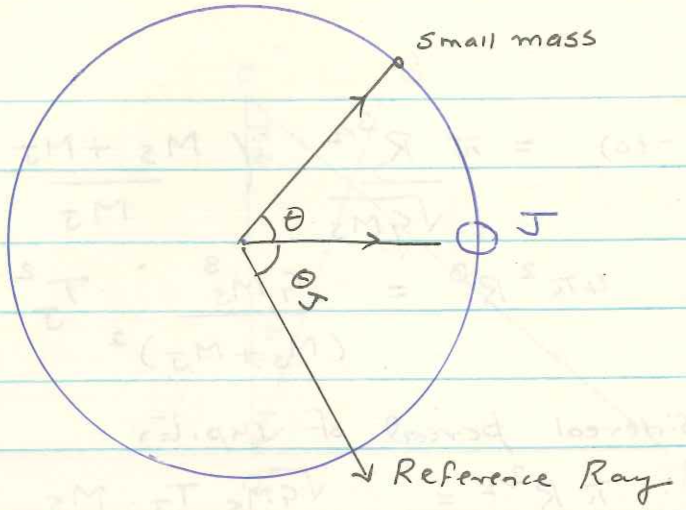
$$\therefore \frac{M_J}{M_S} = (0.007156)^3 / \left(\frac{7.155}{365}\right)^2$$

See p123 of M. Davidson[ ] for the case of Satellites of Mars.

5. The Sidereal period* of Jupiter is 11.86 years find its semi major axis. A body of negligible mass moves around the Sun with the same value of the Semi-Major axis. What is its period? If this body originally drifted away from Jupiter how long would it be before it approached Jupiter again? Assume the orbits are circular and ignore the gravitational attraction of Jupiter.

Sol: Denoting by M_S the mass of the Sun, M_J the mass of Jupiter, T the orbital period of the small mass and T_J the orbital period of Jupiter, we have by Kepler's 3rd law

*For the relation with Synodic period see p127 of M. Davidson.



$$T^2 = T_J^2 \left(M_S / (M_S + M_J) \right)^2$$

Let θ be the angular seperation between Jupiter and the small mass. The recession of the small mass is indicative of $\frac{d\theta}{dt} > 0$ as $t = t_0$ say. Assume that initially the angular seperation is small.

Let θ_J be the angular seperation of Jupiter with some initial reference ray and θ_μ be the same for the small mass

$$\theta_J = \int_{t_0}^{t} d\theta = \int_{t_0}^{t} \dot{\theta}\, dt = \int_{t_0}^{t} \frac{\| \vec{c}_J \|}{R^2}$$

likewise $\theta_\mu = \left(\| \vec{c}_\mu \| / R^2 \right) (t - t_0)$

$$\therefore \theta_J - \theta_\mu = (t - t_0) \left(\| \vec{c}_J \| - \| \vec{c}_\mu \| \right) / R^2$$

$$= \frac{(t - t_0)}{R^2} \left\{ \sqrt{R}\sqrt{\gamma_J} - \sqrt{R}\sqrt{\gamma_\mu} \right\}$$

$$= (t - t_0) R^{-3/2} \left\{ \sqrt{\gamma_J} - \sqrt{\gamma_\mu} \right\}$$

$\therefore (t - t_0)$ must be such that $-\theta_J + \theta_\mu$ $\cancel{\times / \pi}$ $= \pi$

beyond this the angular seperation diminishes

$$\therefore -\pi = (t - t_0) R^{-3/2} \left\{ \sqrt{\gamma_J} - \sqrt{\gamma_\mu} \right\}$$

putting in the values of γ_J, γ_μ

$$(t - t_0) = \pi \frac{R^{3/2}}{\sqrt{GM_S}} \left(\frac{M_S + M_J}{M_J} \right)$$

Now, $4\pi^2 R^3 = \frac{G M_S^3}{(M_S + M_J)^2} T_J^2$

T_J = Sidereal period of Jupiter.

$$\therefore \pi R^{3/2} = \frac{\sqrt{GM_S}\, T_J \cdot M_S}{2(M_S + M_J)}$$

$$\therefore (t - t_0) = \frac{T_J}{2} \frac{M_S}{M_J}$$ Assume $t_0 = 0$

and so the small mass would approach Jupiter after time $\frac{T_J}{2}\left(\frac{M_S}{M_J}\right)$

6. Application to Binary Stars:

A binary system consists of two stars with masses M_1, M_2 (in terms of Solar Mass M_S). The second of these is too faint to be observed but its presence is inferred from the irregular proper motions of the first. If M_1 is found to move around the centre of mass of the system in an ellipse subtending angle a_1'', show that

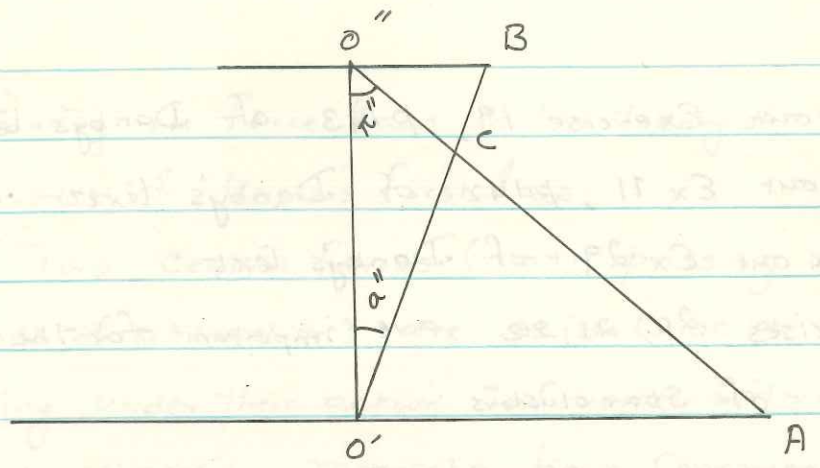
* $$\frac{M_2^3}{(M_1 + M_2)^2} = \frac{a_1''^3}{p''^3 P^2}$$ where P is the period and p'' is the parallax of the system. (See the article of P. Van de Kamp in Encyc. Physics vol 50 pp 187–224)

Sol: Referring to the figure

$\sphericalangle\, O''O'B = a''$

$\sphericalangle\, O'O''A = p''$

*: This is a fundamental eq^n in the study of Binary Stars.



O'A and O''B are the Semi-major axes of the orbit of the Earth and the visible Star. O''A, O'B meet at C

$$4\pi^2 \, O'A^3 = \frac{G M_S^3}{(M_S + M_E)^2} T^2$$

$O'A = 1$ astronomical unit; $T = 1$ sidereal year

$M_S / M_S + M_E \sim 1$.

$$\therefore \quad 2\pi = \sqrt{G M_S} \quad ; \text{again},$$

$$4\pi^2 (O''B)^3 = \frac{G M_2^3}{(M_1 + M_2)^2} P^2$$

$$\therefore \quad G M_S (O''B)^3 = \frac{G M_2^3}{(M_1 + M_2)^2} P^2$$

But if M_S is taken as unity

$$(O''B)^3 = \frac{M_2^3}{(M_1 + M_2)^2} P^2$$

$O''B = O'O''(\tan a'') = (O'O'') a''$ $(\because \tan a'' \sim a'')$

But $O'A = (O'O'') \sin p'' = (O'O'') p''$ $(\because \sin p'' \sim p'')$

$$\therefore \left(\frac{a''}{p''}\right)^3 = \frac{M_2^3 P^2}{(M_1 + M_2)^2} \quad (\because O'A = 1 \text{ A.U})$$

For more technical details Consult Chapters 10, 11 of the book Principles of Astronomy by Peter Van De Kamp, W. H. Freeman & Co, San Fran 1967

7. Work out Exercise 19, p143 of Danby's text
8. Work out Ex 11, p142 of Danby's text
9. Work out Ex 29 of Danby's text.
10. Exercises 20, 21, 22 are important for the Study of Starclusters.

Peter Van de Kamp's book discusses the mathematical details concerning the projection of the actual Kepler orbit onto the celestial sphere and methods for determining orbital elements. See particularly §7, p152 and §9 p156 and the example of Krüger 60 on §11 of Chapter 10. The book contains numerous references to books & papers by the same author. In particular Encyclopedia of Physics, Vol 50 (1958) Springer Verlag may be consulted.

Exercise: Show that Kepler's law of areas is still valid for the projection of the true Kepler Ellipse on the celestial Sphere.

Concerning Astronomical Terms and Data: A clear and elementary account of astronomical terms and data commonly needed may be found in Elements of Mathematical Astronomy by M. Davidson, Hutchinson's Scientific and Technical publications 1947.

§9.4 Duality in force laws in the theory of central forces
Converse of Kepler's first law:

Definition: Two central forces (with possibly different centres of attraction) are said to be dual if particles moving under their ~~action~~ influence with same angular ~~velocity~~ momenta have congruent trajectories.

We shall prove that a central force with magnitude inversely proportional to square of the distance is dual to the force field in which is proportional to the distance.

This is closely related to the Bohlin transformation which linearizes the Kepler problem. The literature on this is vast. See Arnold [ ]; Chandrashekar [ ], the articles by Needham [ ] and Saari [ ].

The converse of Kepler's first law was proved by Newton and also the fact that the two properties (attraction under inverse square law and linear force law) are related [ ].

We begin with the classical proof of Newton, beautifully expounded in Chandrashekar's essay [ ].

Theorem 9.11: (Basic Kinemetic Relation)

Suppose a particle describes a curve $\vec{r}(t)$ in space, its ~~centripetal~~ acceleration along the inward normal is

$$\vec{a}\cdot\hat{n} = \frac{\|\dot{r}\|^2}{\varsigma_{circ}} \quad \text{where} \qquad (9.32)$$

ς_{circ} is the radius of curvature and $\hat{n}$ is the inward normal vector (pointing towards the centre of curvature

proof: Denote by s the arc length parameter along the curve and $\vec{r}'$ denotes $\frac{d\vec{r}'}{ds}$ whereas $\dot{\vec{r}}$ denotes $\frac{d\vec{r}}{dt}$.

Now, $\frac{d}{dt}(\vec{r}') = \frac{d}{dt}\left(\frac{d\vec{r}}{dt} \Big/ \frac{ds}{dt}\right)$

$$= \frac{d}{dt}\left(\frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|}\right)$$

$$= \left(\ddot{\vec{r}}\,\|\dot{\vec{r}}\| - \dot{\vec{r}}\,\frac{d}{dt}\|\dot{\vec{r}}\|\right) \Big/ \|\dot{\vec{r}}\|^2$$

$$= \frac{1}{\|\dot{\vec{r}}\|^2}\left\{\ddot{\vec{r}}\,\|\dot{\vec{r}}\| - \dot{\vec{r}}\,\frac{(\dot{\vec{r}}\cdot\ddot{\vec{r}})}{\|\dot{\vec{r}}\|}\right\}$$

$$= \frac{1}{\|\dot{\vec{r}}\|^3}\left\{\ddot{\vec{r}}\,\|\dot{\vec{r}}\|^2 - \dot{\vec{r}}\,(\dot{\vec{r}}\cdot\ddot{\vec{r}})\right\}$$

By the Serret-Frenet Equation,

$$\frac{dr'}{ds} = +k\hat{n} \quad \therefore \frac{d\vec{r}'}{dt} = \cancel{-k\hat{n}/\|\dot{\vec{r}}\|}$$

$$\therefore \cancel{-k\hat{n}/} = \frac{1}{\cancel{\|\dot{\vec{r}}\|^3}}\left\{\cancel{\ddot{\vec{r}}/\|\dot{\vec{r}}\|^2 - \dot{\vec{r}}(\dot{\vec{r}}/\ddot{\vec{r}})}\right\}$$

$$+K\hat{n} = \frac{d}{ds}(\vec{r}\,')$$

$$= \frac{d}{dt}\vec{r}\,' \Big/ \|\dot{r}\|$$

$$= \frac{1}{\|\dot{\vec{r}}\|^4}\left\{ \ddot{\vec{r}}\,\|\dot{\vec{r}}\|^2 - \dot{\vec{r}}\,(\dot{\vec{r}}\cdot\ddot{\vec{r}}) \right\}$$

So, $$+k = \frac{\ddot{\vec{r}}\cdot\hat{n}}{\|\dot{\vec{r}}\|^2}$$

$$\therefore \quad \vec{a}\cdot\hat{n} = +k\,\|\dot{\vec{r}}\|^2 = +\frac{\|\dot{\vec{r}}\|^2}{\rho_{circ.}}$$

The proof is complete.

Theorem 9.12: <u>Some Properties of the Ellipse</u>

(i) Show that for an ellipse, the radius of curvature ρ_c is given by the formula

$$\rho_c = \frac{1}{ab}\left(\frac{ds}{dt}\right)^3 \qquad (9.33)$$

Where s is the arc length function

(ii) Suppose S is the focus and ϑ_S is the angle between XS and the tangent vector, where X is a point on the ellipse,

$$\rho_c \sin^3 \vartheta_S = b^2/a \ (\text{const.}) \qquad (9.34)$$

(iii) Deduce from (ii) that if $\vec{r}(t)$ is a plane curve such that (9.34) holds, where ϑ_S is the angle between $\vec{r}-S$ and $\hat{T}$ (the unit tangent vector) the locus of $\vec{r}$ is an ellipse with S as a focus.

(iv) Let ϑ_S and ϑ_C be the angles between XS and $\hat{T}$; XC and $\hat{T}$ respectively where C is the centre of the ellipse

Through C Draw a straight line l parallel to $\hat{T}$ cutting SX at E. Show that

$$\frac{\sin\vartheta_S}{\sin\vartheta_C} = \frac{EX}{EX} \quad \text{and that } EX = a = \text{semi-major axis of the ellipse}$$

(See S. Chandrasekar [ ] p 237 ff)

<u>proofs</u>: (i) Take as parametrization

$$\vec{r}(t) = (a\cos t,\ b\sin t)$$

$$\hat{T} = (-a\sin t,\ b\cos t) / (a^2\sin^2 t + b^2\cos^2 t)^{1/2}$$

$$\frac{d\hat{T}}{ds} = \frac{d}{dt}\left(\frac{d\hat{T}}{ds}\right)\left(\frac{ds}{dt}\right)^{-1} = \frac{d}{dt}\left(\frac{d\hat{T}}{ds}\right)(a^2\sin^2 t + b^2\cos^2 t)^{-1/2}$$

$$= \frac{(-a\sin t,\ b\cos t)(b^2-a^2)\sin t\cos t}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} + \frac{(-a\cos t,\ -b\sin t)}{(a^2\sin^2 t + b^2\cos^2 t)}$$

$$\hat{N} = (-b\cos t,\ -a\sin t)/(a^2\sin^2 t + b^2\cos^2 t)^{1/2}$$

$$\frac{d\hat{T}}{ds}\cdot\hat{N} = ab / (a^2\sin^2 t + b^2\cos^2 t)^{3/2}$$

Using the Serret-Frenet Equation

$$\rho = (a^2\sin^2 t + b^2\cos^2 t)^{3/2} / ab = \frac{1}{ab}\left(\frac{ds}{dt}\right)^3$$

(ii) $S = (ae, 0)$ and

$$\sin\vartheta_S = \left|\left(\frac{\vec{r}-S}{\|\vec{r}-S\|}\times\hat{T}\right)\cdot\hat{k}\right|$$

$$= \left|\frac{\vec{r}-S}{\|\vec{r}-S\|}\cdot\hat{n}\right|$$

$$= \frac{\left| (a\cos t - ae, \, b\sin t) \cdot (-b\cos t, \, -a\sin(-)) \right|}{\| (a\cos t - ae, \, b\sin t) \| \; (ds/dt)}$$

$$= \frac{ab(1 - e\cos t)}{a(1 - e\cos t)\left(\frac{ds}{dt}\right)} = b \Big/ \left(\frac{ds}{dt}\right)$$

(These calculations ~~seem~~ seem to suggest a direct geometric approach)

$$\therefore \sin^3 \vartheta_s = b^2 / a \rho_{\text{Circ}} \quad \text{from which (ii) follows.}$$

(iii) Recall the Serret - Frenet Equations:

$$\rho \frac{d\hat{T}}{dt} = \hat{N}$$

$$\rho \frac{d\hat{N}}{dt} = -\hat{T} \quad \text{and} \quad \hat{T} = \frac{1}{\| r'(t) \|} \frac{d\vec{r}}{dt} \qquad (9.34)$$

½ The condition $\rho \sin^3 \vartheta_s = \text{constant}$, gives

since $\sin \vartheta_s = \left| \frac{(\bar{r}(t) - S)}{\| \bar{r}(t) - S \|} \cdot \hat{N} \right|$, a relation

$$\psi(\rho, \hat{N}, \vec{r}(t)) = \text{const}$$

which determines the scalar factor ρ. The system (9.34) is thus a closed system of ODEs and its solution is uniquely determined as soon as initial conditions are prescribed.

Indeed the last equation shows $\| \hat{T} \| = 1$ the first two show that $\hat{N}^2 - \hat{T}^2$ is const and so $\hat{N}^2$ is also const. It follows $\hat{N} \cdot \hat{T}$ is constant

(Differentiating the first and using the second,

$$\rho^2 \ddot{T} + \rho\dot{\rho}\dot{T} + T = 0$$ which decouples the $\hat{N}$ and $\hat{T}$ equations.)

If ρ is known as a function of $\vec{r}$ and $\hat{N}$

(9-34) reads

$$\frac{d\hat{T}}{dt} = \psi_1(\rho, \hat{N}, \vec{r})\,\hat{N}$$

$$\frac{d\hat{N}}{dt} = -\psi_1(\hat{N}, \vec{r})\,\hat{T}$$

$$\frac{d\vec{r}}{dt} = \hat{T}\,\|\vec{r}\|$$

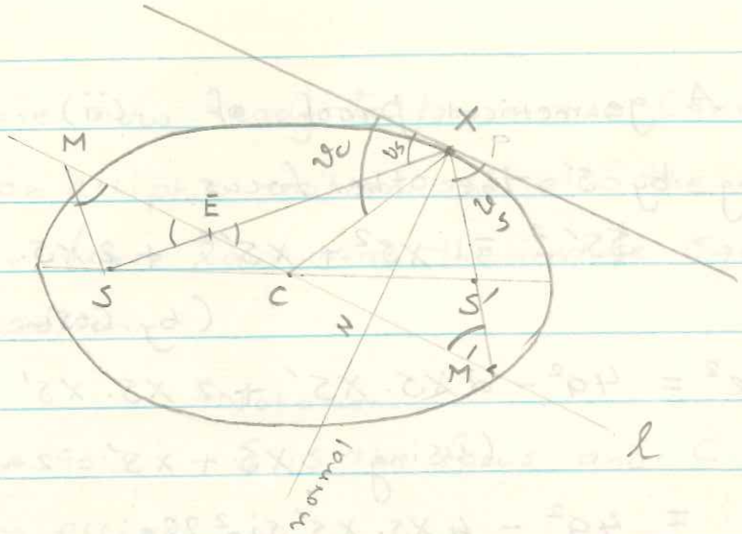
Initial conditions may be so chosen that $\vec{r}(t)$, $\hat{T}(t)$, $\hat{N}(t)$ coincides with the position vector, tangent, normal to an ellipse with focus S and prescribed latus rectum (oriented suitably)

The position vector of this ellipse, unit tgt & normal also satisfies (9·34)

By uniqueness, our curve must coincide with an ellipse.

(iv) proof of this is somewhat not trivial.

A purely analytical approach would be hoplessly complicated.



Construction: Produce XS' to meet l at M' and draw $SM \parallel XS'$ meeting l at M

$$SX + S'X = 2a$$

$$\therefore (SE + EX) + (XM' \mp S'M') = 2a$$

But $\Delta EXM'$ is isosles since the tangent makes equal angles with XS and XS'

$$\therefore \; SE \mp S'M' = 2(a - EX)$$

Our job will be done if we show $SE = S'M'$.

$\Delta CS'M'$ and ΔCSM are Similar

and $SC = CS'$ so the triangles are Congruent

$\therefore \; SM = S'M'$. But ΔSME is also isoscles with $SM = SE$. The proof is complete.

$$\therefore \; EX = a = \text{Semi-major axis.}$$

Normal through X meets l at N

$$\sin v_s = \cos(\angle EXN) = XN/XE$$

$$\sin v_c = \cos(\angle CXN) = XN/XC$$

$$\therefore \frac{\sin v_s}{\sin v_c} = \frac{\cancel{XN}}{\cancel{XN}} \; \frac{CX}{EX} = r/a.$$

Aliter: A geometrical proof of (ii)

Denoting by S' the other focus,

$$SS'^2 = XS^2 + XS'^2 + 2\,XS.\,XS' \cos 2\vartheta_S$$

(by Cosine Rule)

$$\therefore\ 4a^2e^2 = 4a^2 - 2XS.\,XS' + 2\,XS.\,XS' \cos 2\vartheta_S$$

(using $XS + XS' = 2a$)

$$= 4a^2 - 4XS.\,XS' \sin^2\vartheta_S$$

$$\therefore\ XS.\,XS' \sin^2\vartheta_S = b^2.$$

By Apollonius's theorem, $XS^2 + XS'^2 = 2\,XC^2 + 2CS^2$

$$\therefore\ 4a^2 - 2XS.\,XS' = 2r^2 + 2a^2e^2$$

$$\therefore\ XS.\,XS' = 2a^2 - r^2 - a^2e^2$$

$$\therefore\ \sin^2\vartheta_S = b^2/(a^2+b^2-r^2)$$

$$\therefore\ \sin\vartheta_S = b/(a^2\sin^2\xi + b^2\cos^2\xi)^{1/2}$$

$$= b/\left(\frac{ds}{dt}\right)$$ as desired. Now apply (i)

Due to the appearence of ρ in the formula we cannot dispense with calculus.

We are now ready to prove the Converse of Kepler's law on elliptic motion and also the duality between linear force law and the inverse square law of gravitation.

Notation:

Theorem 9.13: S is the focus and C is the centre of an ellipse.

(i) Suppose that a particle describes a trajectory $\vec{R}(t)$ governed by (9.8), that is to say under the action of a central force directed towards S. Then the locus of $\vec{R}(t)$ is an ellipse with focus S if and only if the magnitude of force varies inversely as the square of ~~its~~ the distance of $\vec{R}(t)$ from S.

(ii) Suppose two particles with the same angular momenta describe the same curve under two different laws of force centred at S and C respectively.

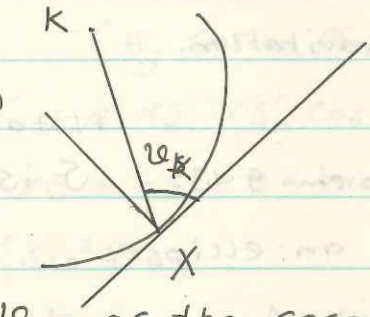
The trajectory is an ellipse (with centre C focus S) iff the laws of force are respectively inverse square law (for the first particle attracted towards S) and a linear force law (force varying linearly as the distance) for the second particle (with centre of attraction C)

Remark: We shall show that (i) $\Rightarrow$ (ii)

pf: adapted from the elegant exposition of
S. Chandrasekar [ ].

~~proof~~ We begin with the momentum equation

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{c}$$



K = Centre of attraction

K = S or C as the case

may be $\vartheta_K = \vartheta_S$ or ϑ_C as the case may be.

v = velocity.

$$\frac{\vec{r}}{\|\vec{r}\|} \times \frac{\left(\frac{d\vec{r}}{dt}\right)}{\left\|\frac{d\vec{r}}{dt}\right\|} = \frac{\vec{c}}{v(KX)}$$ (Referring the origin to K)

(O = Centre of Curvature)

$$\therefore \sin \vartheta_K = \frac{\|\vec{c}\|}{v(KX)} \qquad (9.35)$$

Using 9.11, CF_K = Centripetal force towards K
has magnitude (9.36)

$$CF_K = \frac{v^2}{\rho_{circ}} \frac{1}{\sin \vartheta_K} = \frac{\|\vec{c}\|^2}{(KX)^2} \frac{1}{\rho_{circ} \sin^3 \vartheta_K}$$

ρ_c = Radius of Curvature.

$$\therefore \frac{CF_S}{CF_C} = \left(\frac{\sin \vartheta_C}{\sin \vartheta_S}\right)^3 \left(\frac{CX}{SX}\right)^2 \qquad (9.37)$$

Referring to the figure drawn for Thm 9.12 (iv)

$$\cancel{\frac{EX}{\sin(\measuredangle XEX)}} \cancel{=} \frac{EX}{\sin(\measuredangle XCE)} = \frac{CX}{\sin(\measuredangle CEX)}$$

$$\therefore \frac{EX}{CX} = \frac{\sin(\pi - \upsilon_c)}{\sin \upsilon_s} = \frac{\sin \upsilon_c}{\sin \upsilon_s}$$

$\therefore$ ~~Centrifugal force~~ But $EX = a$

$$\frac{C\cdot F_S}{C\cdot F_C} = \left(\frac{a}{CX}\right)^3 \frac{(CX)^2}{(SX)^2}$$

$$\therefore \frac{(C\cdot F)_S}{(C\cdot F)_C} = \frac{a^3}{(CX)(SX)^2} \qquad (9.38)$$

So if centripetal attraction towards S is proportional to $(SX)^{-2}$ then the centripetal force towards C must be proportional to (CX) if the centre of force is to be transmuted from S to C keeping the same orbit with same angular velocity. Following Needham [ ] let us call this The Principle of Transmutation of Central forces for Keplerian Orbits.

υ_K

Additional Exercises: (Laplace - Runge - Lenz vector)

(i) Show that the vector equation of a conic is

$$\vec{r} \cdot \left(\vec{\varepsilon} + \frac{\vec{r}}{\|\vec{r}\|} \right) = l \qquad (9.39)$$

(see (9.28))

Where the origin is placed at the focus, $\vec{\varepsilon}$ is a vector of length $= e$ (eccentricity) and directed along the ray from focus $\perp$ to the directrix. Constant l is the semi latus rectum.

Sol: See derivation of (9.28).

(ii) Deduce from (9.39) the equation

$$\vec{\varepsilon} = \frac{l}{\|\vec{c}\|^2} (\vec{v} \times \vec{c}) - \frac{\vec{r}}{\|\vec{r}\|} \qquad (9.40)$$

Where $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{c} = \vec{r} \times \vec{v}$.

Hint: Differentiate (9.39) and solve for $\vec{\varepsilon}$.

Sol: $$\vec{v} \cdot (\vec{\varepsilon}) + \frac{d}{dt} \|\vec{r}\| = 0$$

$$\vec{v} \cdot \vec{\varepsilon} + (\vec{r} \cdot \vec{v}) / \|\vec{r}\| = 0$$

$$\vec{v} \cdot \left(\vec{\varepsilon} + \frac{\vec{r}}{\|\vec{r}\|} \right) = 0$$

This shows that $\vec{\varepsilon} + \vec{r}/\|\vec{r}\|$ is orthogonal to $\vec{v}$ and lies in the plane, is orthogonal to $\vec{c}$ as well.

$$\therefore \vec{\varepsilon} + \vec{r}/\|\vec{r}\| = \lambda (\vec{v} \times \vec{c}) \qquad (9.41)$$

where λ is to be determined

Multiplying (9.41) scalarly with $\vec{r}$ and using (9.39)

$$l = \lambda\,(\vec{r}\times\vec{v})\cdot\vec{c} = \|\bar{c}\|^2\lambda$$

$$\therefore\quad \vec{\varepsilon} + \vec{r}/\|\vec{r}\| = (l/\|\vec{c}\|^2)(\vec{v}\times\vec{c})$$

Which is (9·40).

(iii) Prove that the equations (9·40) with the constancy of $\vec{r}\times\vec{v}\ (=\vec{c})$, and $\vec{\varepsilon}$. implies that the locus of (9·40) is ~~an ellips~~ a conic. Find its eccentricity. Can the hypothesis $\vec{r}\times\vec{v}$ be const- be dropped?

Sol: Multiply (9·40) scalarly with $\vec{r}$

$$\vec{r}\cdot\left(\vec{\varepsilon} + \frac{\vec{r}}{\|\vec{r}\|}\right) = l \quad \text{which is (9·39).}$$

Aliter: The hypothesis $\vec{r}\times\vec{v}$ is ~~be~~ constant may be dropped. Multiply (9·40) scalarly with $\vec{v}$

$$\vec{\varepsilon}\cdot\vec{v} = -\vec{r}\cdot\vec{v}/\|\vec{r}\|$$

$$\therefore\quad \frac{d}{dt}\,(\vec{\varepsilon}\cdot\vec{r} + \|\vec{r}\|) = \text{Const}=\lambda$$

$$\therefore\ (er\cos\theta + r) = \lambda$$

which is a conic with semi-latus rectum λ and eccentricity $e = \|\vec{\varepsilon}\|$

So, equation (9·40) does indeed represent a conic with $\vec{\varepsilon}$ the eccentricity vector. Note that we are not even assuming apriori that (9·40) is a plane curve.

Laplace's Proof of Kepler's first Law:

(iv) Consider a point moving in accordance with (9-8) where $f(\vec{R}) = -1/\|\bar{R}\|^3$.

Show that $\frac{1}{\gamma}(\vec{v}\times\vec{c}) - \frac{\vec{R}}{\|\vec{R}\|} = \vec{\lambda}$ (9.42)

is a first integral. This is known as the Laplace - Runge - Lenz vector. Observe that $\vec{\lambda}\cdot\vec{c}=0$. Hence we have again shown that the orbit is ~~an ellipse~~ a conic with semi-latus rectum $\|c\|^2/\gamma$.

This elegant proof is due to Laplace. (see [ ])

Sol: Differentiate (9.42) and use the vector triple product identity.

Rem: This proof is essentially the one given in several modern texts Pollard [ ], Boccaletti and Pucacco [ ] p132 and Danby [ ] p128.

However, the interpretation of the Laplace-Runge-Lenz vector is not clarified. The above set of exercises is adapted from the article by B. Pourciau [ ].

42)

§ 9.5 Bohlin's Theorem and its generalizations: The Levi-Civita Regularization.

We know from the complete integrability of the Kepler problem, the existence of a coordinate system in which the bounded motions are superpositions of Harmonic Oscillations. The theorem of Bohlin elegantly describes this in terms of complex variables and also clarifies the duality between inverse square law and the linear force law. We briefly sketch this connection in this section, referring to the article by Needham [ ], the book of S. Chandrasekhar [ ] (pp 119-125) for more details. We follow here the exposition in Arnold [ ] although we shall be somewhat sketchy. (see note on p 53)

We begin by collecting some elementary facts.

ge-

Thm 9.14: Denote by C_r a circle of radius r centred at the origin

$J(z) = z + 1/z$ (Zhukovskii map)

$B(z) = z^2$

$E_r = J(C_r)$.

(i) $\{E_r\}_{r>0}$ is a family of ellipses with eccentricities $2r/1+r^2$ (for $r \neq 1$): For $r=1$, E_1 is a line segment. The ellipse E_r has major axis $r + 1/r$, minor axes $r - 1/r$ and foci ± 2.

Thus, the family $\{E_r\}_{\substack{r>0 \\ r\neq 1}}$ is a Confocal family of ellipses.

(ii) $B(E_r)$ is an ellipse with focus at the origin and Congruent to E_r.

$B(E_r)$ is simply E_r shifted by 2 units (as sets of course)

(iii) $T_2 \circ J \circ B = B \circ J_{i\pi/2}$, where

$T_2(z) = z+2$

(iv) Any ellipse with focus at the origin is the Square of an ellipse with Centre at the origin.

proof of (iv): Let E be an ellipse with focus at the origin, major axis along the x-axis

λE is a similar ellipse with dist between foci $= 4$. That is to say $T_{-2}(\lambda E)$ is a Zhukovskii ellipse $= E_r$ (with centre at the origin)

$B(E_r)$ is congruent to E_r and has its focus at 0

$\therefore \quad B(E_r) = \lambda E$

$\therefore \ B(E_r/\sqrt{\lambda}) = E$ and $E_r/\sqrt{\lambda}$ also has its Centre at the origin.

If now the ellipse E did not have its major axis along x-axis then

$R_\alpha(E)$ is an ellipse with maj axis along the x-axis and R_α is a rotation through angle α. By what has been proved,

$$R_\alpha(E) = B(\tilde{E})\ ; \quad \tilde{E} \text{ is an ellipse with centre } 0$$

$$\therefore\ E = (R_{-\alpha} \circ B)(\tilde{E})$$

$$= (B \circ R_{-\alpha/2})(\tilde{E})$$

Now $R_{-\alpha/2}(\tilde{E})$ is an ellipse with centre 0.

The proof is complete.

Thm 9.15: (Bohlin: Bulletin Astronomique (28) 1911, 144)

Suppose $w(t)$ traces a trajectory according to

$$\frac{d^2w}{dt^2} + w = 0 \qquad (9.43)$$

(see p 53, (9.47))

There exists a reparametrization $t = \psi(\tau)$ of time such that the point

$Z(\tau) = ((w \circ \psi)(\tau))^2$ describes an ellipse according to

$$\frac{d^2Z}{d\tau^2} = -\text{const}\ \frac{Z}{|Z|^3} \qquad (9.44)$$

and ~~Moreover~~ the law of areas are satisfied for both trajectories.

~~proof: Note that it is a requirement that the laws of areas hold for both systems since both~~

Proof: Since both (9.43) – (9.44) are both central force fields, the law of areas applied to them gives:

$$|w|^2 \frac{d\phi}{dt} = c_1\ ; \quad |Z|^2 \frac{d\phi}{d\tau} = c_2$$

$$\frac{dt}{d\tau} = \frac{c_2}{c_1}\frac{1}{|w|^2}$$

So $$\frac{d}{d\tau} = \left(\frac{c_2}{c_1|w|^2}\right)\frac{d}{dt}$$

$$\frac{dZ}{d\tau} = \frac{c_2}{c_1 \cancel{|w|^2}}\frac{2\dot{w}}{\bar{w}}$$

$$\frac{d^2Z}{d\tau^2} = \frac{2c_2^2}{c_1^2|w|^2}\frac{d}{dt}\left(\frac{\dot{w}}{\bar{w}}\right) = \frac{2c_2^2}{c_1^2|w|^2}\left(\frac{\bar{w}\ddot{w} - \dot{\bar{w}}\dot{w}}{\bar{w}^2}\right)$$

$$= \frac{2c_2^2}{c_1^2|Z|^3}\left(-w\bar{w} - \dot{w}\dot{\bar{w}}\right)w^2$$

$$= \frac{2c_2^2}{c_1^2|Z|^3}(-EZ)\text{; Recalling that}$$

$|w|^2 + |\dot{w}|^2$ is a first-integral of (9.43)

The proof is complete.

Exercise: Suppose that $w(t)$ is a solution of $\ddot{w} = -(\text{Const})\cancel{|w|^{a-1}}\,|w|^{a-1}w$ (9.45)$'$

There exists a reparametrization of time $t = \psi(\tau)$ such that $Z = (w\circ\psi)^{\alpha}$

Solves $\ddot{Z} = -(\text{const})|Z|^{A-1}Z$ (9.45)$''$

if $\alpha = \frac{a+3}{2}$ and $(a+3)(A+3) = 4$ (9.46)

Deduce that self-dual laws are $a = -5, -1$.

The orbits of (9.43) are ellipses with

~~[illegible]~~: centre at the origin. The corresponding orbits of (9.44) are therefore ellipses with a focus at the origin – We have almost proved Kepler's first law.

Note: Our treatment has been somewhat sketchy and does not aim at completeness. We have not accounted for hyperbolic and parabolic orbits.

For more details we refer to Needham's article pp 125 - 127.

In place of (9.43) one may start with

$$\ddot{w} + \Omega w = 0 \qquad (9.47)$$

and we would get

$$\frac{d^2 z}{d\tau^2} = \frac{2c_2^2}{c_1^2 |z|^3}(-Ez) \qquad (9.48)$$

Where, the energy E is given by $|\dot{w}|^2 + \Omega |w|^2 = E$ which may be negative if $\Omega < 0$.

Orbits of (9.47) are ellipses if $\Omega > 0$ hyperbole if $\Omega < 0$ and straight lines if $\Omega = 0$
~~The ellipses and hyperbolas have their axes along the coordinate axis and centre at the origin.~~
~~($\because$ $w \to -w$ is keeps the ODE invariant)~~

By imitating the proof of Thm 9.14 we can account for hyperbolic trajectories.

Exercise: Show that the map $B(z)$ takes

Straight lines not passing through the origin into parabolas with origin as focus.

Show that the Zhukovskii map $\frac{Z}{4}J(z)$ takes rays from the origin to (branches of) hyperbolae. That the Zhukovsii hyperbolas meet the Zhukovsii ellipses orthogonally.

The Zhukovsii hyperbolas (and Ellipses) form a Confocal family

Regularization: Recall that certain solutions of the two body problem with $\vec{c} = 0$ result in collision with the centre of mass (i.e the approach of the two bodies) in finite time. Such a singularity is called a collision singularity.

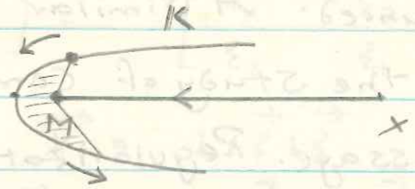
The other type of singularity where the body escapes to infinity in finite time does not occur in the Kepler problem.

For the N-body problem, it is known that ~~tot~~ total collapse (where all bodies come together) is not possible (Sundman-Weirstrass Thm)

So, in particular for the three body problem, triple collisions are ruled out.

Binary Collisions can be handled as for the Kepler problem. At the time of binary collision, in the nbd of the colliding bodies the third body is far away and the problem may be approximated by the Kepler problem.

Heuristics of binary collision and regularization.



Consider a rectilinear orbit XM colliding into the primary M. Approximate the orbit by an elongated conick. The transit time through perihelion is small due to the law of areas. The angular coordinate $\theta(t)$ suffers an "abrupt" change through 2π (i.e. changes from 0 to 2π in a small interval of time)

In the limit one would expect a collision followed by a "rebounce".*

Introducing complex notation $z = x+iy$ with the origin at M, the collision takes place at $r=0$; $r=\sqrt{x^2+y^2}$. to straighten out the singularity (i.e to desingularize) the obvious transformation is to take the square root

i.e set $z = w^2$ which is precisely the transformation obtained by Bohlin.

The above discussion is adapted from D. Saari [ ]

* See J. Milnor's article [ ]. This uses the fact that semi-axis a depends only on E and is indep of $\|\bar{c}\|^2$

and one can let $\vec{c} \to 0$ the elliptical orbits "flatten out".

The problem of eliminating singularities from the ODE through coordinate transformations is called regularization.

Note that real bodies have finite size and in the case of motion of artificial satellites the instabilities caused by singularities, in numerical schemes become pronounced. A similar situation would prevail in the study of comets near the perihelion passage. Regularization is essential for restoration of stability in numerical investigations.

Examples of Regularization:

(i) put $\varrho \frac{d\tilde{u}}{dt} = \sqrt{2E}$: $\frac{d}{dt} = \frac{\sqrt{2E}}{\varrho} \frac{d}{d\tilde{u}}$ (9.49)

in equation (9.23) we get with $\tilde{u}$ as the independent variable, using the Energy integral (9.21)

$$\frac{d^2\varrho}{d\tilde{u}^2} + \varrho = \frac{\gamma}{2E} \qquad (9.50)$$

Thus, the introduction of the variable $\tilde{u}$ (known as the eccentric anomaly) the DE has been desingularized (regularized)

(ii) The Bohlin map $z = w^2$ linearizes the ODE

$$\frac{d^2 z}{dt^2} = -\text{const}\; z/|z|^3$$

after a reparametrization of time.

(iii) The Levi-Civita Transf is a variant of the Bohlin map. Let us rewrite the Bohlin map in Cartesian coordinates:

$$x_1 = \xi_1^2 - \xi_2^2 \; ; \quad x_2 = 2\xi_1\xi_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad (9.51)$$

i.e. $\vec{X} = L(\vec{\xi})\vec{\xi}$

$$\frac{d\vec{X}}{dt} = 2L(\vec{\xi})\frac{d\vec{\xi}}{dt} \quad \therefore \frac{d\vec{\xi}}{dt} = \frac{L^T(\vec{\xi})}{2|\xi|^2}\frac{d\vec{X}}{dt}$$

$$L^T L = |\xi|^2 \; : \quad L^{-1} = |\xi|^{-2} L^T \qquad (9.52)$$

$\cancel{\frac{d\vec{X}}{dt}}$ Introduce the fictitious time s given by

$$dt = \|\vec{X}\|\, ds \qquad (9.53)$$

again $\dfrac{d\vec{X}}{ds} = 2L(\vec{\xi})\dfrac{d\vec{\xi}}{ds}$. But the ODE (9.8) and (9.19)

~~becomes give the harmonic oscillator~~

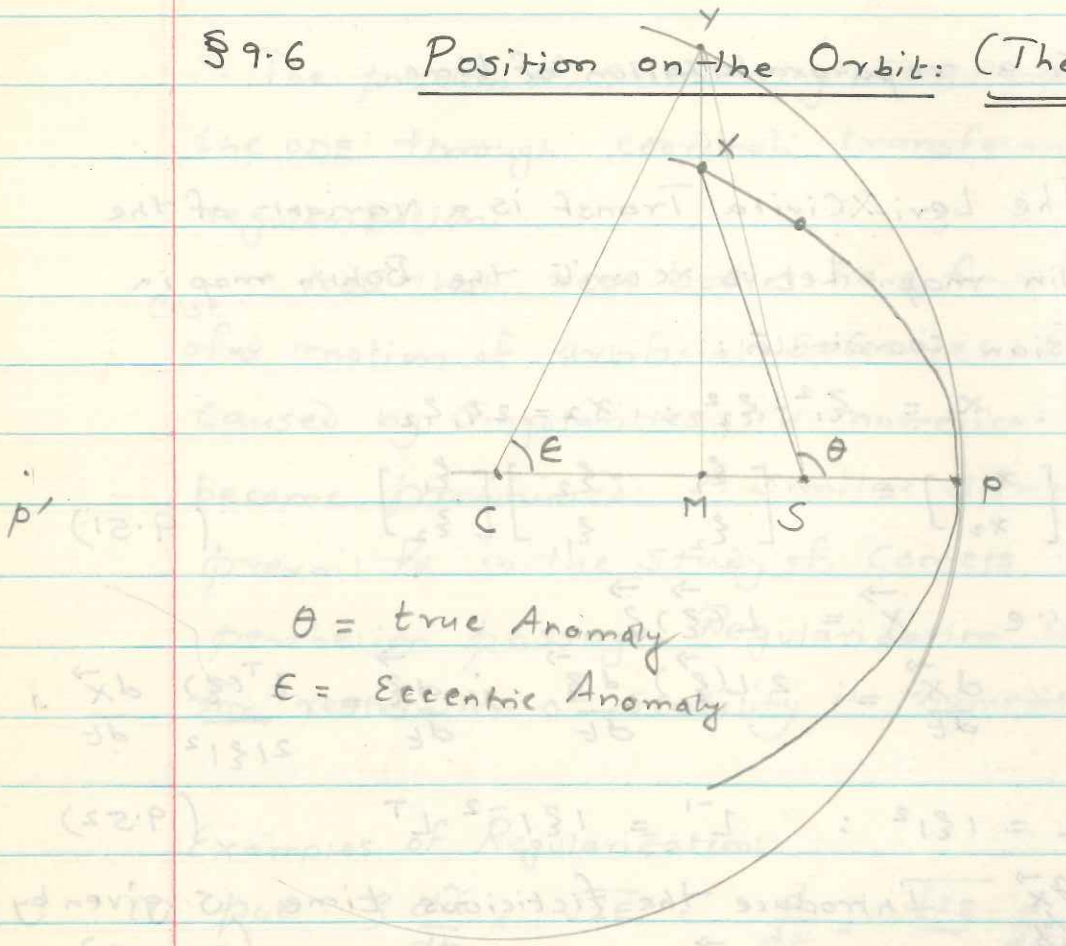
~~$\xi'' - E/\xi \neq 0$~~

gives the harmonic oscillator $\xi'' - E\xi = 0$. (9.54)

The ODEs have been regularized.

The computations omitted here are available on p166 of Boccaletti and Pucacco [ ]. Chapters I, II of E.L. Stiefel & G. Scheifele [ ] contains a detailed and very readable account of this as well as its higher dimensional Analogues.

§9·6 Position on the Orbit: (The Kepler Eqⁿ)



θ = true Anomaly
ϵ = Eccentric Anomaly

Problem: Given the planet's position at time $t=0$ to find its location at later times.

We shall assume that the planet is at the Perihelion at time $t=0$. (P = Perihelion)

X is the position at any instant t

$\measuredangle PSX = \theta$ is called the true Anomaly.

Draw the auxilliary circle (with centre C)

$XM \perp$ major axis at M; MX meets the auxilliary circle at Y (so that X, Y are corresponding points)

$\measuredangle PCY = \epsilon$ = Eccentric Angle called Eccentric ~~Anomaly~~ Anomaly

Exercise: Prove that

$$\frac{\text{Area (Sector PSX)}}{\text{Area (Sector PSY)}} = \frac{\text{Area (Ellipse)}}{\text{Area (Circle)}} \qquad (9.55)$$

$$\therefore \frac{\text{Area (Sector PSX)}}{\text{Area (Ellipse)}} = \frac{\text{Area (Sector PSY)}}{\text{Area (Circle)}}$$

By Kepler's second law

$$\frac{\text{Area (Sector PSX)}}{\text{Area (Ellipse)}} = \frac{t}{T}$$

where t is the time of transit from P to X

$$\therefore \frac{\text{Area (Sector PSY)}}{\text{Area (Circle)}} = \frac{t}{T}\,\cancel{(2\pi a^2)}$$

$$\text{Area (Sector PSY)} = \text{Area (Sector PCY)} - A(\Delta\, CSY)$$

$$= \frac{1}{2} a^2 \epsilon - \frac{1}{2}\, CS \cdot (a \sin \epsilon)$$

$$= \frac{1}{2}(a^2 \epsilon - a^2 e \sin \epsilon)$$

$$\therefore a^2(\epsilon - e \sin \epsilon) = t/T \cdot 2\pi a^2$$

$$\therefore \epsilon - e \sin \epsilon = 2\pi t / T \qquad (9.56)$$

Def: $2\pi t/T$ is called the Mean Anomaly.

$$\boxed{\text{The Kepler Equation: } \epsilon - e \sin \epsilon = \frac{2\pi t}{T}} \qquad (9.56)$$

Exercise: Relation between True ~~Anomaly~~ Anomaly and Eccentric Anomaly

(i) $(1 + e\cos\theta) = (1 - e^2)/(1 - e\cos\epsilon)$ (9.57)'

(ii) $\tan \epsilon/2 = \tan \theta/2 \sqrt{\frac{1-e}{1+e}}$. (9.57)''

$(9.57)'''$

(iii) $\frac{\sin\theta}{\sin\epsilon} = \frac{\sqrt{1-e^2}}{1-e\cos\epsilon}$ (iv) $\frac{d\theta}{d\epsilon} = \sqrt{1-e^2}/(1-e\cos\epsilon)$

Sol: $CM = a\cos\epsilon$; $MX^2 = \rho^2 - MS^2 = \rho^2 - (ae - a\cos\epsilon)^2$

$My = a\sin\epsilon$

Show that $\frac{My}{MX} = \frac{a}{b}$ squaring and simplifying

$\therefore\ a^2 = \rho^2 + 2a^2 e\cos\epsilon - a^2e^2\cos^2\epsilon$

$\therefore\ (1-e\cos\epsilon)(1+e\cos\theta) = 1-e^2$; $\rho = \ell/(1+e\cos\theta)$

~~Use the fact that~~

$(\cos\theta - \cos\epsilon) = e(\cos\theta\cos\epsilon - 1)$

Use now $\cos A = (1-\tan^2 A/2)/(1+\tan^2 A/2)$

$(\tan^2\epsilon/2 - \tan^2\theta/2) = -e(\tan^2\theta/2 + \tan^2\epsilon/2)$

Which gives (9.57) upon simplification. The other relations follow from (9.57)'.

[Kn]owing $\epsilon(t)$ from Kepler Eqn (9.56) gives [thr]ough (9.57) θ as a function of t and hence ~~r(t)~~ $\rho(t)$ via equation (9.29)

The issue is now to Invert the Kepler Eqn. That is to say, given t find $\epsilon(t)$.

The problem has attracted much attention in the last three hundred years. Newton had already applied to (9.56) the method bearing his name for finding approximate roots of algebraic equations. (see Scholium on p 141 of S. Chandrasekhar [ ])

Lagrange obtained a series solution for ϵ in terms of trig polynomials in $2\pi t/T$ and powers of e, that is to say

$$\epsilon(t) = \sum_{j \geq 0} c_j \, e^{\nu} \sin j\mu \qquad (9.58)$$

$$\mu = 2\pi t/T$$

Convergence of (9.58) for all μ is guarenteed only for for small values of eccentricity. Infact for

$$e \leq 0.6627434^{*}$$

Lagrange was not able to provide conclusive evidence and later Laplace tackled the problem. The rigorous proof was given by Cauchy later simplified by Roche. The resolution of (9.56) provided a major impetus for the development of function theory. F.R. Moulten's ~~has~~ 1914 quotation has now become well known, citing the extensive bibliography of 123 papers in Bulletin Astronomique (Jan 1900).

See J.M. Danby's articles (with T.M. Burkardt) The solution of Kepler's Equation - I & II Celestial Mechanics (31), 95-107, 317-328, 1983 vol 40, p 303-312, 1987.

* See the discussion on p 559 of the treatise of G.N. Watson [ ].

Solution of the Kepler Equation (Bessel)

Theorem 9.16: Assume that at time $t=0$ the planet passes through the perihelion, then for each time t, the Kepler equation (9.56) has a unique solution $\epsilon(t)$ which is smooth in t and $\epsilon(t) - \frac{2\pi t}{T}$ is periodic of period T.

Moreover

$$\epsilon(t) = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin\left(\frac{2\pi n t}{T}\right) \quad (9.59)$$

Where $J_n(x)$ is the Bessel's function given by

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - x\sin t)\,dt. \quad (9.60)$$

proof. Let $\phi(u) = u - e\sin u$

$$\phi'(u) = 1 - e\cos u > 0$$

So ϕ is strictly increasing; $\phi(u) \to \pm\infty$ as $u \to \pm\infty$ and so ϕ maps $\mathbb{R}$ bijectively to itself. The solution $\phi(u) = 2\pi t/T$ is smooth by the inverse function theorem.

The transformation $(\epsilon, t) \mapsto (-\epsilon, -t)$ keeps the Kepler Eqn invariant. So

$\epsilon(-t) = -\epsilon(t)$ so the solution is an odd function of t.

If $\epsilon(t)$ solves the Kepler Equation

$\frac{2\pi t}{T} = \epsilon - e\sin\epsilon$ then

$E(t) + 2\pi$ solves the equation

$$\frac{2\pi}{T}\left(t + \frac{T}{2\pi}\right) = (E(t) + 2\pi) - e \sin(E(t) + 2\pi)$$

Since the solution is unique,

$$E(t+T) = E(t) + 2\pi \quad \text{for all } t$$

So the function

$$\Pi(t) = E(t) - \frac{2\pi t}{T}$$

is periodic with period T, and is an odd function of t. It can be written as a sine series

$$\therefore \; E(t) - \frac{2\pi t}{T} = \sum_{j=1}^{\infty} b_j \sin\left(\frac{2\pi t}{T} j\right)$$

Where $b_j = \frac{2}{T} \int_{-T/2}^{T/2} \left(E(t) - \frac{2\pi t}{T}\right) \sin\left(\frac{2\pi t j}{T}\right) dt$

Integrating by parts, using periodicity of $E(t) - \frac{2\pi t}{T}$ w/ period T

$$b_j = \frac{1}{\pi j} \int_{-T/2}^{T/2} \cos\left(\frac{2\pi j t}{T}\right) d\left(E - \frac{2\pi t}{T}\right)$$

$$= \frac{1}{\pi j} \int_{-T/2}^{T/2} \cos\left(\frac{2\pi j t}{T}\right) \frac{dE}{dt}\, dt$$

$$= \frac{2}{\pi j} \int_{0}^{T/2} \cos\left(\frac{2\pi j t}{T}\right) \frac{dE}{dt}\, dt$$

Since $E(t)$ is strictly increasing, we may take E as a new variable.

At $t=0$, $E(t)=0$; at $t = T/2$ the planet is at aphelion and $E(t) = \pi$ and $2\pi t/T = E - e\sin E$

(9.61) $$\therefore \; b_j = \frac{2}{\pi j} \int_0^{\pi} \cos(jE - je\sin E)\, dE = \frac{2}{j} J(je)$$ (9.61)

and (9.59) follows.

Exercise: Prove that $\epsilon(t)$ is a C^∞ function which is periodic. Deduce that the series (9.59) converges rapidly.
Indeed, Show that for any k, $\exists$ a constant C_k such that $|b_j| \le \frac{C_k}{j^k}$ for all j (9.62)

(Do this for a general C^∞ periodic function by integration by parts)

Exercise: (a) Prove $e \sin \epsilon = \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin(2\pi n t/T)$

Exercise: Let $0<e<1$. Show that the function $\phi: \mathbb{R} \to \mathbb{R}$ given by $\phi(y) = y - e \sinh y$ has 2 critical values. Sketch the graph of ϕ and investigate the number of solutions of $\phi(y) = c$ for various c.

(b) Show that for certain purely imaginary values of t the Kepler Eqn (9.56) has one purely imaginary solution; whereas for a range of purely imaginary values of t, there are three branches of solutions.

What conclusions can you draw about the analyticity of $\epsilon(t)$?
Will it be entire? If not, can you give estimates on the radius of convergence of the powerseries $\epsilon(t)$ around the origin?
First of all, is $\epsilon(t)$ holomorphic at all? A detailed

analysis of the ~~sa~~ complex singularities of $\epsilon(t)$ may be found on p ~~558~~ 558ff of Watson's treatise [ ]. (See p562 for the loc. of singularities

Exercise: As an illustration of the rapidity of convergence prove that b_j in (9.61) ~~is~~ satisfies the estimate

$$b_j < 2\mu / j^2$$

$$\mu = \sup_{\theta\in[0,\pi]} \frac{e\sin\theta}{(1-e\cos\theta)^3}$$

First integrate by parts and prove

$$b_j = \frac{2}{\pi j^2}\int_0^\pi \frac{e\sin\epsilon}{(1-e\cos\epsilon)^3}\sin\nu\phi\,d\phi\ ; \text{ where } \epsilon - e\sin\epsilon = \phi.$$

(P.M. Fitzpatrick p98 problem 4)

Cor: Series (9.59) converges at least as rapidly as $\sum 1/n^2$

Exercise: Prove the relation between the true anomaly θ and eccentric anomaly ϵ:

$$1+e\cos\theta = (1-e^2)/(1-e\cos\epsilon) \qquad (9.57)'$$

hence and ~~as~~ prove

$$\frac{\rho}{a} = 1 - e\cos\epsilon = \frac{d}{d\epsilon}(\epsilon - e\,\cancel{\cos}\sin\epsilon) \qquad (9.63)$$

* Prove that ρ is an even function of t. Deduce the

~~BUV~~ $\cancel{\epsilon = 2\pi t/T + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne)\sin\left(2\pi n\frac{t}{T}\right)}$

(* see p66)

$\therefore$ $\cancel{e\sin\epsilon = \epsilon - 2\pi t/T = \sum_{1}^{\infty}\frac{2}{n}J_n(ne)\sin\left(2\pi n\frac{t}{T}\right)}$

Fourier Expansion:

$$\frac{\rho}{a} = 1+\frac{e^2}{2} - 2e\sum_{n=1}^{\infty}\frac{J_n'(ne)}{n}\cos 2\pi t n/T. \qquad (9.64)$$

Proof: Referring to the derivation of (9.57)′ on p60 we get $\frac{\rho}{a} = 1 - e\cos\epsilon$.

Now, by Kepler's 2nd law, ρ is an even function of t if the planet is at the perihelion at time $t=0$; ρ is obviously periodic with period T

$$\therefore \frac{\rho}{a} = \sum_{j=0}^{\infty} A_j \cos\frac{2\pi t j}{T}$$

For $j \geq 1$,

$$A_j = \frac{2}{T}\int_{-T/2}^{T/2} \frac{\rho}{a}\cos\left(\frac{2\pi t j}{T}\right)dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} (1 - e(\cos\epsilon))\cos\frac{2\pi t j}{T}\,dt$$

$$= \frac{-2}{T}\int_{-T/2}^{T/2} e\cos\epsilon\cos\frac{2\pi t j}{T}\,dt$$

integrating by parts,

$$= \frac{1}{\pi j}\int_{-T/2}^{T/2} \cancel{-e\cos\epsilon}\,(e\sin\epsilon)\left(\sin\frac{2\pi t j}{T}\right)\frac{d\epsilon}{dt}\,dt$$

$$= \frac{1}{\pi j}\int_{-\pi}^{\pi} (e\sin\epsilon)\left(\sin\frac{2\pi j t}{T}\right)d\epsilon$$

$$= \frac{2}{\pi j}\int_{0}^{\pi} e\sin\epsilon\,\sin(j\epsilon - je\sin\epsilon)\,d\epsilon$$

$$= \frac{2e}{j}\left(-J'(je)\right)$$

For $j=0$, $A_0 = \frac{1}{T}\int_{-T/2}^{T/2} \rho/a\,dt = \frac{1}{T}\int_{-T/2}^{T/2}(1 - e\cos\epsilon)\,dt$

Now, differentiating Kepler's Eq^n, $\frac{dt}{d\epsilon} = \frac{T}{2\pi}(1 - e\cos\epsilon)$

$$\therefore\ A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - e\cos\epsilon)^2 d\epsilon$$

$$= 1 + \tfrac{1}{2}e^2 \quad \text{proving (9·64).}$$

prove that $A_n = \frac{2e}{\pi n^2}\int_0^{\pi}(\epsilon - \cos\epsilon)(1 - e\cos\epsilon)^{-3}\cos n\phi\, d\phi$

$$\phi = \epsilon - e\sin\epsilon$$

Deduce that the series for ρ/a converges at least as rapidly as $\Sigma\, 1/n^2$. (Fitzpatrick, p98)

We now proceed to determine the true anomaly θ as a function of time. This is non-trivial. We follow the procedure of W. Bessel (1826) see G.N. Watson p 554.

We now make the convention that $\theta(t)$ varies continuously as t increases so that at the perihelion $\theta(0) = 0$, $\theta(T) = 2\pi$, $\theta(2T) = 4\pi$, ... etc; After an elapse of time T, θ changes additively by 2π.

$\theta(t) - 2\pi t/T$ is therefore periodic of period T. This function is an odd function and A_0

$$\theta(t) - \frac{2\pi t}{T} = \sum_{n=1}^{\infty} c_n \sin\frac{2\pi t n}{T} \qquad (9{\cdot}65)$$

We now proceed to determine c_n.

$$\cancel{C_n = \frac{2}{T}\int_0^{\pi}\left(\theta - \frac{2\pi t}{T}\right)\sin\frac{2\pi t n}{T}\,dt}$$

$$C_n = \frac{2}{T}\int_0^{T}\left(\theta(t) - \frac{2\pi t}{T}\right)\sin\frac{2\pi t n}{T}\,dt$$

$$= \frac{2}{T}\int_0^{T}\left(\theta - \frac{2\pi t}{T}\right)\frac{d}{dt}\left(\frac{-\cos 2\pi t n/T}{2\pi n/T}\right)$$

$$= \frac{1}{\pi n}\int_0^{T}\frac{d\theta}{dt}\cos\left(\frac{2\pi t n}{T}\right)dt$$

$$= \frac{1}{\pi n}\int_0^{2\pi}\frac{d\theta}{dE}\cos(nE - e\sin E)\,\cancel{d\theta}\,dE$$

$$= \frac{1}{\pi n}\int_0^{2\pi}\frac{\left(\sqrt{1-e^2}\right)\left(\cos(nE - e\sin E)\right)}{1 - e\cos E}\,dE$$

Claim:
$$\left.\begin{aligned}\frac{\sqrt{1-e^2}}{1-e\cos E} &= 1 + 2f\cos E + 2f^2\cos 2E + 2f^3\cos 3E + \dots \\ \text{where } f &= e/\left(1+\sqrt{1-e^2}\right)\end{aligned}\right\} \quad (9.66)$$

Assume the claim

$$\therefore C_n = \frac{2}{\pi n}\int_0^{\pi}\left[\sum_{j=1}^{\infty} 2f^j\cos jE\cos(nE - e\sin E) + \cos(nE - e\sin E)\right]dE$$

$$= \frac{2}{n}J_n(ne) + \sum_{j=1}^{\infty}\frac{2f^j}{\pi n}\int_0^{\pi}\left[\cos(nE + jE - e\sin E) + \cos(nE - jE - e\sin E)\right]dE$$

$$= \frac{2}{n}\left[J_n(ne) + \sum_{j=1}^{\infty} f^j\left(J_{n-j}(ne) + J_{n+j}(ne)\right)\right]$$

So, $\theta = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{2\pi t n}{T}\right)$

Where $c_n = \frac{2}{n}\left[J_n(ne) + \sum_{j=1}^{\infty}\left(J_{n-j}(ne) + J_{n+j}(ne)\right)\right]$

(see G.N. Watson p554 from which this has been adapted)

Exercise: Prove the claim (9·66)

proof: $1/(1-e\cos z)$ is 2π periodic even function and so may be developed into a Fourier Cosine Series

$$\frac{1}{1-e\cos t} = \sum_{n=0}^{\infty} C_n \cos nt$$

$$C_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos nt \, dt}{1-e\cos nt} \; ; \quad \text{put } z = e^{it}, \quad dt = \frac{1}{iz}dz$$

$$\therefore C_n = \frac{1}{2i\pi}\oint_{|z|=1} \frac{z^n + z^{-n}}{1-\frac{e}{2}(z+1/z)}\frac{dz}{z}$$

$$= \text{sum of Residues of } \frac{z^{2n}+1}{\left(-(z^2+1)e+2z\right)z^n} = F(z)$$

inside the circle $|z|=1$.

Origin is a pole of order n and

$e(z^2+1) - 2z = 0$ has roots $\frac{1}{e}(1 \pm \sqrt{1-e^2})$

$\alpha = \frac{1}{e}(1-\sqrt{1-e^2})$; $\beta = \frac{1}{e}(1+\sqrt{1-e^2})$

$\alpha\beta = 1$,

(9.67) For an elegant approach avoiding Contour $\int$, see J-L-Lagrange [ ] pp 328-330.

$$\operatorname*{Res}_{z=\alpha} F(z) = \operatorname*{Res}_{z=\alpha} \frac{z^{2n}+1}{-z^n e(z-\alpha)(z-\beta)}$$

$$= \frac{\alpha^{2n}+1}{-\alpha^n e(\alpha-\beta)}$$

$$\operatorname*{Res}_{z=0} F(z) = \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1} \frac{z^{2n}+1}{-e(z-\alpha)(z-\beta)}\Bigg|_{z=0}$$

$$= \frac{-1}{e(n-1)!}\left(\frac{d}{dz}\right)^{n} \frac{1}{(z-\alpha)(z-\beta)}\Bigg|_{z=0}$$

$$= \frac{-1}{e(n-1)!}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-1)^r(-1)^{n-1-r}\, r!\,(n-r-1)!}{(-\alpha)^{r+1}(-\beta)^{n-r}}$$

$$= -\frac{1}{e}\sum_{r=0}^{n-1}\frac{1}{\alpha^{r+1}\beta^{n-r}} = -\frac{1}{e}\sum_{r=0}^{n-1}\frac{\alpha^{n-1}}{\alpha^{2r}}$$

$$= \frac{-1}{e(\alpha^2-1)}\,\frac{\alpha^{2n}-1}{\alpha^{n-1}}$$

$$\therefore \text{Sum of Residues} = \frac{-1}{e}\left\{\frac{\alpha^{2n}-1}{\alpha^{n-1}(\alpha^2-1)} + \frac{\alpha^{2n}+1}{\alpha^n(\alpha-1/\alpha)}\right\}$$

$$= \frac{-1}{e}\cdot\frac{1}{\alpha^{n-1}(\alpha^2-1)}\cdot 2\alpha^{2n}$$

$$= \frac{-2}{e}\,\frac{\alpha^{n+1}}{\alpha^2-1} \qquad \text{A little algebra gives}$$

$$= \alpha^n/\sqrt{1-e^2}$$

$$\text{and } \alpha = \frac{e}{1+\sqrt{1-e^2}}$$

The Claim is established.

Exercise: Examine the case $n=0$.

Exercise: Establish the following relations:

$$\rho\cos\theta/a = -\tfrac{3}{2}e + \sum_{n=1}^{\infty} \tfrac{2}{n} J_n'(ne)\cos\frac{2\pi t n}{T}$$

$$\rho\sin\theta/a = \frac{\sqrt{1-e^2}}{e}\sum_{n=1}^{\infty} \frac{2}{n} J_n(ne)\sin\frac{2\pi t n}{T} \qquad (9.67)'$$

$$\cos\epsilon = -\tfrac{1}{2}e + \sum_{n=1}^{\infty} \tfrac{2}{n} J_n'(ne)\cos\frac{2\pi t n}{T}$$

If m is a positive integer

$$\cos m\epsilon = m\sum_{n=1}^{\infty}\frac{1}{n}\left(J_{n-m}(ne) - J_{n+m}(ne)\right)\cdot \cos 2\pi t n/T$$

$$\sin m\epsilon = m\sum_{n=1}^{\infty}\frac{1}{n}\left(J_{n-m}(ne) + J_{n+m}(ne)\right)\sin\frac{2\pi t n}{T} \qquad (9.67)''$$

$$\frac{a}{\rho} = 1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos 2\pi t n/T.$$

$$\cos\theta = -e + \left(\frac{1-e^2}{e}\right)\sum_{n=1}^{\infty} 2J_n(ne)\cos\frac{2\pi t n}{T}$$

$$\sin\theta = \sqrt{1-e^2}\sum_{n=1}^{\infty} 2J_n'(ne)\sin 2\pi t n/T. \qquad (9.67)'''$$

$$\frac{a^2}{\rho^2}\cos\theta = \sum_{n=1}^{\infty} 2nJ_n'(ne)\cos\frac{2\pi t n}{T}$$

$$\frac{a^2}{\rho^2}\sin\theta = \frac{\sqrt{1-e^2}}{e}\sum_{n=1}^{\infty} 2nJ_n(ne)\sin\frac{2\pi t n}{T}.$$

The above are from G.N. Watson pp 554-555

On putting $t = 0, \frac{T}{2}$ or T we may derive special formulae such as

$$\frac{1}{2} + \frac{e}{4} = \sum_{n=1}^{\infty}\frac{J_n'(ne)}{n}.$$ Series of the type described (9.67) are known as Kapteyn Series.

For details on such expansions the authoritative account of G.N. Watson [ ] may be consulted (Chpt 17)

See also Chapter 4 of Plummer's Introductory treatise on Dynamical Astronomy for a good but less elaborate discussion.
and p 328 of J.L. Lagrange [ ].

II: Solving Kepler's Equation (Method of J.L. Lagrange)

Consider the Kapteyn Series for $\epsilon(t)$:

$$\epsilon(t) = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin \frac{2\pi t n}{T}$$

If one expands $J_n(ne)$ as a power series and rearranges the terms into a power series in e:

$$\sum_{j=0}^{\infty} a_j(t)\, e^j$$

the radius of convergence of the resulting series is approximately $0.6627\ldots$
whereas the domain of convergence of the original Kapteyn series is larger.

We now prove a general theorem on theory of functions due to Lagrange.

We recall the inverse function theorem:

If $f: \Omega_1 \to \Omega_2$ is holomorphic and $f'(z_0) \neq 0$ for some $z_0 \in \Omega_1$,

then f is a local analytic diffeomorphism of a neighbourhood of z_0 onto a neighborhood of $w_0 = f(z_0)$. In principle the power series

$$w = f(z) = w_0 + \sum_{j=1}^{n} a_j (z-z_0)^j \qquad (*)$$

may be inverted:

$$z = z_0 + \sum b_j (w-w_0)^j \qquad (**)$$

Substituting $(**)$ into $(*)$ the coefficients $\{b_j\}$ may be recursively computed. However we may employ Cauchy integral formula to give an elegant expression for b_j.

We begin with a proof of the inverse function theorem:

Let $f(z)$ be a non constant holomorphic function and $f(z_0) = w_0$. Since the zeros of $f(z) - w_0$ are isolated,

Imp. $\{$ we can find a circle C centred around z_0 not enclosing any other zero of $f(z) - w_0$. and not passing through any zero of $f(z) - w_0$.

$$\therefore \quad |f(z) - w_0| \geq \delta > 0 \quad \text{for all } z \in C$$

Now, if $|w - w_0| \leq \delta/2$ (Enough to take $|w - w_0| < \delta$)

$$|f(z) - w| \geq |f(z) - w_0| - |w_0 - w| \geq \delta/2 \quad (\gneq 0)$$

So C does not pass through any zero of $f(z) - w$.

The integral

$$\frac{1}{2\pi i}\oint_C \frac{f'(\zeta)\,d\zeta}{f(\zeta) - w}$$ is continuous (infact analytic) function of w and is integer valued for $|w - w_0| \leq \delta/2$. (for $|w - w_0| < \delta$)

So it is constant = Multiplicity of the zero z_0 of $f(z) - w_0$.

If $f'(z_0) \neq 0$ then z_0 is a simple zero and this integer $= 1$ ($|w-w_0| < \delta$ will do)

Thus, for each w in $\{|w-w_0| < \delta/2\} = B$

$\exists!$ point z inside C such that

$f(z) = w$, Moreover, the solution of this equation is given by

$$z = f^{-1}(w) = \frac{1}{2\pi i} \oint_C \zeta \frac{f'(\zeta)\, d\zeta}{f(\zeta) - w}$$

and $f^{-1}(w)$ is analytic in w.

Suppose ψ is analytic ~~on $|w-w_0| < \delta/2$~~ inside C

$$\psi(z) = \psi \circ f^{-1}(w) = \frac{1}{2\pi i} \oint_C \psi(\zeta) \frac{f'(\zeta)\, d\zeta}{(f(\zeta) - w)}$$

$$\frac{d\psi}{dw} = \frac{1}{2\pi i} \oint_C \frac{\psi(\zeta) f'(\zeta)\, d\zeta}{(f(\zeta) - w)^2}$$

$$= \frac{-1}{2\pi i} \oint_C \psi(\zeta) \frac{d}{d\zeta}\left(\frac{1}{f(\zeta) - w}\right) d\zeta$$

$$= \frac{1}{2\pi i} \oint_C \frac{\psi'(\zeta)\, d\zeta}{f(\zeta) - w}$$

$$= \frac{1}{2\pi i} \oint_C \frac{\psi'(\zeta)\, d\zeta}{(f(\zeta) - w_0 + w_0 - w)}$$

$$= \frac{1}{2\pi i} \oint_C \sum_{k=0}^{\infty} \frac{(w-w_0)^k \psi'(\zeta)\, d\zeta}{(f(\zeta) - w_0)^{k+1}}$$

If $|w-w_0| < \delta$ then

now, if $|w-w_0| < \delta/3$ then $|w-w_0|\,|f(\zeta)-w_0|^{-1} < 1$

$|f(\zeta) - w_0| \geq \cancel{2/3}\, \delta$ and the geometric series $\left(1 + \frac{w-w_0}{f(\xi)-w_0}\right)^{-1}$ converges.

$$\therefore \frac{d\psi}{dw} = \sum_{k=0}^{\infty} (w-w_0)^k \frac{1}{2\pi i} \oint_C \frac{\psi'(\xi)d\xi}{(f(\xi)-w_0)^{k+1}}$$

now z_0 is a simple zero of $f(\xi)-w_0$ and so

$$\cancel{f(\xi)-w_0 = (z-z_0)\phi(z)}$$

$$\cancel{\therefore \frac{d\psi}{dw} = \sum_{k=0}^{\infty} (w-w_0)^k \frac{1}{2\pi i} \oint_C \frac{\psi'(\xi)d\xi}{(\xi-z_0)^{k+1}}}$$

Now put $\phi(z) = \frac{z-z_0}{f(z)-w_0}$

which is holomorphic since z_0 is a simple zero of $f(z)-w_0$.

$$\therefore \frac{d\psi}{dw} = \sum_{k=0}^{\infty} (w-w_0)^k \frac{1}{2\pi i} \oint_C \frac{\psi'(\xi)(\phi(\xi))^{k+1}d\xi}{(\xi-z_0)^{k+1}}$$

(okay) $$= \sum_{k=0}^{\infty} \frac{(w-w_0)^k}{k!} \left(\frac{d}{dz}\right)^k \left(\psi'(z)\,\phi(z)^{k+1}\right)\Big|_{z=z_0}$$

$$\therefore \psi = \psi(z_0) + \sum_{k=1}^{\infty} \frac{(w-w_0)^k}{k!} \left\{\left(\frac{d}{dz}\right)^{k-1} \psi'(z)(\phi(z))^k\right\}_{z=z_0} \quad (9.68)$$

which is Lagrange's formula for $\psi \circ f^{-1}(w)$

and $z = f^{-1}(w)$

$$(9.68)' \quad \begin{cases} z_0 + \phi(z)(f(z)-w_0) = z \\ \text{i.e. } z = z_0 + \phi(z)(w-w_0). \end{cases}$$

We have not been particularly careful about the book keeping of domains.

Let us summarize this derivation as a theorem:

Theorem 9.17: (The Lagrange Expansion Theorem)

Suppose $f(z)$ is holomorphic in $\{|z-z_0|<R\}$, continuous on $\{|z-z_0|\le R\}$, such that

(i) $f'(z_0)\neq 0$, $w_0=f(z_0)$ say

(ii) $|z-z_0|=R$ does not enclose any zero of $f(z)=w_0$ except z_0

(iii) $\psi(z)$ is analytic in $\{|z-z_0|<R\}$

$\exists\,\delta>0$ such that ~~on~~ $\{$ for each w with $|w-w_0|<\delta$, $\exists!$ z inside C satisfying

$$f(z)=w$$

The power series expansion for the composite $\psi\circ f^{-1}$:

$$\psi(z)=\psi(f^{-1}(w)) \text{ is given by}$$

$$(9.68)\qquad =\psi(z_0)+\sum_{k=1}^{\infty}\frac{(w-w_0)^k}{k!}\left\{\left(\frac{d}{dz}\right)^{k-1}\psi'(z)(\phi(z))^k\right\}_{z=z_0}$$

where $\phi(z)$ is the unique holomorphic function around z_0 satisfying:

$$z=z_0+\phi(z)(w-w_0)$$

Let us now apply the Lagrange Expansion Thm to solve the Kepler-problem:

$$\epsilon=\frac{2\pi t}{T}+e\sin\epsilon$$

We take $2\pi t/T$ to be const $=z_0$ (assume $t\neq 0, T/2, T$)

and $z=\epsilon$

* ~~$f' \neq 0$ when ϵ real~~

and w is the variable e, ~~so $f(\epsilon) = 1 - e\sin\epsilon$~~

so when $z = z_0$, $w = w_0 = 0$. Lagrange's Expⁿ gives

$$\epsilon = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} \frac{e^n}{n!} \left\{ \left(\frac{d}{d\epsilon}\right)^{n-1} \sin^n \epsilon \right\}_{\epsilon = \frac{2\pi t}{T}}$$

$$\therefore \epsilon = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} \frac{e^n}{n!} \left(\frac{d}{d\mu}\right)^{n-1} \sin^n \mu \qquad (9.69)$$

where μ is the Mean Anomaly $\frac{2\pi t}{T}$. Moreover (see [ ] Henrici p120)

$$D_\mu^{n-1} \sin^n \mu = 2^{1-n} \sum_{k=0}^{[n/2]} \binom{n}{k} (n-2k)^{n-1} \sin\left[(n-2k)\, 2\pi t/T\right]$$

Remarks: (i) Determining the radius of convergence of this series expansion is highly nontrivial.

(**) (ii) Another proof* may be found in Appendix 3 of P.M. Fitzpatrick but the Exercise 1 on p97 is Wrong. Referring to the exercise, $r/M(r)$ may not pass through a maximum value but may steadily decrease from ∞ to 0. E.g. $h(z) = z^2$. Take $a = 0$.

However See E. Goursat [ ] p108;
* Due to Hermite according to ~~Gourat~~ Goursat [ ].

(iii) Series (9.69) converges for $e < 0.6627\ldots$ and as such cannot be used for Comets with highly eccentric orbits.

* Exercise: Identify $f(\epsilon)$. Ans: $f(\epsilon) = e$

Diff the Kepler Eqⁿ, $\frac{de}{d\epsilon} \sin\epsilon = 1 - e\cos\epsilon$

when $\epsilon = \frac{2\pi t}{T}$, we see that $\frac{de}{d\epsilon} \neq 0$ so the conditions of applicability of Lagrange's thm are satisfied.

(**) The symbolic method of Laplace may be found in §189, p404 of Goursat, vol I.

Exercise: Apply Lagrange's Expansion theorem to the root of $w = 2(z-\mu)/(z^2-1)$ which reduces to

μ when $w=0$. Deduce that

$$\frac{1}{\sqrt{1-2\mu w+w^2}} = \sum_{n=0}^{\infty} w^n P_n(\mu) \qquad (9.70)$$

$$P_n(\mu) = \frac{1}{2^n n!}\left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n \qquad (9.71)$$

(Rodrigues Formula)

$P_n(\mu)$ = nth Legendre Polynomial. (E.T Copson p151)

We have derived the important

~~Theorem 9.18: (Rodrigues Formula) The generating function for~~ $\cancel{1/\sqrt{1-2\mu w+w^2}}$ ~~is~~

$$\cancel{\sum_{n=0}^{\infty} \frac{w^n}{2^n n!} D^n(\mu^2-1)^n}$$

Solution: The relevant root is given by

$$z = \frac{1}{w}\left\{1 - \sqrt{1-2\mu w+w^2}\right\}$$

Rationalizing $z = \dfrac{2\mu + w}{1+\sqrt{1-2\mu w+w^2}}$ which reduces to μ when $w=0$.

$$\phi(z) = \frac{z-z_0}{w-w_0} = \frac{z-\mu}{w} = \frac{z^2-1}{2}$$

$$\therefore z(w) = \mu + \sum_{k=\phi}^{\infty} \frac{w^k}{k!\,2^k} D^{k-1}(z^2-1)^k \Big|_{z=\mu}$$

$$\therefore z(w) = \mu + \sum_{k=1}^{\infty} \frac{w^k}{k!\,2^k}\left(\frac{d}{d\mu}\right)^{k-1}(\mu^2-1)^k.$$

$$\therefore 1 - \sqrt{1-2\mu w+w^2} = \mu w + \sum_{k=1}^{\infty} \frac{w^{k+1}}{k!\,2^k}\left(\frac{d}{d\mu}\right)^{k-1}(\mu^2-1)^k.$$

Diff w.r.t μ and we get Rodrigues' formula.

III : Solving Kepler's Eq^ns (Method of Cauchy)
See H. C. Plummer p41 § 43.

IV : Newton's graphical Method: See H. C. Plummer p 25, § 28.

Exercise: Suppose $f(z) - w = 0$ has a unique simple zero z_w inside the contour γ (for each w in a range $|w - w_0| < \delta$, say) and $\psi(z)$ is holomorphic within and on γ ~~in a nbd of~~ then

$$\psi(z_w) = \frac{1}{2\pi i} \oint_\gamma \frac{\psi(z) f'(z) dz}{f(z) - w} \qquad (9.72)$$

(Assume γ is a simple closed curve)

Hint: Deform the contour to a circle C with centre at z_w.

$$f(z) - w = (z - z_w) g(z)$$

$$g(z_w) = f'(z_w)$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{\psi(z) f'(z) dz}{f(z) - w} = \frac{1}{2\pi i} \oint_C \frac{\psi(z) f'(z) dz}{(z - z_w) g(z)}$$

$$= \psi(z_w) f'(z_w) / g(z_w) = \psi(z_w)$$

by Cauchy integral theorem.

We now recall Rouché's theorem from Complex Analysis:

Thm (9.18) (Rouché - 1861) If $f(z)$, $g(z)$ are two analytic functions in a simply connected region Ω and

$|g(\xi)| < |f(\xi)|$ for all $\xi \in \gamma$, a simple closed curve lying in Ω then

$f+g$ and f have the same number of zeros inside γ

pf: $\displaystyle\operatorname*{Inf}_{\xi\in\gamma} \{ |f(\xi)| - |g(\xi)| \} = \mu \gneq 0$

For $0 \le \lambda \le 1$ consider the function

$f(\xi) + \lambda g(\xi)$ and let us estimate its absolute value:

$$|f(\xi) + \lambda g(\xi)| \geq |f(\xi)| - \lambda |g(\xi)|$$

$$\cancel{\geq |f(\xi)| + \lambda |g(\xi)|}$$

$$\geq |f(\xi)| - |g(\xi)| \gneq 0$$

So the integral

$$I(\lambda) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(\xi) + \lambda g'(\xi)}{f(\xi) + \lambda g(\xi)} \, d\xi$$

is continuous and integer valued

$\therefore I(0) = I(1)$. The proof is complete once we recall the argument principle.

Exercises: Show that the equation

$$z e^{a-z} = 1, \quad a > 1$$

has precisely one root in the circle $|z| \le 1$.

Explain why this root must be real positive.

(Z. Nehari p 135) More gen, show that if $k > 1$, $z^k e^{a-z} = 1$ has all its k roots in $|z| < 1$ (Copson p 151)

(b) Show that all five roots of $z^5 + 15z + 1$ lie inside $|z| = 2$ but only one lies inside $|z| < \frac{3}{2}$

(c) Show that the solution of the equation

$$z = a + we^z$$

which reduces to a when $w = 0$ is

$$z = a + we^a + \frac{w^2}{2!} 2e^{2a} + \dots + \frac{w^n}{n!} n^{n-1} e^{na} + \dots$$

(Exp^n is valid if $|w| < e^{-1 - \operatorname{Re} a}$)

Determine the radius of convergence of the series.

(d) Let z be the root of $z^3 - z^2 - w = 0$ which $\to 1$ as $w \to 0$

Show that

$$z = 1 + \sum_{1}^{\infty} \frac{(-1)^{n-1}(3n-2)!\, w^n}{n!\,(2n-1)!}$$

(p228 of Polya-Latta) Complex variables; John Wiley 1974.

for $|w| < 4/27$

Determine the critical values of $f(z) = z^3 - z^2$ and examine the relevance to the above expansion.

(e) Show that the solution of the cubic equation

$z^3 - 3z - w = 0$ which tends to zero when $w \to 0$ is given by $z = \frac{1}{3} w \, F\left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; -w^2/4\right)$

F is the hypergeometric function. More generally expand in powers of w the solution of the trinomial equation $z^n + nz - w = 0$ which tends to zero with w.

Note: (d) - (e) are from P. Henrici, vol I, p118

For more examples see p121 of Henrici - P [ ] ~~p22~~

The cardano formulae may be recovered (see p121 of [ ])

Consider the problem of solving the equation

$$z = z_0 + (w - w_0)\phi(z) \qquad (9.73)$$

for z, given w, such that $w = w_0$ when $z = z_0$; inside a given contour γ enclosing z_0.

Here $\phi(z)$ is a holomorphic function in a simply connected region Ω containing γ.

Write $$M = \sup_{z \in \gamma} \left| \frac{\phi(z)}{z - z_0} \right| \qquad (9.74)$$

Then $$|w - w_0| < \frac{1}{M} \qquad (9.75)$$

implies

$$|(w - w_0)\phi(z)| = \left|(w - w_0)\frac{\phi(z)}{z - z_0}(z - z_0)\right| < |z - z_0|$$

By Rouché's theorem the functions

$$(w - w_0)\phi(z) + (z - z_0)$$

and $z - z_0$ have the same number of zeros inside γ and so (9.73) is uniquely solvable for $z = z_w$ inside γ for each w satisfying (9.75)

Theorem 9.19: Given a simple closed curve enclosing z_0 and a holomorphic function ϕ defined in a simply connected Ω containing γ,

For each w satisfying $|w - w_0| < 1/M$ there exists a unique $z = z_w$ inside γ such that

$$z_w = z_0 + (w - w_0)\phi(z).$$

Here $$M = \sup_{\gamma} \left| \frac{\phi(z)}{z - z_0} \right|.$$

$$\psi(z_w) = \psi(z_0) + \sum_{k=1}^{\infty} \frac{(w-w_0)^k}{k!} \left(\frac{d}{dz_0}\right)^{k-1} \left(\psi'(z_0)(\phi(z_0))^k\right)$$

holds for $|w-w_0| < 1/M$. for any holomorphic ψ in Ω

pf: The existence of z_w is established using Rouche's theorem. The validity of Lagrange expansion can throughout (9·75) can be established by a critical examination of our earlier proof.

The geometric series on p75 converges in the region (9-75) because of the estimate (9·75)

Let us now reexamine the Kepler equation

$$\mu = \frac{2\pi t}{T} = \epsilon - e\sin\epsilon \qquad (9\cdot 76)$$

where $\epsilon = \epsilon(e,t) = \epsilon(e)$

supressing the t-dependence.

Take $z_0 = \frac{2\pi t}{T}$ real, $w_0 = 0$, $w = e$ and $\phi(z) = \sin z$ and γ is the circle $z = z_0 + Re^{i\theta}$

$$|\sin^2 z| = \cosh^2(R\sin\theta) - \cos^2(z_0 + R\cos\theta)$$

$$|\sin z| \leq \cosh R = \tfrac{1}{2}(e^R + e^{-R})$$

and $$M = \sup_{\gamma} \left|\frac{\phi(z)}{z-z_0}\right| = \frac{1}{2R}(e^R + e^{-R}) \qquad (9\cdot 77)$$

So the Lagrange expansion is valid for atleast

$$|e| < \frac{2R}{e^R + e^{-R}} = f(R) \qquad (9\cdot 78)$$

~~now~~ hitherto R was arbitrary: we now select R so that $f(R)$ is largest possible.

$$\psi'(R) = 2(e^R+e^{-R})^{-2}\left((e^R+e^{-R}) - R(e^R-e^{-R})\right)$$

$\psi'(R)=0$ gives the equation

$$(e^R+e^{-R}) = R(e^R-e^{-R}) \qquad (9\cdot79)$$

Exercise: Show that $(9\cdot 79)$ has a unique positive real root R_0. Determine the root using numerical techniques. Show that this corresponds to a ~~local~~ the global maximum of $\psi(R)$. Determine $\psi(R_0)$ and hence estimate the ~~optimal~~ radius of convergence.

Ans: The radius of convergence turns out to be $0\cdot 6627\ldots$ Caratheodory [ ] ascribes this to T. Stieltjes (p).

We now show that the bound obtained is optimal.

Take $z_0 = \pi/2$ and take $\operatorname{Arg} z = \pi/2$ on the circle so that $z = \frac{\pi}{2} + iR$ along γ.

~~$\sin z = \frac{1}{2i}(e^R - e^{-R})$, $\cos z =$~~

$$\sin z = \cos(iR) = \frac{1}{2}(e^R+e^{-R})$$

$$\cos z = \frac{1}{2i}(e^R - e^{-R})$$

$$w(z) = \frac{z-z_0}{\sin z} = \frac{2iR}{e^R+e^{-R}} = i\psi(R)$$

Differentiating the ~~implicit~~ Eq^n $z - z_0 - w\sin z = 0$

$$1 - w'(z)\sin z - w\cos z = 0$$

$$-w'(z)\cosh R = w\cos z - 1 = \frac{-1}{\cosh R}\psi'(R)$$

so for $R = R_0$, $\psi'(R_0) = 0$

and $w'(\frac{\pi}{2} + iR_0) = 0$

The inverse function has a singularity at $\frac{\pi}{2}$

$$w(\pi/2 + iR_0) = i\psi(R_0)$$

and so the radius of convergence = dist to the nearest singularity $= \psi(R_0)$.

Notes and comments: The Literature on Kepler's Equation is vast and in this connection Moulton [ ] is often quoted (see Moulton [ ], p). However, information seems quite scattered. Here we are not interested in the technicalities of orbit computation but merely ~~focus~~ focus on the underlying mathematical analysis in the plane. The above discussion is adapted from C. Caratheodory [ ].

Comments on p 108 of Goursat [ ] ~~is~~ are important (See also P.M. Fitzpatrick, Ex 1 p 97)

Exercises have been taken from the classic texts of E.T. Copson [ ], ~~&~~. The first volume of P. Henrici's two volume work, is a gold mine of exercises, references, ~~and~~ interesting alternate proofs and most importantly details of special functions. For the symbolic treatment of Lagrange Expansion theorem (also attributed to Bürman) see p55 of [ ].

However, G.N. Watson treatise [ ] has remained one of the authoritative accounts on Bessels function.

It would be interesting to see an account of the development of complex analysis inspired by the Kepler equation.

§9.7 Topology of the Kepler Problem.

With the above title, an article of W. Kaplan appeared in American Math. Monthly (1942) and so the following is a summary of it. See also Boccaletti - Puccaco [ ], but the latter has left a few points to be clarified concerning the Laplace - Runge - Lenz vector.

In what follows we assume that the angular momentum vector $\vec{c}$ given by (9.14) is parallel to $\hat{k}$ and so the motion takes place in the xy plane. The governing ODEs (9.7)₁ - (9.7)₂ assume the form

$$\frac{d^2x}{dt^2} = \frac{-\gamma x}{\|\vec{x}\|^3} \; ; \; \frac{d^2y}{dt^2} = \frac{-\gamma y}{\|\vec{x}\|^3} \qquad (9.80)$$

The integral of energy is (taking $\gamma = 1$ WLOG)

$$u^2 + v^2 = 2E + 2/\rho \qquad (9.81)$$

where $u = \dot{x}$, $v = \dot{y}$ and $\rho^2 = x^2 + y^2$

Momentum equation reads

$$xv - yu = c \qquad (9.82)$$

Thm 9.20: The intersection of (9.81) - (9.82) is real if and only if $c^2 \leq -1/2E$ for $E < 0$ (9.83)

proof: $c^2 = (xv - yu)^2 = x^2v^2 + y^2u^2 - 2xyuv$

$$= (x^2+y^2)(u^2+v^2) - (x^2u^2 + y^2v^2 + 2xyuv)$$
$$\leq \rho^2 (2E + 2/\rho) = 2E(\rho^2 - \rho/_{2E})$$
$$= 2E(\rho + 1/_{2E})^2 - 1/_{2E}$$
$$\leq -1/_{2E} \text{ if } E < 0.$$

λ

Remark: for the case $E>0$, we only get $C^2 \geqslant -1/2E$ which has no substance. in particular c may be zero and $E>0$ which is so when the initial velocity is high enough for one of the particles to escape to infinity.

In what follows we shall assume $E<0$ and restrict ~~ourself~~ ourselves to Elliptic motions only. (Blanket Assumption)

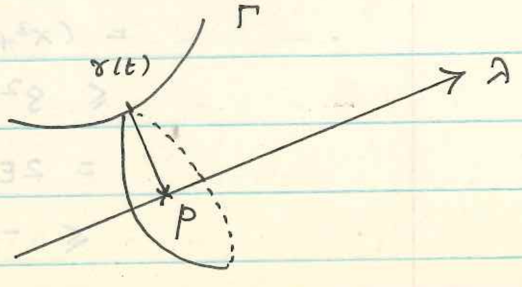
We now show that the surfaces $E=$ const and $c=$ const are homeomorphic to $\mathbb{R}^3$ minus a straight line.

We now recall the equation of a surface of revolution in $\mathbb{R}^3$

Consider a surface Σ obtained by revolving a curve $\gamma(t)$ about a line λ.

a plane orthogonal to λ meets the surface along a circle. This can be taken as the definition of a surface of revolution

Let P be the foot of the $\perp$ from $\gamma(t)$ to the line λ, and $\hat{e}$ be a unit vector along λ



The vector $(\gamma(t) - p) \times \hat{e}$ is orthogonal to both $\hat{e}$ and $\vec{\gamma}(t) - \vec{p}$ and of the same length as the latter

$\vec{p} + (\vec{\gamma}(t) - p)\cos\theta + ((\gamma(t) - p) \times \hat{e})\sin\theta$ is the equation of the circle of cross section, passing through $\gamma(t)$ with centre at p

and as t varies we get the parametrization for the surface of revolution

$$G(t,\theta) = (\vec{\gamma}(t) - p)\cos\theta + (\gamma(t) - p) \times \hat{e}\,\sin\theta + \vec{p} \qquad (9.84)$$

In particular, if λ is the x-axis and Γ is the graph $y = f(x)$ then

$\gamma(t) = (t, f(t))$, $P = (t, 0)$; $\hat{e} = \hat{i}$

$$G(x,\theta) = f(t)\cos\theta\,\hat{j} + t\hat{i} - k\,f(t)\sin\theta$$
$$= (t,\ f(t)\cos\theta,\ -f(t)\sin\theta);\quad (t = x)$$

In cartesian form,

$$y^2 + z^2 = (f(x))^2$$

That is to say $r = f(x)$, $\qquad (9.85)$

is the equation of surface of revolution of $y = f(x)$ about the x-axis.

<u>Exercise</u>: Show that if $\hat{u}, \hat{v}$ are two orthogonal unit vectors, (in $\mathbb{R}^n$) $\hat{u}\cos t + \hat{v}\sin t$ is a circle and that $\hat{u}\cos t + \hat{v}\cos t$ is an ellipse in $\mathbb{R}^n$ if ~~$\hat{u}$, $\hat{v}$ are not orthogonal but are~~ $\hat{u}, \hat{v}$ are unit vectors which are not orthogonal.

The above argument leading to (9.84) and finally (9.85) can be adapted to $\mathbb{R}^4$

We now consider a 2 dimensional surface Γ

First, the cross product of 2 vectors cannot be defined in $\mathbb{R}^4$ but the fact that $(\gamma(t)-p)\times\hat{e}$ in the last example is proportional to $\hat{k}$ could have been anticipated even without cross products!

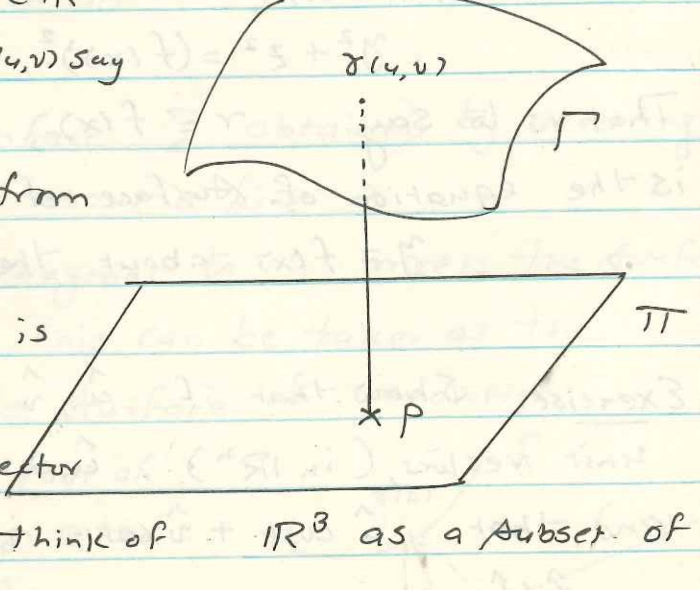
$\gamma(t)-p$ and $\hat{e}$ are both in the plane and so $\hat{k}$ is the unique vector (up to multiples) orthogonal to both.

~~Now, p + ($\gamma(u,v)$~~ Consider now a surface $\Gamma\subset\mathbb{R}^3$ parametrized by (u,v) and not meeting the plane $\Pi\subset\mathbb{R}^3$

Γ is given by $\gamma(u,v)$ say

Let P be the foot of the $\perp$ from $\gamma(u,v)$ to Π

Then $\gamma(u,v)-P$ is orthogonal to Π



and so is the vector

$\hat{f} = (0,0,0,1)$ (think of $\mathbb{R}^3$ as a subset of $\mathbb{R}^4$)

So the circle

$$C(u,v,\theta) = P + (\gamma(u,v)-P)\cos\theta + \hat{f}\sin\theta\ \|\gamma(u,v)-P\|$$

with centre P is orthogonal to Π.

as u, v, θ vary $C(u, v, \theta)$ defines a parametrized hypersurface in $\mathbb{R}^4$ – the surface of revolution of Γ about the plane Π.

<u>Exercise</u>: (i) Show that if a half plane Γ is rotated about the plane Π passing through $\partial\Gamma$ and orthogonal to Γ (in $\mathbb{R}^3$) the resulting 3-dimensional hypersurface in $\mathbb{R}^4$ is homeomorphic to $\mathbb{R}^3$ – straight line.

(ii) The paraboloid $z - c = x^2 + y^2$ in $\mathbb{R}^3$ minus its vertex is revolved around the plane $z = 0, w = 0$ in $\mathbb{R}^4$. Show that the resulting hypersurface is homeomorphic to $\mathbb{R}^3$ minus a line. $(c > 0)$

Will the surface of revolution be homeomorphic to $\mathbb{R}^3$ minus a line if the vertex were ~~included~~ removed?

<u>Ans</u>: $(x, y, \theta) \mapsto (x, y, (c + x^2 + y^2)\cos\theta, (c + x^2 + y^2)\sin\theta)$

For each (x_0, y_0), the projection
$$\left.\begin{array}{l} \\ ((c + x_0^2 + y_0^2)\cos\theta,\ (c + x_0^2 + y_0^2)\sin\theta) \end{array}\right\} *$$
is a circle since $c > 0$. So the space is homeomorphic to $\mathbb{R}^2 \times S^1$. Since $\mathbb{R}^2$ is homeomorphic to the half plane H, $\mathbb{R}^2 \times S^1$ is homeomorphic to the space $\mathbb{R}^3$ minus a line.

*: Write out a homeomorphism ($c > 0$ is crucial)

Suppose now Γ is given by $z = f(x,y)$, show that the Cartesian equation of the surf of revolution of Γ about xy plane is :

$$f(x,y) = \sqrt{z^2+w^2} \quad (\text{Assume } f > 0 \text{ everywhere})$$

i.e $f(x,y) = \varsigma$.

Let us now consider the system of first order equations:

$$\dot{x} = u, \quad \dot{y} = v; \quad \dot{u} = \frac{-\gamma x}{\varsigma^3}, \quad \dot{v} = -\frac{\gamma y}{\varsigma^3} \qquad (9.86)$$

The energy integral $u^2+v^2 = 2E + \frac{1}{\varsigma}$

Consider the inversion in the xy plane given by

$$u' = u, \quad v' = v; \quad x' = \frac{x}{x^2+y^2}, \quad y' = \frac{y}{x^2+y^2} \qquad (9.87)$$

Note that the points $x = 0 = y$ are excluded from the phase space (singularities of the Kepler problem) and the transformation (9.87) makes perfect sense.

Under the transformation (9.87), the energy equation transforms to

$$u'^2 + v'^2 = 2E + \varsigma' \qquad (9.88)$$

Which is an equation of a surface of revolution

~~since $f(x,y) = (2E + \varsigma')^2$~~

~~$= 4E^2$~~

Comparing it with the surface of revolution in Cartesian coordinates we see that the graph of

$$f(u',v') = u'^2 + v'^2 - 2E$$

is revolved about the $x'y'$ plane.

and the graph of f is a paraboloid and so (9.88) is homeomorphic to $\mathbb{R}^3$ minus a straight line.

We now show that the momentum equation is also a surface of revolution

<u>Theorem 9.21</u>: The level sets of Energy and momentum are both homeomorphic to $\mathbb{R}^3$ minus a line

They intersect along a two dimensional torus on which the trajectories live as closed curves.

The phase space thus foliates into invariant tori.

<u>proof</u>: Consider the momentum surface

$$xv - yu = c \qquad (9.89)$$

which is a quadric surface.

Perform rotations in the x-v plane and u-y planes and reduce (9.89) to

$$x^2 - v^2 - y^2 + u^2 = c.$$

$$\therefore x^2 + u^2 = c + y^2 + v^2 \qquad (9.90)$$

~~Note that $c \neq 0$ so y and x cannot be~~

The transformation

$$x = x', \quad u = u'; \quad y = \frac{y'}{(y'^2+v'^2)^{1/4}}; \quad v = \frac{v'}{(y'^2+v'^2)^{1/4}}$$

is a homeomorphism of $\mathbb{R}^4$ (the last two expressions are defined to be zero for $y' = v = 0$) - Check this by directly computing the inverse -

and transforms (9.90) to

$$x^2 + u^2 = c + \sqrt{y'^2 + v'^2} \quad \text{which is}$$

congruent to (9.88) and so represents a

space homeomorphic to $\mathbb{R}^3$ minus a line

We now sketch the proof of the fact that the intersection of the energy and momentum level sets, is topologically a torus.
The energy hyper surface is

$$u^2+v^2 = 2E + \frac{2}{\rho}$$

Momentum surf: $xv - yu = c$

Let us perform the homeomorphic change of variables

$$x = x'/(x'^2+y'^2) \,; \quad y = y'/(x'^2+y'^2)$$

Dropping the primes we get respectively

$$u^2+v^2 = 2E + 2\rho \qquad (9.91)_1$$

$$xv - yu = c\rho^2 \qquad (9.91)_2$$

note that the latter is a quadric cone that contains the plane of singularities $x = y = 0$

writing $x = \rho\cos\theta$, $y = \rho\sin\theta$

$$u^2+v^2 = 2E + 2\rho$$

$$v\cos\theta - u\sin\theta = c\rho$$

Eliminating ρ we get

$$\left(u + \frac{\sin\theta}{c}\right)^2 + \left(v - \frac{\cos\theta}{c}\right)^2 = 2E + \frac{1}{c^2}$$

which shows that for a fixed θ_0 the locus of (u, v) is a right circular cylinder of radius INDEPENDENT of ρ.

The special case $2E + 1/c^2 = 0$ gives

$$u = -\frac{\sin\theta_0}{c}, \quad v = \frac{\cos\theta_0}{c}, \text{ and then} \qquad (9.93)$$

$2\rho + 2E = 1/c^2$ so that ρ is constant
The locus of (u, v, x, y) is a circle. $\Big\}$ (9.94)

If $c = 0$ the momentum equation simply reads

$$(u, v) = \alpha(x, y) \qquad (9.95)$$

Subst into the energy equation, $u^2 + v^2 = 2E + 2\rho$,

$$\alpha^2\rho^2 - 2\rho - 2E = 0$$

This shows that (for real ρ) $0 < \alpha^2 \leq -1/2E$ and ρ varies between the larger and smaller roots of the quadratic equation.

The map $\pi: (u, v, x, y) \mapsto (x, y)$ is bijective thanks to (9.95).

The inverse map is $(x, y) \mapsto (\alpha(x, y), x, y)$ and so the intersection of the energy and momentum loci is homeomorphic to an annulus.

Excluding these extreme cases, the equation of (u, v) namely

$$\left(u + \frac{\sin\theta_0}{c}\right)^2 + \left(v - \frac{\cos\theta_0}{c}\right)^2 = 2E + 1/c^2 \qquad (9.96)$$

is a right-circular cylinder meeting the "plane"

$$v\cos\theta_0 - u\sin\theta_0 = c\rho \qquad (9.97)$$

along an ellipse. The centre of the ellipse is on the axis of the cylinder and on the plane (9.97). If the coordinates of the centre are $(x, y, ~~x_3, x_4~~\ \frac{-\sin\theta_0}{c}, \frac{\cos\theta_0}{c})$, then subst into (9.97),

$\rho = {}^1/_{c^2}$ and $x = \rho\cos\theta_0$, $y = \rho\sin\theta_0$ are the polar coordinates

$$\text{Centre} = \left(\frac{1}{c^2}\cos\theta_0, \frac{1}{c^2}\sin\theta_0, -\frac{1}{c}\sin\theta_0, \frac{1}{c}\cos\theta_0\right)$$

$$= \cos\theta_0\left(\frac{1}{c^2}, 0, 0, \frac{1}{c}\right) + \sin\theta_0\left(0, \frac{1}{c^2}, \frac{-1}{c}, 0\right) \qquad (9.98)$$

which is a point on a circle with centre at the origin.

So the ellipse of intersection of (9.96)-(9.97) has its centre along the circle (9.98)

as θ varies from 0 to 2π the ellipse closes itself and traces out a torus.

Note that nothing more can be said about the foliation of these tori by phase curves without a knowledge of one more first integral.

The first order system (9.86) has zero divergence and so phase volume is preserved by the flow (Liouville's theorem). The classical theorem on last integral of Jacobi enables one to explicitly determine an additional first integral.

Instead of following this route we use the Laplace-Runge-Lenz vector Λ (Equation (9.42)).

$\vec{\Lambda}\cdot\vec{c} = 0$ and so the three components of $\vec{\Lambda}$ do not provide independent first integrals.

Also from (9.40) and (9.42) we know that

$$\|\vec{\Lambda}\| = \|\vec{\varepsilon}\| = e$$

We have expressed e in terms of $\|c\|$ and E (9.30) and so only the direction of $\vec{\Lambda}$ is the additional information given by $\vec{\Lambda}$ ($\vec{\Lambda}\cdot\vec{c} = 0$ merely gives that $\vec{\Lambda}$ is in the plane spanned by $\vec{R}$ and $\vec{v}$)

To determine this, from (9.42) we get

$$\Lambda_1 = \vec{\Lambda}\cdot\hat{i} = \vec{v}\cdot(\vec{c}\times\hat{i}) - x/\rho$$
$$= c\vec{v}\cdot\hat{j} - x/\rho = cv - x/\rho$$
$$\Lambda_2 = \vec{\Lambda}\cdot\hat{j} = -cu - y/\rho$$

Check: $\Lambda_1^2 + \Lambda_2^2 = 1 + 2Ec^2$ in accordance with (9.30).

$\therefore$ inclination of $\vec{\Lambda}$

$$c_3 = \mathrm{Arg}(\vec{\Lambda}) = \mathrm{Arg}\left\{(cv - x/\rho) + i(-cu - y/\rho)\right\}$$
$$= \mathrm{Arg}\left\{-ic(u+iv) - \frac{1}{\rho}(x+iy)\right\}$$
$$= \mathrm{Arg}\left\{-ic\dot{R} - \hat{e}_R\right\} = c_3$$

Then, true anomaly $\theta_T = \theta - c_3$

and the equation of trajectory is

$$\rho = l/(1 + e\cos\theta_T)$$
$$= l/(1 + e\cos(\theta - c_3))$$

Under the homeomorphism $x = x'/(x'^2+y'^2)$; $y = y'/(x'^2+y'^2)$

this equation transforms to (suppressing the primes)

$$\rho = l^{-1} \left(1 + e\cos(\theta - c_3)\right)$$

Which is a limaçon

Each value of θ determines a unique value of ρ and two values of (u, v).

At perihelion, these two values of the velocity vector differ by sign and so correspond to the different senses in which the orbit is traced.*

Since E, $\|\vec{c}\|$ fix only the torus, the value of c_3 distinguishes the various trajectories on the torus. That is to say the foliation on of the torus is parametrized by the 3rd first-integral c_3.

We abandon this topic after in this somewhat sketchy treatment.

The paper is well worth rewriting clearly.

<u>Que:</u> $(9.91)_2$ is a quadric cone in $\mathbb{R}^4$. What is its ~~generating~~ signature. Does this lead to a proof of the toroidal ~~character~~ nature of the invariant manifolds ?

Re: The two branches of velocity, do they permute when θ goes from 0 to 2π ?

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* This point needs to be investigated more closely.

The explanation in terms of sense of orientation is not complete ...

~~Project: Discuss Newton's theorem on non-algebraic nature of the area of an oval cut off by a line~~

Project: Discuss Newton's theorem: If the area of segment of an oval cut off by a line is algebraic function of the coefficients of the line, the oval cannot be smooth.

It would be nice to give a rigorous proof along the lines suggested in Arnold's book using the theory of covering spaces etc.,

Project: Discuss those developments of complex function theory inspired by Kepler's equation.

Write a report on the methods developed for solving the Kepler equation.

Additional Exercises:

Position on Parabolic Orbits:

Show that the equation of a parabolic orbit is $\rho = q \sec^2 \theta/2$ where q is the perihelion distance. Use Kepler's second law to derive the equation

$$\tan \theta/2 + \frac{1}{3} \tan^3 \theta/2 = \text{const.}(t - T)$$

T = time of perihelion passage (assume $T = 0$ for instance). Show that for each t there is a unique solution of the cubic for $\tan \theta/2$. Assume that $T=0$ and show $\rho \sim q t^{2/3}$ for t large.

Position on hyperbolic Orbits:

Define (by analogy) $\rho = a(e\cosh \epsilon - 1)$

prove that $n(t-T) = e \sinh \epsilon - \epsilon$

~~where~~ and the relation between ϵ and true anomaly is given by

$$\tan \theta/2 = \sqrt{\frac{e+1}{e-1}} \tanh \epsilon/2$$

(Gudermannian)

Comet Barnard (1889 – III) was found to have an orbit with perihelion distance 1·102 A.U. $e = 0.957$ and period = 218·3 ~~days~~. years. Calculate the position of the comet 3 years after perihelion passage assuming the trajectory to be a parabola. How does this compare with the result one obtains ~~for~~ assuming elliptic orbit.

(McCuskey p59)

When observed in 1929, Comet Schwassman-Wachmann ~~Wattern~~ II had an elliptic orbit of period $P = 6.42$ yrs and eccentricity 0.40. At what value of eccentric angle ϵ would the comet be found 2.00 yrs after it has passed the perihelion? What would be its radius vector then? McCuskey p50
(McCuskey S.W.: Introduction to Celestial Mechanics
Addison Wesley 1963).

Prove Hamilton's theorem (see J. Milnor)
If a planet describes an ellipse, the velocity vector $\dot{R}$ describes a circle. What is the centre and radius of this circle?

The law of areas is valid even after projection of the orbit on any other plane. This has practical applications to the study of Binary Stellar Systems.