Laplace Transform

Instructor G. K. Srinivasan

Functions of exponential type

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A function $f : (0, \infty) \longrightarrow \mathbb{R}$ is said to be of exponetial type if

(i) The integral $\int_0^t |f(s)|ds$ exists for each $t > 0$ as a proper or improper Riemann intgegral.

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That is to say, $f(t)$ does not grow too rapidly. Thus we are NOT imposing any strong regularity restrictions on the function but only Riemann intgrability.

Some exampled of functions of exponential type

Let us illustrate the definition through many examples.

(i) $f(t) = \sin t$ and $f(t) = \cos t$

(ii) $f(t) = \exp(1000t)$

(iii) $f(t) = t^{10}$. More generally ALL polynomials.

(iv) $f(t) = 1/\sqrt{t}$.

(v) $f(t) = |t|$

Note that in the last example the function is unbounded on $(0, t)$ for EVERY $t > 0$ but the integral of $|f(t)$ is finite over each $(0, t)$. The function is bounded on $[1, \infty)$.

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For the case of $f(t) = \tan t$, the function is NOT integrable on $(0, t)$ for any $t > \pi/2$ and the function becomes infinite at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ and so we cannot assert

$$|f(t)| \leq \exp(at), \quad t \geq b.$$

Exercises:

(1) Which one of the following functions are of exponential type?

  (i) $f(t) = \log t$
  (ii) $f(t) = t \cot t$.
  (iii) $f(t) = \exp(-x^2)$

  Provide a brief justification for each case.

(2) Are the functions $\sin^2 t$ and $\cos^2 t$ of exponential type?

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Let us observe that the integral makes perfect sense. It ofcourse makes sense on $[0, b]$ (recall the b in the definition of functions of exponential type).

Comments on the definition

Now we examine what is happening on $[b, \infty)$. Note that since $|f(t)| \leq \exp(at)$ for $t \geq b$,

$$|f(t)|e^{-st} \leq \exp(-t(s-a))$$

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would converge absolutely whenever $s > a$.
We see that the Laplace transform $F(s)$ is meaningful for $s > a$.

Abscissa of convergence and absolute convergence

The smallest real number ρ such that the integral

$$\int_0^\infty f(t)e^{-st}dt$$

converges for all $s > \rho$ is called the the abscissa of convergence and the smallest real number ρ_0 such that the improper integral converges absolutely is called the abscissa of absolute convergence.

Examples

(i) Consider the simplest example $f(t) = 1$. We compute

$$F(s) = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

and we see that the abscissa of convergence is 0 and is the same as the abscissa of absolute convergence.

(ii) Let us consider $f(t) = \sin t$. Then

$$F(s) = \int_0^\infty e^{-st} \sin t dt$$

The integral converges absolutely for all $s > 0$ and diverges if $s \leq 0$. Thus the abscissa of convergence and absolute convergence are both 0. Also a direct computation shows that

$$F(s) = \frac{1}{1 + s^2}$$

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To compute the integral, Put $t(s-k) = u$ assuming $s > k$ and we get

$$F(s) = \int_0^\infty \frac{u^n e^{-u} du}{(s-k)^{n+1}} = \frac{n!}{(s-k)^{n+1}}$$

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$$F(s) = \frac{1}{1 + (s-k)^2}$$

Can you generalize this to a theorem?

First shifting theorem

Theorem 3

Suppose $F(s)$ is the Laplace transform of a function $f(t)$ of exponential type then for any $k \in \mathbb{R}$, the function $e^{kt}f(t)$ is also of exponential type and its Laplace transform is $F(s-k)$

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Proof:

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Proof: Quite routine! By inspection !

$$\int_0^\infty f(t)e^{kt}e^{-st}dt = \int_0^\infty f(t)e^{-t(s-k)} = F(s-k).$$

This function has a tendency to pop up at unexpected places as we shall see.

Definition 4

The gamma function is given by

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt, \quad a > 0.$$

Observe that

$$\Gamma(1) = 1, \quad \Gamma(k+1) = k!, \quad k \in \mathbb{N}.$$

(7) Show that $\Gamma(1/2) = \sqrt{\pi}$.

Some Laplace transform formulas

Let us now compute the Laplace transform of the function $f(t) = t^a$ where $a > -1$. The condition $a > -1$ ensures that

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exists for each $t > 0$ (the first requirement for it to be of exponential type).

$$F(s) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s}.$$

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$$F(s) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s}.$$

And hence

$$F(s) = \frac{\Gamma(a+1)}{s^{a+1}}.$$

Generalizing a formula we obtained earlier.

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We shall have occasion to use the special case $s = -1/2$.
The Laplace transform of $1/\sqrt{t}$ is given by

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We shall later an application of Laplace transform to a mechanical problem studied by Abel namely, The Tautochrone problem. In this connection the above formula would be employed.

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(8) Calculate the Laplace transform of $t^2 \sin t$.

Laplace transform and derivatives

The idea in the last slide can be generalized to a theorem

Theorem 5

Suppose that $F(s)$ is the Laplace transform (with abscissa of convergence ρ) of a function $f(t)$ of exponential type then,

(i) *$F(s)$ is differentiable infinitely often on (ρ, ∞)*

(ii) *$tf(t)$ is also of exponential type and its Laplace transform is given by*

$$-\frac{d}{ds}(F(s)).$$

The proof is almost automatic. Simply differentiate under the integral sign. However some care is needed to justify differentiation under the integral sign. We shall not discuss the details here.

Laplace transform of derivatives

We shall now come to one of the most important result on Laplace transforms that make it very useful for the study of the initial value problem for differential equations.

Theorem 6

Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is differentiable and $f'(t)$ is of exponential type then

$$\mathcal{L}(f')(s) = s(\mathcal{L}f)(s) - f(0).$$

For all $s > b$ where b is as in the definition of functions of exponential type.

To prove this we integrate by parts once:

$$\mathcal{L}(f')(s) = \int_0^\infty e^{-st} f'(t)dt = -\int_0^\infty f(t)\frac{d}{dt}(e^{-st})dt + e^{-st}f(t)\Big|_0^\infty$$

Since $s > b$, $e^{-st}f(t)$ vanishes at infinity and we are left with the boundary term at 0 namely $-f(0)$.

Laplace transform of derivatives contd...

The previous result easily generalizes to higher order derivatives. Assume that the function together with the first k derivatives are of exponential type. Then applying the formula to $f^{(k-1)}$ in place of f.

$$\begin{aligned}(\mathcal{L}f^{(k)})(s) &= s(\mathcal{L}f^{(k-1)})(s) - f^{(k-1)}(0)\\ &= s^2(\mathcal{L}f^{(k-2)})(s) - sf^{(k-2)}(0) - f^{(k-1)}(0)\\ &= s^3(\mathcal{L}f^{(k-3)})(s) - s^2f^{(k-3)}(0) - sf^{(k-2)}(0) - f^{(k-1)}(0)\end{aligned}$$

etc.,

The Heaviside unit step function

This is denoted by **u** and defined as follows

$$\mathbf{u}(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases}$$

The function has a jump discontinuity at the origin.

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The function has a jump discontinuity at the origin. Now even if $f(t)$ is defined on the entire real line and is of exponential type as described earlier, the Laplace transform of f is defined as

$$\mathcal{L}f(s) = \int_{-\infty}^{\infty} f(t)\mathbf{u}(t)e^{-st}dt.$$

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In other words we clip the signal off or make it zero for $t < 0$ and integrate over the real line. We shall see later that it is better to think of the Laplace integral as an integral over the entire real line with the Heaviside function thrown in.

The Riemann Lebesgue Lemma

Let us comparte the Laplace transforms of $\sin t$ and $\cos t$:

$$(\mathcal{L} \sin t)(s) = \frac{1}{1+s^2}$$
$$(\mathcal{L} \cos t)(s) = \frac{s}{1+s^2}$$

Note that the first decays like $1/s^2$ whereas the second line $1/s$ as $s \longrightarrow 0$.

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To understand the reason why the Laplace transform of $\sin t$ decays faster, observe that $\mathbf{u}(t)\sin t$ is continuous at the origin and has one sided derivatives as well. However, $\mathbf{u}(t)\cos t$ has a jump discontinuity at the origin.

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Regularity of the function and decay of the Laplace transform

$$\mathcal{L}\mathbf{u}(t)(s) = \frac{1}{s}$$
$$\mathcal{L}t\mathbf{u}(t)(s) = \frac{1}{s^2}$$
$$\mathcal{L}t^2\mathbf{u}(t)(s) = \frac{2}{s^3}$$
$$\mathcal{L}t^3\mathbf{u}(t)(s) = \frac{6}{s^4}$$

Notice that the as we go down the list, the function $\mathbf{u}(t)f(t)$ gets increasingly regular and the Laplace transform decays faster at infinity.

Riemann Lebesgue Lemma contd...

We now state the Riemann Lebesgue lemma:

Theorem 7

If $f : [0, \infty) \longrightarrow \infty$ *is of exponential type, then* $\mathcal{L}f(s) \longrightarrow 0$ *as* $s \longrightarrow \infty$.

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As an application, let us find the Laplace transform of the function $\sin t/t$.

$$F(s) = \int_0^\infty \frac{(\sin t)e^{-st}dt}{t}$$

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$$F(s) = \int_0^\infty \frac{(\sin t)e^{-st}dt}{t}$$

Differentiating with respect to s we get

$$F'(s) = -(\mathcal{L}\sin t)(s) = -1/(1+s^2).$$

Hence $F(s) = C - \tan^{-1} s$. To find C use the Riemann Lebesgue lemma which asserts that $F(s) \longrightarrow 0$ as $s \to \infty$ and this would force $C = \frac{\pi}{2}$.

Exercises

(9) Find the Laplace transform of $(1-\cos t)/t$

(10) Is the function

$$\log\left(\frac{s+3}{2s-1}\right)$$

the Laplace transform of a function of exponential type?

(11) Calculate the Laplace transform of $(1-\cos t)/t^2$.

(12) Calculate the Laplace transform of $(1-e^{-t})/t$.

An example

Let us see some examples on how to use this result.
Calculate the Laplace transform of $\sin^2 t/t^2$.
Solution: Denoting by $F(s)$ the Laplace transform of $\sin^2 t/t^2$, we get

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F''(s) &= \frac{1}{2} \int_0^\infty (1 - \cos 2t) e^{-st} dt
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F''(s) &= \frac{1}{2}\int_0^\infty (1-\cos 2t) e^{-st} dt \\
F''(s) &= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)
\end{aligned}$$

Example contd...

So we have,

$$F''(s) = \frac{1}{2}(\frac{1}{s} - \frac{s}{s^2 + 4})$$

One integration gives,

$$F'(s) = \frac{1}{2}(\log s - \frac{1}{2}\log(s^2 + 4)) + C$$

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The constant of integration must be zero by Riemann Lebesgue Lemma. One more integration gives the desired result:

$$\begin{aligned} F(s) &= \frac{1}{2}(s\log s - s - \frac{s}{2}\log(s^2+4)) + \frac{1}{2}\int \frac{s^2 ds}{s^2+4} \\ &= \frac{1}{2}(s\log s - \frac{s}{2}\log(s^2+4) - 2\tan^{-1}(s/2)) + C_1 \end{aligned}$$

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$$F(s) = \frac{1}{2}(s \log s - \frac{s}{2}\log(s^2 + 4) - 2\cot^{-1}(s/2))$$

and the job is done!

A word of warning:

It is NOT true that every function (even very nice functions - infinitely differentiable function) decaying to zero as $s \to \infty$ is the Laplace transform of another function.

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The problem of describing precisely the set of functions that are Laplace transforms of functions of exponential type is a highly non-trivial problem. We shall not enter into these discussions. The point is to draw your attention to fact that the theory of Laplace transforms can be quite subtle.

Uniqueness theorem of Lerch

Observe that if we alter the value of a Riemann integrable function at a point or at finitely many points we would again get a Riemann integrable function and the integrals of the two functions would be identical. Thus two such functions would have the SAME Laplace transform !

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We are NOT saying that the uniqueness result CANNOT be formulated for discontinuous functions. All we are trying is to obtain a statement that is simple (at this level) and devoid of technicalities. To go beyond continuous functions and formulate a uniqueness result would demand that we get into the theory of Lebesgue integrals.

Lerch's Theorem:

Theorem 8

Suppose $f(t)$ and $g(t)$ are CONTINUOUS functions of exponential type on $[0,\infty)$ such that $\mathcal{L}f(s)$ and $\mathcal{L}g(s)$ are defined on the range $c < s < \infty$ and

$$\mathcal{L}f(s) = \mathcal{L}g(s), \qquad \text{for all } s > c,$$

then,

$$f(t) = g(t), \qquad \text{for all } t \geq 0.$$

Examples:

Let us find a function whose Laplace transform is

$$\log\left(\frac{s+1}{s+2}\right)$$

Let us denote this function by $F(s)$ and we are looking for a function $f(t)$ such that

$$\mathcal{L}(f(t)) = F(s)$$

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By the uniqueness theorem,

$$f(t) = \frac{1}{t}(e^{-2t} - e^{-t}).$$

Theorem 9

Let $f(t)$ be a function of exponential type. Assume $c > 0$ then

$$\mathcal{L}(f(t-c)\mathbf{u}(t-c)) = e^{-cs}(\mathcal{L}f)$$

Proof:

$$\begin{aligned}
\mathcal{L}(f(t-c)\mathbf{u}(t-c)) &= \int_0^\infty f(t-c)\mathbf{u}(t-c)e^{-st}dt \\
&= \int_c^\infty f(t-c)e^{-st}du \\
&= \int_0^\infty f(u)e^{-s(c+u)}du \\
&= e^{-cs}\int_0^\infty f(u)e^{-su}du
\end{aligned}$$

Finding the original function

Is there a function of exponential type whose Laplace transform is e^{-s}/s?

Finding the original function

Is there a function of exponential type whose Laplace transform is e^{-s}/s?
Ans: $f(t) = \mathbf{u}(t-1)$.

(13) Find a function whose Laplace transform is $(e^{-s} - e^{-2s})/s$. Discuss the continuity of the original function.

(14) Find a function whose Laplace transform is $(e^{-s} - e^{-2s})/s^2$. Discuss the continuity and differentiability of the original function.

(15) Find a function whose Laplace transform is $(e^{-s} - e^{-2s})/s^3$. Discuss the continuity and differentiability of the original function.

Finding the original function contd...

(16) Find a function whose Laplace transform is

$$\frac{1}{s^4 - 4}$$

(17) Find a function whose Laplace transform is

$$\frac{1}{s^4 + 4}$$

Finding the original function contd...

(16) Find a function whose Laplace transform is

$$\frac{1}{s^4 - 4}$$

(17) Find a function whose Laplace transform is

$$\frac{1}{s^4 + 4}$$

Let us take up the first example:

$$F(s) = \frac{1}{(s^2-2)(s^2+2)} = \frac{1}{4}\left(\frac{1}{s^2-2} - \frac{1}{s^2+2}\right)$$

$$= \frac{1}{8\sqrt{2}}\left(\frac{1}{s-\sqrt{2}} - \frac{1}{s+\sqrt{2}}\right) - \frac{1}{4}\frac{1}{s^2+2}$$

Discussion of example 16 contd...

$$F(s) = \frac{1}{8\sqrt{2}}\Big(\frac{1}{s-\sqrt{2}} - \frac{1}{s+\sqrt{2}}\Big) - \frac{1}{4}\frac{1}{s^2+2}$$

This can be rewritten as

$$\begin{aligned} F(s) &= \frac{1}{8\sqrt{2}}\Big(\mathcal{L}e^{\sqrt{2}t} - \mathcal{L}e^{-\sqrt{2}t}\Big) - \mathcal{L}\Big(\frac{\sin\sqrt{2}t}{4\sqrt{2}}\Big) \\ &= \frac{1}{4\sqrt{2}}\mathcal{L}(\sinh\sqrt{2}t - \sin\sqrt{2}t) \end{aligned}$$

The original function is thus

$$\frac{1}{4\sqrt{2}}(\sinh\sqrt{2}t - \sin\sqrt{2}t)$$

and this happens to be continuous. By Lerch's theorem this is the only one which is continuous.

Discussion of example 17 contd...

For the example (17) use the Partial Fractions decomposition:

$$\frac{1}{s^4+4} = \frac{1}{(s^2+2s+2)(s^2-2s+2)}$$

A matter of notation

Suppose that $f(t)$ has Laplace transform $F(s)$ we shall find it convenient to write

$$\mathcal{L}^{-1}F = f$$

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Suppose that $f(t)$ has Laplace transform $F(s)$ we shall find it convenient to write

$$\mathcal{L}^{-1}F = f$$

WARNING: We are NOT asserting that the operator $\mathcal{L}$ is invertible and we are NOT discussing the range of the operator $\mathcal{L}$. We are NOT describing the image of the Laplace transform operator. Also two functions may have the same Laplace transform without being equal ! For example tamper the value of a function $f(t)$ at ONE or say FINITELY many points to get a new function $g(t)$. Then f and g would have the same Laplace transform. However if f is continuous then g would certainly become discontinuous! However, there cannot be TWO continuous function with the SAME Laplace transform.

Solving Initial Value Problems for Ordinary Differential Equations

It is better to illustrate the technique through a simple example.

$$y'' + y = \sin t, \quad y(0) = 0, y'(0) = 1.$$

Denoting by $Y(s)$ the Laplace transform of $y(t)$ we get

$$(s^2 + 1)Y - 1 = \frac{1}{s^2 + 1}$$

which gives

$$Y(s) = \frac{1}{s^2 + 1} + \frac{1}{(s^2 + 1)^2}$$

The first piece is the Laplace transform of $\sin t$. Let us find a function whose Laplace transform is the second piece.

Example contd...

$$
\begin{aligned}
-\frac{d}{ds}\Big(\frac{s}{s^2+1}\Big) &= \frac{s^2-1}{(s^2+1)^2} \\
&= \frac{s^2+1-2}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}
\end{aligned}
$$

Hence

$$\frac{1}{(s^2+1)^2} = \frac{1}{2}\mathcal{L}\sin t - \frac{1}{2}\mathcal{L}(t\cos t)$$

Thus,

$$Y(s) = \mathcal{L}(\frac{3}{2}\sin t - \frac{1}{2}t\cos t)$$

which means

$$y(t) = \frac{3}{2}\sin t - \frac{1}{2}t\cos t$$

as the solution of the initial value problem.

The General Idea

Note that the procedure is quite clear. One assumes that the differential equation is of the form

$$P(D)y = f(t)$$

where $P(D)$ is a polynomial in the derivative $\frac{d}{dt}$ which means its coefficients are constants. The function $f(t)$ on the RHS is of exponential type whose Laplace transform is available in closed form. One then takes the Laplace transform of both sides and we would get

$$Y(s) = R(s)$$

where $R(s)$ is known and in case $f(t)$ is a linear combination of $t^k e^{ct}$, $t^k \sin ct$ and $t^k \cos ct$ then $R(s)$ is a rational function. One often uses tables of Laplace transform to recover $y(t)$ from the last displayed equation.

Case of rational functions

We decompose the rational function into partial fractions and we then have to deal with finding a function $f(t)$ whose Laplace transform is one of the following:

$$\frac{1}{(s-c)^{k+1}}, \frac{s-c}{((s-c)^2+b^2)^{k+1}}, \frac{b}{((s-c)^2+b^2)^{k+1}}$$

The first is easy:

$$\mathcal{L}(t^k e^{ct}) = \frac{1}{(s-c)^{k+1}}$$

The last two can be quite tedious.

$$\mathcal{L}(e^{ct}\sin bt) = \frac{b}{(s-c)^2+b^2}$$

But,

$$-\frac{d}{ds}\left(\frac{b}{(s-c)^2+b^2}\right) = \frac{2b(s-c)}{((s-c)^2+b^2)^2}$$

So we see that

$$\frac{s-c}{((s-c)^2+b^2)^2} = \mathcal{L}\left(\frac{1}{2b}te^{ct}\sin bt\right)$$

Again,

$$\begin{aligned} -\frac{d}{ds}\Big(\frac{s-c}{(s-c)^2+b^2}\Big) &= \frac{(s-c)^2-b^2}{((s-c)^2+b^2)^2} \\ &= \frac{1}{(s-c)^2+b^2} - \frac{2b^2}{((s-c)^2+b^2)^2} \end{aligned}$$

Thus,

$$\mathcal{L}(bte^{ct}\cos bt) = \mathcal{L}(te^{ct}\sin bt) - \frac{2b^3}{((s-c)^2+b^2)^2}$$

This gives us the function whose Laplace transform is

$$\frac{1}{((s-c)^2+b^2)^2}$$

Clearly, this is not going to be such a merry exercise with the denominators

$$((s-c)^2+b^2)^3$$

and higher powers!

A more elegant and perhaps efficient way would be to completely factorize into linear factors with real or complex roots.
As an example let us take up

$$\frac{1}{((s-1)^2+1)^2}$$

The partial fraction decomposition is $(\lambda = 1 + i)$

$$\frac{1}{((s-1)^2+1)^2} = \frac{a}{s-\lambda} + \frac{b}{(s-\lambda)^2} + \frac{\alpha}{s-\overline{\lambda}} + \frac{\beta}{(s-\overline{\lambda})^2} \tag{2.1}$$

Since original the rational function has real coefficients, we have

$$\alpha = \overline{a},\ \beta = \overline{b}$$

Multiplying (2.1) by $(s-\lambda)^2$ and setting $s=\lambda$ we get the pair

$$b=\beta=-\frac{1}{4}$$

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Multipying (2.1) by $s-\lambda$ and letting $s\to\infty$ we get the equation

$$a+\alpha=0.$$

We need one more equation and for this we set $s=0$ in (2.1):

$$\frac{1}{4}=-\left(\frac{a}{\lambda}+\frac{\alpha}{\overline{\lambda}}\right)-\frac{1}{2}\mathrm{Re}\left(\frac{1}{\lambda^2}\right)$$

From which we get the pair

$$a=-\frac{i}{4},\quad \alpha=\frac{i}{4}.$$

So (2.1) reads

$$\frac{1}{((s-1)^2+1)^2} = -\frac{i}{4}\Big(\frac{1}{s-\lambda} - \frac{1}{s-\overline{\lambda}}\Big) - \frac{1}{4}\Big(\frac{1}{(s-\lambda)^2} + \frac{1}{(s-\overline{\lambda})^2}\Big)$$

We can now use the formula

$$\frac{1}{s-c} = \mathcal{L}e^{ct}, \quad \frac{1}{(s-c)^2} = \mathcal{L}(te^{ct})$$

which holds for all complex values c.

(18) Complete the job of finding the function $f(t)$ whose Laplace transform is $1/((s-1)^2+1)^2$.

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For a discussion on Real Inversion theorem see the book G. Fodor, Laplace transforms in Engineering, Akademiai Kiado, 1965.

Solving a system of first order differential equations

Example: Consider the system of ODEs:

$$\frac{dx}{dt} = -y - t, \quad \frac{dy}{dt} = x - 1$$

with initial conditions $x(0) = 0, y(0) = 0$. Taking the Laplace transform of the system

$$sX = -Y - \frac{1}{s^2}, \quad sY = X - \frac{1}{s}.$$

Solving these simultaneously for X and Y we get

$$X = 0, \quad Y = -1/s^2$$

which gives immediately $x(t) = 0, \quad y(t) = -t$.

Needless to say this example was designed to illustrate the idea and keep the computational complexity at a minimum. In general we get a pair of equations for X and Y which we solve and then recover $x(t)$ and $y(t)$ from these. The difficulty lies in recovering the original function from the Laplace transforms.

Example: Next, let us take up the example

$$\frac{dx}{dt} = y + \sin t, \; \frac{dy}{dt} = -x + \cos t, \quad x(0) = -1, \; y(0) = 0.$$

Taking the Laplace transforms of the two equations:

$$sX - Y = \frac{-s^2}{s^2+1}, \quad X + sY = \frac{s}{s^2+1}$$

Solving for X and Y:

$$X = \frac{-s(s^2-1)}{(s^2+1)^2}, \quad Y = \frac{2s^2}{(s^2+1)^2}$$

(19) Do the partial fractions decomposition and find $x(t)$ and $y(t)$.

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(19) Do the partial fractions decomposition and find $x(t)$ and $y(t)$.

Ans: $x(t) = t\sin t - \cos t$ and $y(t) = t\cos t + \sin t$

What are the merits of the Laplace transform technique over other techniques such as the method of variation of parameters and the method of undetermined coefficients?

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(1) The Laplace transform method incorporates the initial conditions in the solution process. One does not have to determine the values of constants later by solving a system of linear algebraic equations.

(2) It handles linear systems with constant coeffients most efficiently - it makes no difference whether the matrix of coefficients is diagonalizable or not.

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(2) It handles linear systems with constant coeffients most efficiently - it makes no difference whether the matrix of coefficients is diagonalizable or not.

Let us take a moment and review the conventional process to understand comment (2) above.

Solving a linear system

We consider the system of ODEs

$$\frac{dx}{dt} = Ax \tag{2.2}$$

where A is an $n \times n$ matrix of constants. The system is coupled system of equations in the sense that all the variables $x_1, \ldots, x_n$ would appear in all the equations. In the exceptional case where A is a diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ down the main diagonal, equation (2.2) reads:

$$\frac{dx_1}{dt} = \lambda_1 x_1, \ldots \frac{dx_n}{dt} = \lambda_n x_n \tag{2.3}$$

The system (2.3) is said to be a decoupled system.

Clearly we can solve (2.3) quite easily but as such it is not clear how to negotiate the coupled system (2.2).

Decoupling a coupled system of equation

Clearly we can solve (2.3) quite easily but as such it is not clear how to negotiate the coupled system (2.2).

Using basic ideas from linear algebra one can decouple the system (2.2) in certain cases

Diagonalizable case

Assume that the matrix is diagonalizable

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Assume that the matrix is diagonalizable which means there is a basis of $\mathbb{R}^n$ consisting of eigen-vectors of A. Let the basis of eigen vectors be $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Put

$$P = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

Then P is an invertible matrix and

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

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Then P is an invertible matrix and

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

Put $x = Py$ and the system of ODEs becomes

$$P\dot{y} = APy$$

which means

$$\dot{y} = P^{-1}APy = \operatorname{diag}(\lambda_1 y_1, \lambda_2 y_2, \ldots, \lambda_n y_n)$$

So the transformed system is the decoupled system

$$\dot{y}_1 = \lambda_1 y_1, \ldots, \dot{y}_n = \lambda_n y_n$$

After obtaining $y(t)$ we can recover $x(t)$ as

$$x(t) = Ay(t).$$

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Note that if A is not diagonalizable, the problem becomes much more complicated and the solution process would get quite involved.
However the method of Laplace transform goes through whether or not A is diagonalizable!

Using the Laplace transform to compute integrals

The Laplace transform can be used to compute certain integrals efficiently. We shall take up some time to illustrate this VERY practical application!

(20) Compute

$$I(t) = \int_0^\infty \frac{\sin tx}{x}, \quad t > 0$$

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$$\begin{aligned} \mathcal{I} &= \int_0^\infty \mathcal{L}(\frac{\sin tx}{x}), \quad t > 0 \\ &= \int_0^\infty \frac{dx}{x^2 + s^2} = \frac{\pi}{2s} \end{aligned}$$

Hence $I(t) = \frac{\pi}{2}$.

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Hence $I(t) = \frac{\pi}{2}$. This integral plays an extremely important role in Fourier Analysis and is called the Dirichlet integral.

The Frullani Integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx, \quad 0 < a < b.$$

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$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx, \quad 0 < a < b.$$

We shall have occasion to use this integral later today! Let us introduce a parameter and denote

$$I(t) = \int_0^\infty \frac{e^{-tax} - e^{-tbx}}{x} dx$$

Taking the Laplace transform we get

$$\begin{aligned}(\mathcal{L}I)(s) &= \int_0^\infty \frac{1}{x}\Big(\frac{1}{s+ax} - \frac{1}{s+bx}\Big) dx \\ &= \int_0^\infty \frac{b-a}{(s+ax)(s+bx)} dx\end{aligned}$$

The Frullani integral contd...

$$(\mathcal{L}I)(s) = \frac{b-a}{ab}\int_0^\infty \frac{1}{(x+\frac{s}{a})(x+\frac{s}{b})}dx$$

(21) Do the partial fraction decomposition and evaluate the integral and recover $I(t)$. Put $t = 1$ and complete the evaluation of the Frullani integral.

The Frullani integral contd...

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Ans: $\log(b/a)$

The Cauchy distribution

This appears naturally in statistics. One is interested in the Fourier transform of the Cauchy distribution and this is the integral

$$\frac{2}{\pi} \int_0^\infty \frac{\cos x\xi \, dx}{1 + x^2}$$

(22) Use Laplace transform to compute this integral.

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(22) Use Laplace transform to compute this integral.

(23) Determine the Laplace transform of

$$I(t) = \int_0^\infty \frac{\sin txdx}{x^2+1}$$

Ans for (22): $e^{-\xi}$
Ans for (23): $\log s/(s^2-1)$ There is no explicit formula for the function $I(t)$.

Euler's constant

Theorem 10

The sequence

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1) \tag{3.1}$$

converges as $n \to \infty$ *and its limit is denoted by* γ*, known as Euler's constant.*

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$$\int_0^\infty e^{-jt} dt = \frac{1}{j}, \quad j = 1, 2, 3, \ldots$$

Adding over $j = 1, 2, \ldots, n$ we get summing the finite geometric series,

Euler's constant contd...

$$\int_0^\infty \frac{e^{-t}(1-e^{-nt})}{1-e^{-t}}dt = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

It is useful to write this as

$$\int_0^\infty \frac{e^{-t}(1-e^{-nt})}{t}\frac{t}{1-e^{-t}}dt = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

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Subtracting we get

Euler's constant contd...

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n+1)=\int_0^\infty(1-e^{-nt})\left\{\frac{1}{1-e^{-t}}-\frac{1}{t}\right\}e^{-t}dt \quad (3.2)$$

Euler's constant contd...

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n+1) = \int_0^\infty (1-e^{-nt})\left\{\frac{1}{1-e^{-t}}-\frac{1}{t}\right\}e^{-t}dt \quad (3.2)$$

Now using L'Hospital's rule, the expression within braces tends to $1/2$ as $t \to 0+$.

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Now using L'Hospital's rule, the expression within braces tends to $1/2$ as $t \to 0+$. This shows that the integral is not improper at the lower end (it is obviously so at the upper end). Let us now let $n \longrightarrow \infty$ in (3.2) and take the limit inside the integral. It is not difficult to justify this but doing so would be a detour for which we have no time.

Euler's constant contd...

We thus see that the limit in (3.1) exists and IMPORTANTLY, the limit equals

$$\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) e^{-t} dt$$

This integral can be further transformed to a much simpler form as we shall now see

Euler's constant contd...

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$$\gamma = -\int_0^\infty e^{-t} \log t dt.$$

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$$\gamma = -\int_0^\infty e^{-t} \log t dt.$$

(25) Determine the Laplace transform of $\log t$. Ans: $-\gamma/s - (\log s)/s$.

Laplace transforms of Periodic functions

Theorem 11

Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is periodic with period p. Then

$$\mathcal{L}f = \frac{e^{ps}}{e^{ps}-1}\int_0^p f(t)e^{-st}dt. \tag{3.3}$$

We use the second shifting theorem. Assume that $p > 0$. Observe that

$$\mathbf{u}(t)f(t) - \mathbf{u}(t-p)f(t-p)$$

is zero outside $[0, p]$. This can be seen analytically but it is easier to see this graphically.

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is zero outside $[0, p]$. This can be seen analytically but it is easier to see this graphically. Now over the interval $[0, p]$ the above function equals $f(t)$ so multiplying by e^{-st} and integrating we get invoking the second shifting theorem,

$$\int_0^p f(t)e^{-st} = F(s) - F(s)e^{-ps}$$

From this the theorem follows.

Laplace transform of periodic functions

Applying this result to $|\sin t|$ which is periodic with period π we see that

$$\mathcal{L}|\sin t| = \frac{e^{\pi s}}{e^{\pi s}-1}\int_0^{\pi} e^{-st}\sin t dt$$

(26) Calculate the integral and find the answer.

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Next, let us take up the $2\pi-$periodic square wave function

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi. \end{cases}$$

Laplace transforms of periodic functions

Using the formula we get

$$\mathcal{L}f = \frac{e^{2\pi s}}{e^{2\pi s} - 1} \int_0^{2\pi} e^{-st} dt.$$

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We shall now take a very different approach to compute this Laplace transform and compare results.

Bringing in the Fourier series! Mittag-Leffler expansions

Let us calculate the Fourier series of the above function:

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Taking the Laplace transform of the above equation we get:

$$\begin{aligned}
\mathcal{L}f &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + s^2} \\
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\frac{1}{s}\tanh\left(\frac{\pi s}{2}\right) &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + s^2} \qquad (3.4)
\end{aligned}$$

Special values of zeta function.

The identity (3.4) is very interesting. Letting $s \to 0+$ we infer

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

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(29) Carry out a similar calculation for $|\sin t/2|$ which is periodic with period 2π.

Ans to problem 28: $\pi^2/6$

Convolution of two intergable functions

Assume that $f(t)$ and $g(t)$ are two functions which are absolutely integrable:

$$\int_{-\infty}^{\infty} |f(t)|dt \quad \text{and} \quad \int_{-\infty}^{\infty} |g(t)|dt$$

are both finite.

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are both finite.

Definition 12

The convolution of f and g denoted by $f * g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

One immediately sees that

$$f * g = g * f$$

Theorem 13

The convolution $f * g$ is absolutely integrable.

$$|f * g(x)| \leq \int_{-\infty}^{\infty} |f(x-t)||g(t)|dt$$

Integrating with respect to x we get

$$\int_{-\infty}^{\infty} |f * g(x)|dx \leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x-t)||g(t)|dt$$

Interchanging the integrals

$$\int_{-\infty}^{\infty} |f * g(x)|dx \leq \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(x-t)|dx$$

In the inner integral put $x - t = y$ and infer that the right hand side is the product

$$\int_{-\infty}^{\infty} |f(t)|dt \int_{-\infty}^{\infty} |g(y)|dy$$

so by comparison test we see that $f * g$ is absolutely intgerable.

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so by comparison test we see that $f * g$ is absolutely intgerable.
In particular, the Laplace integral

$$\int_{0}^{\infty} |f * g(x)| e^{-xs} dx$$

exists for all $s > 0$.

Theorem 14

If $f(t)$ and $g(t)$ are of exponential type then the convolution

$$(\mathbf{u}f) * (\mathbf{u}g)(x)$$

is also of exponential type.

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is also of exponential type.

To see this first note that the last display equals

$$\int_0^x f(y)g(x-y)dy.$$

and we must now estimate this. By hypothesis there exist positive constants A and T such that

$$|f(t)| < e^{At}, |g(t)| < e^{At}, \qquad \text{for all } t \geq T.$$

Thus, if we take $x > 2T$, then

$$\begin{aligned}
|(\mathbf{u}f) * (\mathbf{u}g)(x)| &\leq \int_0^x |f(y)||g(x-y)|dy \\
&\leq \int_0^T |f(y)||g(x-y)|dy + \int_T^x |f(y)||g(x-y)|dy \\
&\leq C + \int_T^x |f(y)||g(x-y)|dy
\end{aligned}$$

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But then $x - y > T$ if $T \leq y \leq x$ and so the last integral can be dealt as follows:

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But then $x - y > T$ if $T \leq y \leq x$ and so the last integral can be dealt as follows:

$$\int_T^x |f(y)||g(x-y)|dy < \int_T^x e^{Ay} e^{A(x-y)} dy < xe^{Ax}$$

All we need to observe is that

$$C + xe^{Ax} < e^{(A+1)x}$$

for all x sufficiently large and the proof is complete.

Theorem 15

Suppose $f(t)$ and $g(t)$ are of exponential type then

$$\mathcal{L}((\mathbf{u}f) * (\mathbf{u}g)) = (\mathcal{L}f)(\mathcal{L}g)$$

In other words the Laplace transform converts a convolution into a product.

Theorem 15

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For example, The Renewal Equation - See W. Feller, Introduction to the theory of Probability theory, Vol - II, Chapter 6.

Proof of the convolution theorem

$$
\begin{aligned}
\mathcal{L}((\mathbf{u}f) * (\mathbf{u}g)) &= \int_0^\infty e^{-st} (\mathbf{u}f) * (\mathbf{u}g) dt \\
&= \int_0^\infty e^{-st} dt \int_{-\infty}^\infty (\mathbf{u}(x) f(x) \mathbf{u}(t-x) g(t-x)) dx \\
&= \int_0^\infty e^{-st} dt \int_0^\infty (f(x) \mathbf{u}(t-x) g(t-x)) dx \\
&= \int_0^\infty f(x) dx \int_0^\infty \mathbf{u}(t-x) g(t-x) e^{-st} dt
\end{aligned}
$$

In the inner integral put $t - x = y$.

$$
\mathcal{L}((\mathbf{u}f) * (\mathbf{u}g)) = \int_0^\infty f(x) dx \int_{-x}^\infty \mathbf{u}(y) g(y) e^{-s(x+y)} dy
$$

Proof of convolution theorem contd...

Sine $x > 0$ the presence of $\mathbf{u}(y)$ would imply that for the integral over $[-x, \infty)$, the range of integration may be replaced by $[0, \infty)$. Thus,

$$\begin{aligned}
\mathcal{L}((\mathbf{u}f) * (\mathbf{u}g)) &= \int_0^\infty f(x)dx \int_0^\infty \mathbf{u}(y)g(y)e^{-s(x+y)}dy \\
&= \int_0^\infty e^{-sx}f(x)dx \int_0^\infty g(y)e^{-sy}dy \\
&= (\mathcal{L}f)(\mathcal{L}g)
\end{aligned}$$

This completes the proof of the theorem.

An example

Let a and b be positive real numbers. Let us compute the convolution

$$\mathbf{u}(t)t^{a-1} * \mathbf{u}(t)t^{b-1}$$

Well, the convolution is zero on the negative real axis as first display below indicates.

$$\begin{aligned}
\mathbf{u}(t)t^{a-1} * \mathbf{u}(t)t^{b-1}(x) &= \int_{-\infty}^{\infty} \mathbf{u}(t)t^{a-1}\mathbf{u}(x-t)(x-t)^{b-1}dt \\
&= \int_{0}^{\infty} t^{a-1}\mathbf{u}(x-t)(x-t)^{b-1}dt \\
&= \int_{0}^{x} t^{a-1}(x-t)^{b-1}dt
\end{aligned}$$

Proof of convolution theorem contd...

Put $t = xz$ and the integral transforms into

$$x^{a+b-1} \int_0^1 z^{a-1}(1-z)^{b-1} dz$$

which is $B(a, b)x^{a+b-1}$ and zero if $x < 0$. Thus the convolution is

$$B(a, b)\mathbf{u}(x)x^{a+b-1}$$

The Beta-Gamma relation

Theorem 16

If a and b are positive real numbers then

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b).$$

To prove this we apply the convolution theorem to the equation

$$\mathbf{u}(t)t^{a-1} * \mathbf{u}(t)t^{b-1} = B(a,b)\mathbf{u}(t)t^{a+b-1}$$

The result is

$$\frac{\Gamma(a)}{s^a}\frac{\Gamma(b)}{s^b} = B(a,b)\frac{\Gamma(a+b)}{s^{a+b}}$$

Canceling the factor s^{a+b} gives the result.

Integral equations of convolution type

These arise in many applications such as Birth and Death processes, The Renewal Equation in Probability theory and also in the integral representations of solutions of many initial value problems for ordinary differential equations.

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An integral equation of convolution type is an equation of the form

$$y(t) = f(t) + \int_0^t y(t-x)g(x)dx$$

where f and g are given functions.

Example of an integro-differential eqn of conv. type

Let us consider the equation

$$y(t) = 1 + \int_0^t y(t-x) \sin x dx.$$

Taking the Laplace transform we see that

$$Y(s) = \frac{1}{s} + \frac{Y(s)}{s^2+1}$$

which gives

$$Y(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

The solution $y(t)$ is then

$$t + \frac{t^2}{2}$$

Integro-differential equations

Solve the following integro-differential equation of convolution type:

$$y'(t) = 1 + \int_0^t y(t-x)\cos x dx, \quad y(0) = 0.$$

Take the Laplace transform of both sides and we get

$$sY = \frac{1}{s} + \frac{sY}{s^2+1}$$

Upon simplifying we get

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

The solution is given by

$$y(t) = t + \frac{t^3}{6}$$

Tautochrone property of a cycloid

Problem: Consider points $P = (0, 0)$ and $Q = (x, y)$ in the plane with Q in the first quadrant and a curve C joining Q and P. A bead slides down along C from an intermediate position $R = (u, v)$ to the origin namely P. How to shape the curve C so that the time of descent from R to P is independent of the point R ?

Note that a generic point on the curve is (u, v).

In other words, the time of descent must be constant say T irrespective of the intermediate point R from where the bead was released.

The tautochrome property of a cycloid contd...

The energy equation gives:

$$mg(y - v) = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

which can be re-written as

$$-\sqrt{2g}\sqrt{y - v} = \frac{ds}{dt} = \frac{ds}{dv}\frac{dv}{dt}$$

Integrating with respect to t we get,

$$\sqrt{2g}\,T = \int_0^y \frac{ds}{dv}\frac{dv}{\sqrt{y - v}} \tag{1}$$

Let us now assume the curve C to be given by $u = f(v)$ so that

$$\frac{ds}{dv} = \sqrt{1 + (f'(v))^2} = \phi(v) \text{ say} \tag{2}$$

The tautochrome property of a cycloid contd...

Equation (1) now reads:

$$\sqrt{2g}\,T = (\phi(v)H(v)) * \left(\frac{H(v)}{\sqrt{v}}\right)$$

Taking Laplace transform we get

$$\frac{\sqrt{2g}\,T}{s} = \frac{\sqrt{\pi}}{\sqrt{s}}(\mathcal{L}\phi)$$

which gives

$$\mathcal{L}\phi = \frac{T\sqrt{2g}}{\pi}\sqrt{\frac{\pi}{s}} = \frac{\sqrt{2g}}{\pi}\mathcal{L}t^{-1/2}$$

The function $\phi(v)$ is given by

$$\phi(v) = \frac{A}{\sqrt{v}}, \quad \text{where } A = \frac{\sqrt{2g}\,T}{\pi}$$

The tautochrome property of a cycloid contd...

Substituting this in (2) we get

$$f'(v) = -\sqrt{\frac{A^2}{v} - 1}$$

Put $v = A^2 \sin^2\theta = \frac{A^2}{2}(1 - \cos 2\theta)$ and we get immediately

$$f(v) = \frac{BA^2}{2} - A^2 \int 2\cos^2\theta d\theta = \frac{A^2}{2}(B - 2\theta - \sin 2\theta)$$

and the curve $u = f(v)$ is given parametrically by

$$\frac{A^2}{2}(B - 2\theta - \sin 2\theta, 1 - \cos 2\theta).$$

B is a constant of integration. The curve is an inverted cycloid.

The cycloid:

A circle with a point marked on it, rolls without slipping on a straight line. The Cycloid is the locus of the marked point.

Exercise: Derive the parametric equations of a cycloid starting with its geometric definition.

The image in the next page has been taken from:
https://commons.wikimedia.org/wiki/File:Cycloid-tangent.jpg

Huygen's Isochronous pendulum

The evolute of a cycloid is another (congruent) cycloid but this cannot be discussed here. You need to look into books on elementary differential geometry.

A proof can be found on page 77 of:

J. McCleary, Geometry from a differential viewpoint, Cambridge university press, 1997.

This was discovered by C. Huygens in 1665 and he used it to construct his famous Isochronous Pendulum.

1. Solve the integral equation

$$y(t) = e^{-t} + \int_0^t \sin(t-x)y(x)dx, \quad \text{Ans:}2e^{-t} + t - 1.$$

2. Solve $y'' + ty' - 2y = 4$ with $y(0) = 1, y'(0) = 0$. Note that this is a variable coefficient ODE. Ans: $y(t) = 3t^2 + 1$.

3. An ODE with continuous and non-differentiable RHS:

$$y'' + y = \begin{cases} \sin t & t \in [0, \pi] \\ 0 & t \geq \pi \end{cases}$$

   Solve the initial value problem with initial conditions $y(0) = 0, y'(0) = 0$. Ans: $\frac{1}{2}(\sin t - t\cos t)$ on $[0, \pi]$ and $-(\pi/2)\cos t$ on $t \geq \pi$.

4. Compute $\mathcal{L}f$ where

$$f(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t\sin\theta)d\theta$$

Find a second order ODE satisfied by this function $f(t)$.

5. Find the Laplace transform of the function

$$\int_0^t \frac{\sin u du}{u}, \quad \text{Ans:} s^{-1}\cot^{-1} s.$$

6. Apply the Laplace transform to the ODE

$$t^2y'' + 2ty' - (t^2 + p(p+1))y = 0.$$

Identify the resulting ODE. The original equation is related to the Bessel equation.

7. Apply the Laplace transform to the Laguerre equation

$$ty'' + (1-t)y' + py = 0.$$

Solve the resulting first order ODE. Show that the Laguerre equation has polynomial solutions when $p \in \mathbb{N} \cup \{0\}$. These polynomials are called Laguerre polynomials.

8. Interpret the identity

$$s^{-2}e^{-(a+b)s} = (s^{-1}e^{-as})(s^{-1}e^{-bs})$$

in the light of the convolution theorem.

9. Calculate the Laplace transform of $\sin^2 t/t^2$. Ans: $(s \log s - \log(s^2+4) - 2\cot^{-1}(s/2))$

10. Calculate the Laplace transform of the greatest integer function $f(t) = [t]$. Use of shift theorem may be easier.

11. Suppose $f : [0, \infty) \longrightarrow \mathbb{R}$ is a non-negative function of exponential type show that the Laplace transform $F(s)$ satisfies

$$(-1)^n F^{(n)} \geq 0, \qquad \text{for all } n = 0, 1, 2, \ldots \tag{1}$$

A function F satisfying the condition (1) is said to be *completely monotone.*

For example

$$\frac{1}{(n+x)^2}$$

is completely monotone for each $n = 0, 1, 2, \ldots$. What about

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2}$$

Note: The function $f(x)$ is the second derivative of $\log(\Gamma(x))$.
Bernstein's theorem: A function is completely monotone if and only of it is the Laplace transform of a positive Borel measure. For example if $f(t)$ is a non-negative function of exponential type then $f(t)dt$ is a positive Borel measure.

12. Let $F(s)$ be the Laplace transform of $1/(1+t^2)$. Show that $F(s)$ satisfies the ODE

$$F'' + F = \frac{1}{s}$$

Deduce that

$$F(s) = \int_0^\infty \frac{\sin \lambda d\lambda}{(\lambda + s)}$$

The first part is easy. Let us turn to the last part. Applying the method of variation of parameters we get

$$F(s) = c_1 \cos s + c_2 \sin s + \int_\alpha^s \frac{\sin(s-u)du}{u}$$

where $\alpha > 0$. Expanding $\sin(s-u)$ this can be written as:

$$\cos s(c_1 - \int_\alpha^s \frac{\sin udu}{u}) + \sin s(c_2 + \int_\alpha^s \frac{\cos udu}{u})$$

By Riemann Lebesgue lemma, we infer

$$c_1 = \int_{\alpha}^{\infty} \frac{\sin u du}{u}, \quad c_2 = -\int_{\alpha}^{\infty} \frac{\cos u du}{u}$$

So that

$$F(s) = \int_{s}^{\infty} (\cos s \sin u - \sin s \cos u) \frac{du}{u} = \int_{s}^{\infty} \frac{\sin(u-s) du}{u}$$

Now put $u - s = \lambda$ and you get the result.

Aliter: Suggested by several students of MA 108.

$$F(s) = \int_0^\infty \frac{e^{-st}dt}{1+t^2}$$

However writing $1/(1+t^2)$ as the Laplace transform of sine,

$$F(s) = \int_0^\infty e^{-st}dt \int_0^\infty (\sin\lambda)e^{-\lambda t}d\lambda$$

Switching the order of integration we get the result.

Mittag-Leffler expansion for cosech s and coth s

Consider the saw tooth wave

$$f(t) = t, \quad -\pi < t < \pi$$

extended as a $2\pi-$periodic function. Let us compute the Laplace transform of $f(t)$ denoted by $F(s)$:

$$F(s) = (1 - e^{-2\pi s})^{-1} \int_0^{2\pi} f(t)e^{-st}dt.$$

The integral can be computed and we find:

$$F(s) = \frac{1}{s^2} - \frac{\pi}{s}\text{cosech}(\pi s)$$

Let us now write the Fourier series for $f(t)$:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Since $f(t)$ is an odd function $a_0 = 0$ and $a_n = 0$ for $n = 1, 2, 3, \ldots$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin ntdt = \frac{(-1)^{n-1}2}{n}$$

whereby

$$f(t) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nt}{n}$$

Exercise: Put $t = \pi/2$. What do you get?

Mittag-Leffler expansion for cosech s and coth s contd...

Taking Laplace transform of the last equation we get another expression for $F(s)$ namely:

$$F(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + s^2}$$

Comparing the two expressions for $F(s)$ we deduce:

$$\pi \text{cosech}(\pi s) = \frac{1}{s} + 2\sum_{n=1}^{\infty} \frac{(-1)^n s}{n^2 + s^2}$$

which is equivalent to

$$\text{cosech}(s) = \frac{1}{s} + 2\sum_{n=1}^{\infty} \frac{(-1)^n s}{n^2\pi^2 + s^2}$$

To obtain the Mittag-Leffler expansion for coth s we proceed as follows:

Mittag-Leffler expansion for cosech s and coth s contd...

$$e^{-s}\text{cosech } s = \frac{2e^{-s}}{e^s - e^{-s}} = \frac{(e^s + e^{-s}) - (e^s - e^{-s})}{e^s - e^{-s}} = \coth s - 1.$$

Now in order to get the extra factor of e^{-s} we must resort to the shift theorem and use shift the graph of our original function to the right by π units which means we must employ the function

$$g(t) = t - \pi, \quad 0 < t < 2\pi$$

extended as a $2\pi-$periodic function. The Laplace transform $G(s)$ of g is easily computed:

$$G(s) = \frac{1}{s^2} - \frac{\pi}{s}\frac{(1 + e^{-2\pi s})}{1 - e^{-2\pi s}} = \frac{1}{s^2} - \frac{\pi}{s}\coth \pi s, \quad \text{voila!}$$

Mittag-Leffler expansion for cosech s and coth s contd...

The Fourier series for $g(t)$ is (check)

$$g(t) = -2\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

whereby

$$G(s) = -2\sum_{n=1}^{\infty} \frac{1}{n^2 + s^2}.$$

Comparing the two expressions for $G(s)$ we get:

$$\frac{\pi}{s}\coth(\pi s) = \frac{1}{s^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2 + s^2}$$

which is equivalent to

$$\coth s = \frac{1}{s} + 2\sum_{n=1}^{\infty} \frac{s}{n^2\pi^2 + s^2}$$

We used this to derive the Jacobi theta function identity!