MA 205 (Complex Analysis)

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Autumn 2010

Four Credits.
Course Ends with the Midsem break.

Complex Numbers.

A complex number z is an ordered pair of real numbers:

$$z = (x, y) \quad ; \quad x, y \in \mathbb{R}$$

Addition and Multiplication are defined as follows:

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$
$$= (x_1 + x_2, \; y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, \; x_1 y_2 + y_1 x_2)$$

Note that $\quad (0,1)\cdot(0,1) = (-1, 0)$

The addition and multiplication so defined satisfy the usual rules of arithmetic.

$$z_1 + z_2 = z_2 + z_1 \; ; \; z_1 z_2 = z_2 z_1 \quad \text{(Commutativity)}$$

$$\left.\begin{aligned} z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ z_1 (z_2 z_3) &= (z_1 z_2) z_3 \end{aligned}\right\} \text{Associativity}$$

$$z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \text{Distributivity}$$

Furthermore $(0,0)$ is the additive identity

$$z + (0,0) = z = (0,0) + z.$$

Hereafter we simply denote $(0,0)$ by 0.

$(1,0)$ is the multiplicative identity:

$$z\,(1,0) = z = (1,0)\,z.$$

Hereafter we denote $(1,0)$ by 1

The additive inverse of $z = (x,y)$ is $(-x,-y)$ denoted by $-z$
That is,

$$(x,y) + (-x,-y) = 0$$

Let us denote $(0,1)$ by i
Then,

$$i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$$

and

$$\begin{aligned}(x,y) &= (x,0) + (0,y)\\ &= (x,0) + (y,0)\,(0,1)\end{aligned}$$

Now, denoting $(x,0)$ by x we write

$$z = (x,y) = (x,0) + (0,1)(y,0)$$
$$= x + iy$$

The rules of addition and multiplication assume the familiar form

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)\cdot(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Modulus or Absolute Value of a Complex number.

Let $z = x + iy$: $x, y \in \mathbb{R}$

We define the modulus or absolute value of z as

$$|z| = \sqrt{x^2 + y^2}$$

x is called the Real part of z $(x \in \mathbb{R})$

y is called the imaginary part of z $(y \in \mathbb{R})$

Notations: $x = \operatorname{Re} z$

$y = \operatorname{Im} z$

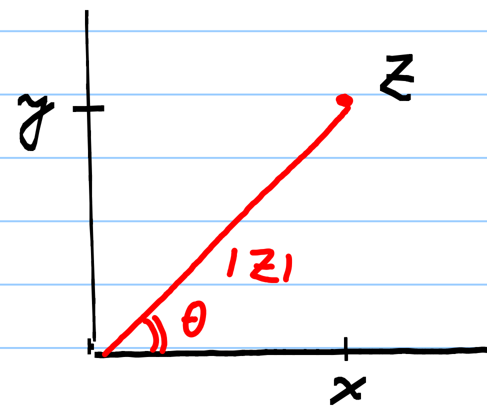
Clearly

$$|\operatorname{Re} z| \leq |z|$$
$$|\operatorname{Im} z| \leq |z|$$

Now assume $z \neq 0$ so that $|z| > 0$

Then

$$\frac{x}{|z|} \quad \text{and} \quad \frac{y}{|z|}$$ both lie in the interval $[-1, 1]$.



Suppose $y > 0$. Then

There exists a UNIQUE value $\theta \in (0, \pi)$ such that

$$\frac{y}{|z|} = \sin\theta \quad \text{and}$$

$$\frac{x}{|z|} = \cos\theta$$

On the otherhand if $y < 0$, there is a UNIQUE value $\theta \in (-\pi, 0)$ such that

$$\frac{y}{|z|} = \sin\theta \; ; \quad \frac{x}{|z|} = \cos\theta$$

If $y = 0$ we take $\theta = 0$ if $x > 0$

$\qquad\qquad\qquad\quad = \pi$ if $x < 0$

Thus we have for each $z \neq 0$ a UNIQUE angle $\theta \in (-\pi, \pi]$ such that

$$x = |z| \cos\theta \; ; \quad y = |z| \sin\theta$$

This angle θ is called the principal argument of z and denoted by

$$\mathrm{Arg}(z)$$

Now suppose θ is the principal argument and k is an integer then

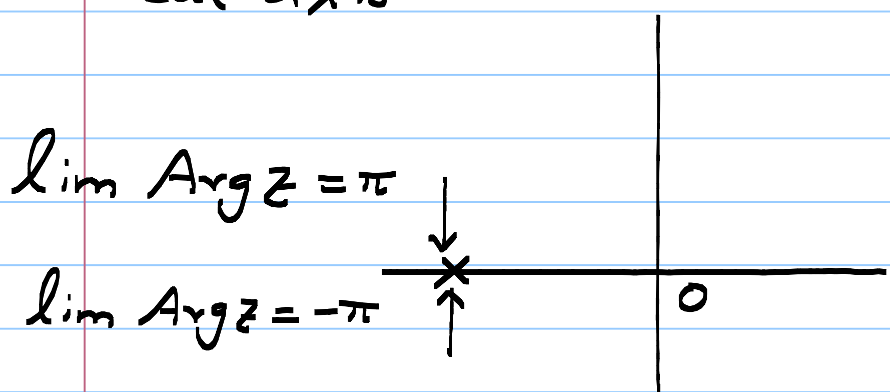
$$x = |z| \cos(\theta + 2\pi k)$$
$$y = |z| \sin(\theta + 2\pi k)$$

$\theta + 2\pi k$ is called a general argument of z denoted by

$$\arg z.$$

Thus, if $\phi = \arg z$ and
$\phi \in (-\pi, \pi]$ then
$\phi = \operatorname{Arg} z$.

The function $\operatorname{Arg} z$ is a single-valued function of z and is discontinuous along the negative real axis



Whereas the "function" $\arg z$ is multi-valued (!)

Now if $\theta = \arg z$ then

$$z = |z|(\cos\theta + i\sin\theta)$$

Note: Unlike real numbers there is no Order relation in the set of Complex numbers compatible with the Operation of addition and multiplication.

Exercise: Determine the fifth roots of unity in radicals

Sol: $z^5 - 1 = 0$

$z = 1$ or $z^4 + z^3 + z^2 + z + 1 = 0$

Continue!

Announcement:

Quiz I: Aug 14 (Saturday)

Time 11:00 am

Venue: Convocation Hall.

Quiz II: Sept. 8 (Wednesday)

Time. 8:30 am

Venue: Convo Hall

$$z^4 + z^3 + z^2 + z + 1 = 0$$

Dividing by z^2 we get

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = 0$$

$$\left(z + \frac{1}{z}\right)^2 - 2 + \left(z + \frac{1}{z}\right) + 1 = 0$$

put $\quad z + \frac{1}{z} = w$

$$w^2 + w - 1 = 0 \quad \text{so}$$

$$w = \left(-1 \pm \sqrt{5}\right)/2$$

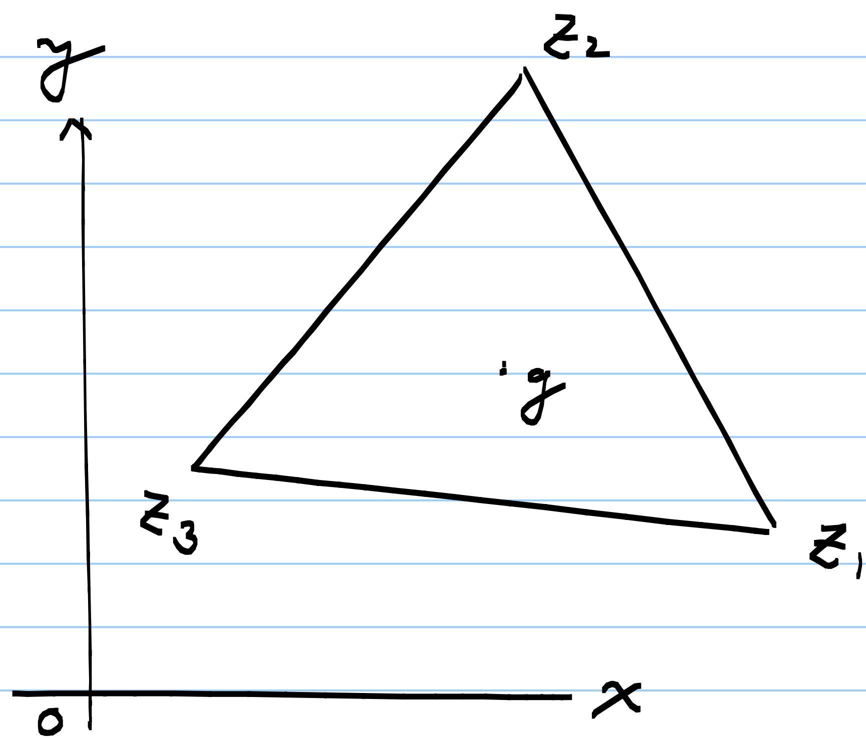
Now Solve $\quad z + \frac{1}{z} = \left(-1 \pm \sqrt{5}\right)/2$

using the Quadratic Formula.

<u>Exercise:</u> Given Complex numbers

z_1, z_2, z_3, Show that these form the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0.$$



The centroid g is the point given by the complex number

$$\frac{1}{3}(z_1 + z_2 + z_3) = 0$$

Observe that the Condition

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3 = 0$$

is invariant under translation so we may shift the coordinate axis so as to place the origin at the centroid of the triangle.

That is to say, we may assume that

$$z_1 + z_2 + z_3 = 0$$

The equation

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

is also invariant under rotations. A rotation of the plane is simply the mapping $T: \mathbb{C} \to \mathbb{C}$ given by

$$T(z) = z(\cos\theta + i\sin\theta)$$

Multiplication by $\cos\theta + i\sin\theta$ is equivalent to a counter-clockwise rotation through angle θ

Now we may assume z_3 real and positive.

Finally, the equation

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

is invariant under scaling

(Multiplication by a positive real number)

So we may assume $z_3 = 1$

Then $z_1 + z_2 = -1$ and

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - (z_1 + z_2) z_3 = 0$$

reduces to

$$z_1^2 + z_2^2 - z_1 z_2 + 2 = 0$$

Now find z_1, z_2 and finish the problem.

Note that we have used the symmetry in the defining equations to obtain the result.

Symmetries $\Rightarrow$ Invariance under suitable transformations

Lecture II

Complex Differentiability and the Cauchy Riemann Equations

Let Ω be an open region in the plane.

Example: (i) $\Omega = \{ z \in \mathbb{C} \,/\, |z| < 1 \}$

(ii) $\Omega = \{ x+iy \,/\, x, y \in \mathbb{R}$ and

$$\frac{x^2}{4} + \frac{y^2}{9} < 1 \}$$

(iii) $\Omega = \{ z \in \mathbb{C} \,/\, \operatorname{Im} z + \operatorname{Re} z > 7 \}$

We shall consider functions

$f : \Omega \to \mathbb{C}$ namely

$$f(x+iy) = u(x+iy) + i\, v(x+iy)$$

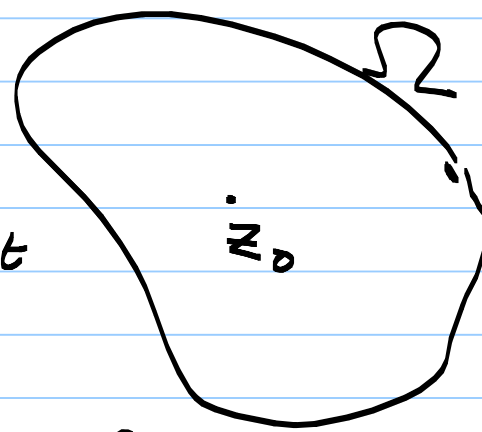
where $x, y \in \mathbb{R}$ and $u, v \in \mathbb{R}$

Thus a Complex valued function
$f : \Omega \to \mathbb{C}$ gives rise to
<u>two</u> Real-valued functions

$$u : \Omega \to \mathbb{R} \text{ and } v : \Omega \to \mathbb{R}.$$

f is continuous if and only if u and v are continuous.

Let $z_0 \in \Omega$. We say that



f is Complex diff at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{1}{h} \left(f(z_0 + h) - f(z_0) \right)$$

exists. We call the limit $f'(z_0)$ and refer to it as the derivative of f at z_0.

Now observe that h above is Complex and $h \to 0$ through Complex values in general. Two noteworthy special cases are

(i) $h \to 0$ through real values

(ii) $h \to 0$ through <u>purely</u> imaginary values

We now examine carefully these two cases and compare the results

$$\left(f(z_0+h) - f(z_0)\right)\frac{1}{h} \qquad (h = t \in \mathbb{R})$$

$$= \frac{1}{t}\left(u(x_0+t, y_0) - u(x_0, y_0)\right)$$

$$+ \frac{i}{t}\left(v(x_0+t, y_0) - v(x_0, y_0)\right)$$

Letting $t \to 0$ we get the result:

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$$

Exercise. Take the case when $h \to 0$ through purely imaginary values $h = it$, $t \in \mathbb{R}$ and $t \to 0$ and deduce that

$$f'(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

Comparing the two results we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (CR)$$

This pair of equations is called The Cauchy-Riemann Equations

Conversely one can prove using the Mean value theorem that if

$u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are a pair of real valued functions such that

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist in a neighborhood of z_0, are continuous at z_0 and satisfy the CR equations then

$$f(z) = u(x+iy) + i\,v(x+iy)$$ is

Complex differentiable at z_0.

We shall not prove the converse but take this for granted

Definition: If $f : \Omega \to \mathbb{C}$ is

Complex differentiable at every point of Ω then f is said to be

Analytic or Holomorphic on Ω.

In the tutorial sheets the word Analytic is employed though in many books you will see the modern terminology "holomorphic"

Example: Let $u = x^2+y^2$
$v = x^2-y^2$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\cancel{\partial y}}$ and so the CR equations

fails (except at the origin).
So
$(x^2+y^2)+i(x^2-y^2)$ is not holomorphic on $\mathbb{C}$.

Que: Is the function Complex differentiable at the origin?

Now let us assume that
$f = u+iv$ is holomorphic on Ω
and further that u, v have Continuous Second order partial derivatives

(It is actually a theorem that if $f: \Omega \to \mathbb{C}$ is holomorphic then

u, v, the real and imaginary parts of f are differentiable infinitely often)

We draw Some interesting Conclusions from the CR equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Diff. the first w.r.t x and Second w.r.t y and add

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$
$$= 0$$

Thus u Satisfies the Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Likewise, it is easy to check that

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Remark: The theory of Laplace's equation is atleast a 100 years older to the theory of function of a Complex variables

The Laplace's equation features prominantly in Electrostatics and so it is hardly surprising that Complex function theory finds its uses in electromagnetic theory.
It is also quite useful in Fluid Mechanics

Exercise: Let $u = \sin x \cosh y$
$v = \cos x \sinh y$
Check that the pair (u, v) satisfies the CR equations whereby we conclude

$$f(z) = (\sin x \cosh y) + i(\cos x \sinh y)$$

$(z = x + iy)$ is analytic.

Que: Consider an arbitrary vertical line L in the xy plane. Find the image of L in the

(u, v) plane under the map f.

Likewise determine the images

of horizontal lines.
At what angle do the image curves meet?

Recall the definition of a Convex region.

A subset S of $\mathbb{C}$ is said to be Convex if given two points $p, q \in S$, the line segment given by the set of points

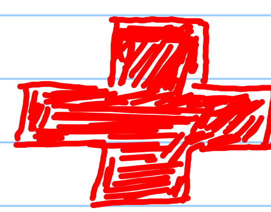
$$tp + (1-t)q : \quad 0 \leq t \leq 1$$

lies entirely in S.

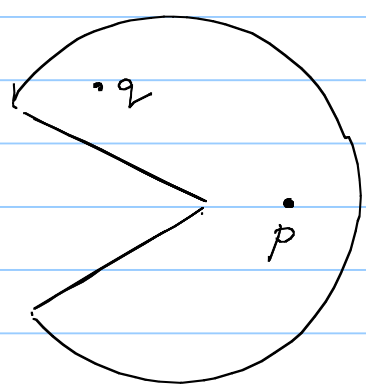
For example the points lying within a triangle or a parallelogram form a Convex Set.

Exercise: Prove that the Set of points inside an ellipse forms a Convex Set.

The set depicted is NOT Convex.



Def: A domain Ω is said to be starshaped with respect to a point $p \in \Omega$ if for every $q \in \Omega$ the line segment joining p and q lies entirely in Ω.



Picture depicts a domain that is star-shaped but not convex.

Recall now from MA 105

If $\vec{X}$ is a conservative vector field then
$\operatorname{Curl} \vec{X} = 0$

But $\operatorname{Curl} \vec{X} = 0$ does not imply that $\vec{X}$ is conservative

Example: $\vec{X} = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$

Check that $\operatorname{Curl} \vec{X} = 0$. Now if we let $\gamma(t) = (\cos t, \sin t)$; $0 \le t \le 2\pi$

then $\oint_{\gamma} \vec{X} \cdot d\vec{r} \neq 0$ showing

that $\vec{X}$ is <u>not</u> Conservative.

However on the domain
$\{ (x, y) \in \mathbb{R}^2 \,/\, x > 0 \}$ the
vector field <u>is</u> Conservative. Since it
admits a Scalar potential

$$\phi(x, y) = \tan^{-1}\left(\frac{y}{x} \right)$$

<u>Que</u>: Explain why $\phi(x,y)$ fails to
Serve as a Scalar potential on
$\mathbb{R}^2 - (0,0)$.

We have the following theorem that
provides a Sufficient (but not
necessary) Condition in order to
Conclude

$$\text{Curl } \overline{X} = 0 \implies \vec{X} \text{ is Conservative.}$$

Theorem: Suppose Ω is star-shaped and $\vec{X}$ is a smooth vector field on Ω then

$$\operatorname{Curl} \vec{X} = 0 \Rightarrow \vec{X} \text{ is conservative}$$

and hence $\exists$ a scalar potential ϕ

$$\vec{X} = \nabla \phi .$$

After this digression into vector calculus we return back to the theory of functions of a complex variable:

Assume $u : \Omega \to \mathbb{R}$ such that u is smooth and

$$\Delta u = 0 .$$

<u>Que:</u> Does there exist a smooth $v : \Omega \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Such a v (if it exists) is called a harmonic conjugate of u.

The harmonic conjugate of u is <u>NOT</u>

unique but if v_1, v_2 are two harmonic conjugates of u then

$$v_1 - v_2 \text{ is constant.}$$

We seek v such that

$$\nabla v = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right) = \vec{X}$$

Exercise: Verify Curl $\vec{X} = 0$.
So if Ω is star-shaped such a v exists and provides us the sought harmonic conjugate. We have proved

Theorem: If u is a smooth real-valued function on a starshaped domain Ω such that

$$\Delta u = 0 \quad \text{then}$$

there exists a v such that

$$u + iv \text{ is holomorphic on } \Omega$$

Note: The condition that Ω be

Star-Shaped may be weakened but cannot be dispensed off completely.

Example: $u(x,y) = \frac{1}{2}\log(x^2+y^2)$

on $\mathbb{C} - \{0\}$.

Check that $\Delta u = 0$ but a harmonic conjugate v does not exist on $\mathbb{C} - \{0\}$

Example: Let $u(x,y) = \sin x \cosh y$

It is easy to see that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let us look for a harmonic conjugate $v(x,y)$. That is, we seek a $v(x,y)$ such that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sin x \sinh y \quad (1)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \cos x \cosh y \quad (2)$$

From (1) $\quad v = \cos x \sinh y + \psi(y) \quad (3)$

Where $\psi(y)$ is a function of y alone

Which may have vanished while computing $\frac{\partial}{\partial x}$.

From (3),

$$\partial_y v = \cos x \cosh y + \psi'(y) \qquad (4)$$

Comparing (4) and (2) we see that

$$\psi'(y) = 0 \quad \text{or}$$

$$\psi \text{ is a constant } C$$

So $v(x,y) = \cos x \sinh y + C$ is a harmonic conjugate of $u(x,y)$.

Example: Show that

$$\frac{1}{2} \log\left(\frac{(x+1)^2 + y^2}{(x-1)^2 + y^2} \right)$$ is harmonic and find a harmonic conjugate in the ring

$$2 < x^2 + y^2 < 3$$

Problems for tut class on Friday

Exercise list I: 10, 12, 13, 15
Exercise list II: 2, 3.

Chapter III

Infinite Series; Power Series

Given a sequence $a_1, a_2, a_3, \ldots$ the infinite series $a_1 + a_2 + a_3 + \ldots$ is defined as the limit

$$\lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n) \text{ if it}$$

exists.
If this limit exists say ℓ then we shall write

$$\sum_{n=1}^{\infty} a_n = \ell.$$

and we say the series converges.

If $\lim\limits_{n\to\infty} (a_1 + \ldots + a_n)$ doesn't exist we say the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Example: Consider the series

$b + br + br^2 + \ldots\ldots$ (Geometric Series)

It converges if $b = 0$

When $b \neq 0$, it Converges precisely when $|r| < 1$ and diverges when $|r| \geq 1$

Example: Let us prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{diverges}$$

Proof: Suppose not

$$\lim \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \text{ exists and}$$

So in particular it is bounded. Let M be its bound. That is to say

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq M \quad \text{for Every } n$$

In particular take $n = 2^m$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$+ \left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}\right) \leq M$$

Replace $\frac{1}{3}$ by $\frac{1}{4}$

Replace $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ by $\frac{1}{8}$ and we see

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$+ \dots + \left(\frac{1}{2^m} + \frac{1}{2^m} + \dots + \frac{1}{2^m}\right) \leq M$$

i.e $\quad 1 + \frac{m}{2} \leq M$

Since m was arbitrary we have a Contradiction.

Exercise: Use the method of grouping as above to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \quad \text{Converges.}$$

One Can generally Show that

$1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ Converges if $p > 1$
diverges if $p \leq 1$

We shall refer to $1+\frac{1}{2}+\frac{1}{3}+\dots$ as the harmonic series.

Theorem: If $a_1+a_2+a_3+\dots$ converges then $\lim_n a_n = 0$

The converse is NOT true as is shown by the harmonic series.

However there is one important special case where the converse IS true we state it as a theorem

Theorem (Alternating Series test)
Suppose that a_1, a_2, a_3 are all non-negative,

$$a_1 \geq a_2 \geq a_3 \geq \dots$$

and $\lim_n a_n = 0$

Then $a_1 - a_2 + a_3 - a_4 + \dots$ converges
Moreover its sum lies between

$$a_1 - a_2 \quad \text{and} \quad a_1.$$

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ Converges and its sum lies between $\frac{1}{2}$ and 1.

Infact its sum is $\log 2$

Absolute and Conditional Convergence

Que: Suppose given an infinite series $a_1 + a_2 + a_3 + a_4 + \dots$ which is known to converge to say l

After permuting its terms would it remain convergent?

If so would its sum remain the same?

Answers to both these questions is unfortunately NO, Not necessarily

Suppose we permute the terms of a series

$$a_1 + a_2 + a_3 + a_4 + \dots \quad as$$

$$a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + a_9 + a_{11} + \dots$$

Let us look at its sequence of "partial sums"

$a_1 + a_2 + a_3 + \dots$ | $a_1 + a_3 + a_2 + a_5 + \dots$

a_1 | a_1

$a_1 + a_2$ | $a_1 + a_3$

$a_1 + a_2 + a_3$ | $a_1 + a_3 + a_2$

$a_1 + a_2 + a_3 + a_4$ | $a_1 + a_3 + a_2 + a_5$

$\vdots$ | $\vdots$

The resulting sequences are entirely different and so it should not be surprising if they have different limits

Example: We have seen that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots = l \; ; \quad \frac{1}{2} < l < 1.$$

Let us permute its terms as follows:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\dots$$

But $1-\frac{1}{2}=\frac{1}{2}$

$$\frac{1}{3}-\frac{1}{6}=\frac{1}{6} \; ; \quad \frac{1}{5}-\frac{1}{10}=\frac{1}{10}$$

So the result of permuting the terms is to produce the series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\dots$$

$$= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\right) = \frac{l}{2}$$

Que: Under what conditions can one rearrange the terms of an infinite series without altering its value??

Definition: A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem: $\sum_{n=0}^{\infty} |a_n|$ converges

$\Rightarrow \sum a_n$ converges

Def: If $\sum_n a_n$ converges but $\sum_n |a_n|$ diverges then we say $\sum_n a_n$ is conditionally convergent.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is conditionally convergent

Theorem (Dirichlet).

If $\sum a_n$ is absolutely convergent, its terms can be rearranged resulting in a series that converges to the same limit.

On the other hand, we have the following spectacular theorem of B. Riemann.

Theorem (Riemann): Given a Conditionally Convergent Series and <u>any</u> arbitrary real number α,

there exists a rearrangement such that the resulting series converges to α.

<u>Tests for Convergence</u>

Suppose $\sum a_n$ and $\sum b_n$ are two series such that ~~for some~~ n_0,

$$|a_n| \leq b_n \quad \text{for all } n \geqslant n_0$$

we shall say

$$\sum b_n \text{ dominates } \sum a_n.$$

<u>Theorem</u>: If $\sum b_n$ dominates $\sum a_n$ and

$\sum b_n$ converges then $\sum a_n$ converges absolutely;

If $\sum a_n$ diverges then $\sum b_n$ also must diverge.

Theorem (D'Alembert): Suppose $\sum a_n$ is an infinite series such that $a_n \neq 0$ for any n

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \text{ exists } = \ell \text{ or is } +\infty$$

Then (i) $\sum a_n$ Converges absolutely if $\ell < 1$

(ii) $\sum a_n$ diverges if $\ell > 1$ or $\ell = +\infty$

Test fails if $\ell = 1$

WARNING: It is not enough that $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all n.

One MUST take the limit of $\left|\frac{a_{n+1}}{a_n}\right|$ before applying the ratio test.

Theorem (Cauchy): Suppose $\sum a_n$ is an infinite series and

$$\lim_{n\to\infty} |a_n|^{1/n} \text{ exists } = \ell$$

(or is $+\infty$). Then

(i) $\sum a_n$ Converges absolutely if $\ell < 1$

(ii) $\sum a_n$ diverges if $\ell > 1$ or $\ell = +\infty$

Test fails if $\ell = 1$.

Example. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ Converges absolutely for every complex z.

Proof: Let us apply the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{n+1}}{(n+1)!}\right| \Big/ \left|z^n/n!\right|$$

$$= \frac{|z|}{n+1}$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ and so the series Converges for all Complex z.

Definition: The function $\exp z$ is defined as the sum of the series

$$\sum_{n=0}^{\infty} z^n/n!$$

Also denoted by e^z.

Discuss the Convergence of the following infinite series:

(1) $\sum z^n/n^2$ (Sum is called the dilogarithm)

(2) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$ Sum is denoted by $\sin z$

(3) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$ Sum is called $\cos z$

(4) $z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$

(5) $z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

(6) For what values of z does

$$\sum \frac{(3n)!}{n!^3} \cdot z^n \text{ converge?}$$

(7) Discuss the convergence of

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} z^n.$$

The Cauchy Product of two Series:

Consider two infinite Series

$$u_0 + u_1 + u_2 + \ldots$$

$$v_0 + v_1 + v_2 + \ldots$$

Their Cauchy product is defined as

$$u_0 v_0 + (u_0 v_1 + u_1 v_0) + (u_0 v_2 + u_1 v_1 + u_2 v_0) + \ldots$$

Theorem (Mertens): If the two Series

$u_0 + u_1 + u_2 + \ldots$

$v_0 + v_1 + v_2 + \ldots$ Converge to say u and v and atleast one of them Converges absolutely then their Cauchy Product also Converges and its sum is uv.

Example: Consider the two Series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = e^z \text{ and}$$

$$1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots = e^w$$

They both converge absolutely as we have seen. So their Cauchy product converges to
$$e^z \cdot e^w.$$
Let us now compute the Cauchy product.

Recall that the Cauchy product is

$u_0 v_0 + (u_0 v_1 + u_1 v_0) + \cdots$ whose general term is
$$\sum_{k=0}^{n} u_k v_{n-k}$$
In this case,
$$\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n = e^{z+w}$$

We have established the following

Exponential Addition Theorem:

$$e^{z+w} = e^z \cdot e^w$$

Now use the definition of $\sin z$, $\cos z$ to show that

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{iz}\right)$$

and prove that

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$

$$e^{iz} = \cos z + i \sin z.$$

For $z = x+iy$, $x, y \in \mathbb{R}$

$$e^z = e^x \cdot e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x \cos y + i e^x \sin y$$

Cor: e^z does not vanish

For if $e^{z_0} = 0$ then

$$e^{z_0 - z_0} = e^{z_0} \cdot e^{-z_0} = 0$$

But $e^{z_0 - z_0} = e^0 = 1$ and we get a contradiction.

Remark: The non vanishing of the exponential function has serious consequences.

Def: A series of the form

$a_0 + a_1 z + a_2 z^2 + \dots$ is called a power series. Examples are

(1) $\quad 1 + z + \frac{z^2}{2!} + \dots$

(2) $z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

(3) $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{4^n (n!)^2}$

(The sum of this series is called the Bessel's function of order zero)

(4) $\sum \frac{(3n)!}{(n!)^3} \cdot z^n$

(5) $\sum_{n=1}^{\infty} n^n \cdot z^n$

(6) $\sum_{n=1}^{\infty} n! \, z^n$

(7) $\sum_{n=0}^{\infty} z^{2^n}$

Exercise: Apply the ratio test in each case and find the values of z for which the series converges absolutely.

In all the examples Considered we see that for a power Series

$$\sum_{n=0}^{\infty} a_n z^n$$ the set of Complex values for which it Converges is a disc Centered at the origin

This disc may be the whole of $\mathbb{C}$ or may sometimes be the single point $\{0\}$

These three situations are illustrated by the examples

(i) $\sum_{n=0}^{\infty} z^n$ Converges absolutely on $|z|<1$

(ii) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ Converges absolutely for all values of z

(iii) $\sum_{n=0}^{\infty} n!\, z^n$ which Converges only when $z=0$

The following result holds

Theorem: Given a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

there exists a real number $R \geq 0$ such that the series converges absolutely on $|z| < R$ and diverges on $|z| > R$.

In general nothing can be said as to the behaviour of the series on the circle of convergence.

R is called the Radius of Convergence of the power series

Note: Sometimes one may have to consider a power series with center at z_0:

$$\sum a_n (z - z_0)^n$$

Again the region of Convergence of the power Series is a disc

$$|z - z_0| < R$$

with centre at z_0.

The Series Converges absolutely on $|z-z_0|<R$
diverges on $|z-z_0|>R$

and on the circle $|z-z_0|=R$ nothing Can be Said in general.

<u>Differentiation Theorem:</u>

The Sum of a power Series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is an analytic function on the disc of Convergence $|z-z_0|<R$

Denoting the Sum by $f(z)$ we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$ That is to

Say, the power series may be differentiated term by term within the disc of convergence.

The series of derivatives

$$\sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} \quad \text{also}$$

converges on the disc $|z-z_0| < R$.

Example: $e^z = 1 + z + \frac{z^2}{2!} + \cdots$

The disc of convergence is $\mathbb{C}$ and

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} n \cdot \frac{z^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z .$$

Example: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

The power Series Converges for all values of $z \in \mathbb{C}$.

Term by term differentiation gives

$$\frac{d}{dz}(\sin z) = \cos z \, ; \quad \frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\sin^2 z + \cos^2 z) = 2\sin z \cos z - 2\cos z \sin z$$
$$= 0$$

$$\therefore \sin^2 z + \cos^2 z = \text{constant}$$

putting $z = 0$ we get that

$$\sin^2 z + \cos^2 z = 1$$

<u>Exercise</u>: Check that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

Now $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$
$$= \sin x \cosh y + i \cos x \sinh y$$

Note that

$$\left.\begin{aligned} \sinh z &= z + \frac{z^3}{3!} + \cdots \\ \cosh z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \end{aligned}\right\} (\text{Def})$$

Hence $\cos iz = \cosh z$
$\sin iz = i\sinh z$

$$|\sin z|^2 = |\sin x \cosh y + i\cos x \sinh y|^2$$
$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2 x \cosh^2 y + \cos^2 x (\cosh^2 y - 1)$$
$$= \cosh^2 y - \cos^2 x$$

Observe that as $y \to \pm\infty$

$|\sin z| \longrightarrow +\infty$

Thus $\sin z$ is unbounded.

Exercise: Prove that $|\cos z| \to +\infty$ as $\operatorname{Im} z \to \pm\infty$

<u>Def:</u> Suppose a power series
$\sum_{n=0}^{\infty} a_n z^n$ Converges for all values of $z \in \mathbb{C}$,
the sum of the power series defines an analytic function on $\mathbb{C}$.

Such an analytic function is said to be

an Entire Function.

Examples: e^z, $\sin z$, $\cos z$ are entire functions

Polynomials are entire functions.

Here is yet another example

$$\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n} \frac{(-1)^n}{n!^2}$$

The sum of this power series is denoted by $J_0(z)$ called the Bessel's function of order ZERO.

Theorem: If an entire function is bounded then it is a constant.

This theorem is called Liouville's thm.

Exercise. Use Liouville's theorem to prove that a non-constant polynomial has a complex root.

The little Picard theorem

An entire function which misses two complex values is a constant.

<u>Note</u>: The exponential function e^z misses the value 0

A non constant polynomial assumes <u>all</u> complex values

<u>Ex</u>: Show that $\sin z$ <u>does not</u> miss any complex value.

Do this in two ways

(i) $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

(ii) Using the little Picard theorem

<u>Ex</u>: Determine the zeros of $\sin z$ and $\cos z$

Use $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\cos z = \frac{1}{2}\left(e^{zi} + e^{-iz}\right)$

Find the largest domain on $\mathbb{C}$ on which $e^z/(\sin z + \cos z)$ is analytic.

Prove that if x is a real number in $(0, \pi/2)$

$$\sin x \geq \frac{2}{\pi} x$$

Find the coefficient of z^n in the power series for $e^z \cos z$.

Prove that

$$\left(\sin \frac{\pi}{k}\right)\left(\sin \frac{2\pi}{k}\right) \cdots \left(\sin \frac{(k-1)\pi}{k}\right) = \frac{k}{2^{k-1}}$$

<u>Hint</u>: let $\xi_1, \ldots \xi_k$ be the roots of

$z^k = 1$ with $\xi_1 = 1$

Evaluate the limit

$$\lim_{\xi \to 1} \frac{\xi^k - 1}{\xi - 1}$$ in two ways.

Exercise: Compute the n-th term of the Cauchy product $\left(\sum_{j=0}^{\infty} \frac{1}{2^j}\right)\left(\sum_{j=0}^{\infty} \frac{1}{2^j}\right)$

Compute the n-th term of the Cauchy product of

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{\sqrt{j+1}}$$ with itself.

Does the Cauchy Product Converge?

Find the sum of the series

$$1 + \cos\theta + \frac{1}{2!}\cos 2\theta + \cdots$$ and

$$\sin\theta + \frac{1}{2!}\sin 2\theta + \cdots$$

The logarithm and Inverse Trig. Functions

Given a non zero complex number a we can find a complex number b such that

$$e^b = a$$

This b is called a logarithm of a.

Well, taking the absolute values

$$|e^b| = |a|$$

$\therefore$ $e^{\operatorname{Re} b} = |a|$ so $\operatorname{Re} b = \ln|a|$ the usual real logarithm of $|a|$

Note: $a \neq 0$ so $|a| > 0$ and $\ln|a|$ is defined.

So $\dfrac{e^b}{e^{\operatorname{Re} b}} = \dfrac{a}{|a|}$; put $b = x+iy$, $\operatorname{Re} b = x$

$$\therefore \quad e^{iy} = \frac{a}{|a|}$$

$$\therefore \quad \cos y + i\sin y = \frac{a}{|a|}.$$

Hence $y = \arg a$ and we have

$$b = (\mathrm{Re}\, b) + i(\mathrm{Im}\, b) \qquad e^b = a$$

$$b = \ln|a| + i \arg a \iff b = \log a$$

Now since $\arg a$ is <u>not</u> a single-valued function, $\log a$ is <u>also</u> multi-valued.

If we take the principal value $\operatorname{Arg} a$ the corresponding logarithm will be denoted by $\operatorname{Log} a$ (with a capital L)

and called the principal logarithm or the principal value of the logarithm or the principal branch of the logarithm.

<u>Is the function $\operatorname{Log} z$ holomorphic?</u>

Let $z = x + iy$

Then $\ln|z| = \frac{1}{2} \ln(x^2+y^2)$

We have seen that $\frac{1}{2}\ln(x^2+y^2)$ is harmonic and its harmonic conjugate on $\mathbb{C} - (-\infty, 0]$ is $\operatorname{Arg} z$.

Thus $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ is holomorphic on $\mathbb{C} - (-\infty, 0]$.

Let us calculate $\frac{d}{dz}(\operatorname{Log} z)$

Well, $f'(z) = u_x + i v_x$

$$= u_x - i u_y$$

here $u = \frac{1}{2}\ln(x^2+y^2)$

So $u_x = \dfrac{x}{x^2+y^2}$; $u_y = \dfrac{y}{x^2+y^2}$

$$\therefore f'(z) = \frac{x - iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

In otherwords $\frac{d}{dz}(\operatorname{Log} z) = \frac{1}{z}$

The power functions

When $a \in \mathbb{C}$ we define

$$z^a = \exp a \log z.$$

The multivaluedness of $\log z$ implies that z^a is also multivalued

But taking the principal value $\operatorname{Log} z$ we get the principal value for the power function namely

$$\exp(a \operatorname{Log} z)$$

This is holomorphic on $\mathbb{C} - (-\infty, 0]$

$$\frac{d}{dz}(z^a) = \frac{d}{dz}(\exp a \operatorname{Log} z)$$

$$= \exp a \operatorname{Log} z \cdot \frac{d}{dz}(a \operatorname{Log} z)$$

$$= (\exp a \operatorname{Log} z) \cdot \frac{a}{z}$$

$$= a\,(\exp a\,\mathrm{Log}\,z)\,\exp(-\mathrm{Log}\,z)$$

$$= a\,\exp((a-1)\,\mathrm{Log}\,z)$$

Thus $\frac{d}{dz}(z^a) = a z^{a-1}$ with the understanding that both sides are assigned principal values

<u>Binomial Theorem:</u>

Let $a \in \mathbb{C} - \{0, 1, 2, 3, \ldots\}$

$$1 + az + \frac{a(a-1)}{2!} z^2 + \cdots$$

has radius of convergence 1

The sum of the series gives the principal value of $(1+z)^a$; $|z| < 1$.

When $a = 0, 1, 2, \ldots$ the series terminates giving a polynomial and we recover the usual binomial theorem for positive integer powers

Example: Determine the principal value of i^i

Determine all the values of i^i

How many values does $i^{7/4}$ have?

How many values does $i^{\sqrt{2}}$ have?

Now suppose z traces a simple closed curve encircling the origin once counter clock-wise returning to its starting position the value of the argument changes by 2π

Hence the value of $\log z$ changes additively by $2\pi i$

However since $\sqrt{z} = \exp \frac{1}{2}(\log z)$, the value of $\sqrt{z}$ changes

Multiplicatively by a factor

$$\exp \tfrac{1}{2}(2\pi i) = -1$$

Now suppose z makes one more circuit around the curve the argument would change additively by $4\pi i$ and $\sqrt{z}$ would change multiplicatively by $\exp \frac{1}{2}(4\pi i) = 1$

Thus each circuit results in the introduction of a multiplicative factor of (-1) in $\sqrt{z}$

In a sense the two values of the square are "conneted" and are not "disjoint entities".

Theorem: Suppose $|z|<1$ the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$ converges to

$\text{Log}(1+z)$

Proof: The radius of convergence of the series is 1.

Let $f(z)$ be the sum of the power series

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \; ; \; |z|<1$$

Term by term differentiation (valid on $|z|<1$) gives

$$f'(z) = 1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z}$$

$$= \frac{d}{dz}\left(\text{Log}(1+z)\right)$$

$\therefore$ $f(z) - \text{Log}(1+z)$ is a constant.

Setting $z=0$ we see that the constant must be zero. The proof is complete

The inverse trigonometric functions:

Given $w \in \mathbb{C}$, we seek a $z \in \mathbb{C}$ such that

$$\sin z = w$$

Clearly such a z is NOT unique thereby we get a <u>multivalued</u> function

$\sin^{-1} w$. Let us investigate this closely.

$$\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = w$$

$$\therefore \quad e^{iz} - e^{-iz} = 2iw$$

$$\therefore \quad e^{2iz} - 2iwe^{iz} - 1 = 0 \quad \text{or}$$

$$e^{iz} = iw + \sqrt{1-w^2}$$

Let us check that RHS $\neq 0$ (since the exponential function misses the value 0)

$$iw + \sqrt{1-w^2} = 0 \Rightarrow iw = -\sqrt{1-w^2}$$
$$\Rightarrow -w^2 = 1-w^2$$
$$\Rightarrow 0 = 1 \quad \text{Contradiction}$$

So $\quad iw + \sqrt{1-w^2} \neq 0$ and

$$z = \frac{1}{i} \log\left(iw + \sqrt{1-w^2}\right)$$

That is

$$\sin^{-1} w = \frac{1}{i} \log\left(iw + \sqrt{1-w^2}\right)$$

If we select the principal value for the logarithm and square root, the special value of the inverse sine will be denoted by $\mathrm{Sin}^{-1} w$ (with a capital S)

$$\mathrm{Sin}^{-1} w = \frac{1}{i} \mathrm{Log}\left(iw + \sqrt{1-w^2}\right)$$

Exercise: Determine

$$\cos^{-1} w \quad \text{and} \quad \cosh^{-1} w$$

Exercise: Suppose that $f(z) = u + iv$ is holomorphic. Check that

$U = \frac{1}{2} \log(u^2+v^2)$ is harmonic provided $f \neq 0$ on Ω

In particular if Ω is convex or star-shaped, a harmonic conjugate V exists and

$F(z) = U + iV$ is holomorphic

One checks that

$$\exp F(z) = f(z) \text{ and so}$$

$\log f(z)$ is a single valued holomorphic function on Ω.

Thus $\sqrt{f(z)} = \exp \frac{1}{2}(\log f(z))$ is a single-valued holomorphic function on Ω if $f \neq 0$ on Ω and Ω is star-shaped.

<u>Que:</u> On what domain in $\mathbb{C}$ is $\sin^{-1} w$ a single valued holomorphic function?

Que: Specify domains in $\mathbb{C}$ on which $\sqrt{1-z^2}$ is single valued and holomorphic

Exercise: Determine $\tan^{-1} z$ and $\operatorname{Tan}^{-1} z$. Specify domains on $\mathbb{C}$ on which these are single-valued and holomorphic.

Exercise: On what domain on $\mathbb{C}$ is $\sin\sqrt{z}$ holomorphic?

Prove that $\cos\sqrt{z}$ is entire.

What about $\dfrac{\sin\sqrt{z}}{\sqrt{z}}$?

Is the function

$$f(z) = \cosh z^{1/3} + \cosh \omega z^{\frac{1}{3}} + \cosh \omega^2 z^{1/3}$$

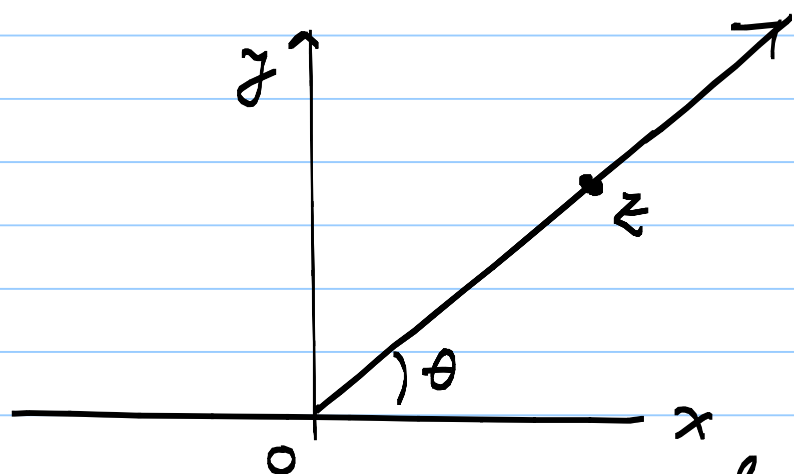
entire?

(All three occurrences of $z^{1/3}$ are assigned the same value)

Prove that $\cot \pi z$ is bounded along the sides of the rhombus with vertices $N+\frac{1}{2}$, $iN+\frac{i}{2}$, $-N-\frac{1}{2}$, $-iN-\frac{i}{2}$ (with a bound independent of N)

Exercise: Determine $\lim_{|z|\to\infty} \tan z$ as z tends to infinity along a ray emanating at the origin making an angle θ with positive x-axis.

$0<\theta<\pi$



Exercise: Show that $\tan z$ fails to take the values $\pm i$

The Cauchy Integral Theorem:

Recall: A simple closed curve γ in the plane is a continuous and piecewise smooth map

$$\gamma : [a,b] \to \mathbb{R}^2 \text{ such that}$$

$$\gamma(a) = \gamma(b) \quad \text{and for}$$

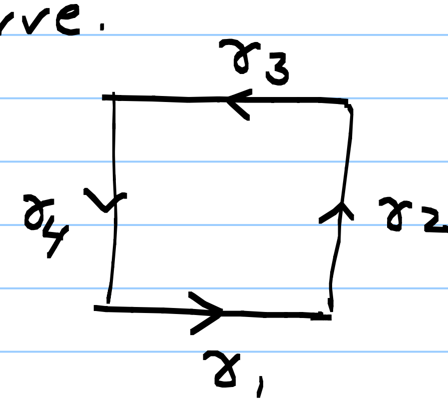
$$a \le t_1 < t_2 \le b, \quad \gamma(t_1) = \gamma(t_2)$$
$$\Rightarrow t_1 = a, \; t_2 = b.$$

Ex: The parametrized curve

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t); \quad 0 \le t \le 1$$

is a simple closed curve.

Ex: The unit square made up of the juxtaposition of the four line segments $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is a simple closed curve



Numerous examples may be constructed
the boundary of a triangle,
An ellipse etc;

Given a parametrized curve γ one
can compute the line integral

$$\int_\gamma P\,dx + Q\,dy \quad \text{as}$$

$$\int_a^b \left(P(\gamma(t)) \frac{d\gamma_1}{dt} + Q(\gamma(t)) \frac{d\gamma_2}{dt} \right) dt$$

Where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and

$\gamma : [a,b] \to \mathbb{R}^2$.

Now, if γ is the juxtaposition
of $\gamma_1, \gamma_2, \gamma_3, \ldots$ (finite) then

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \cdots$$

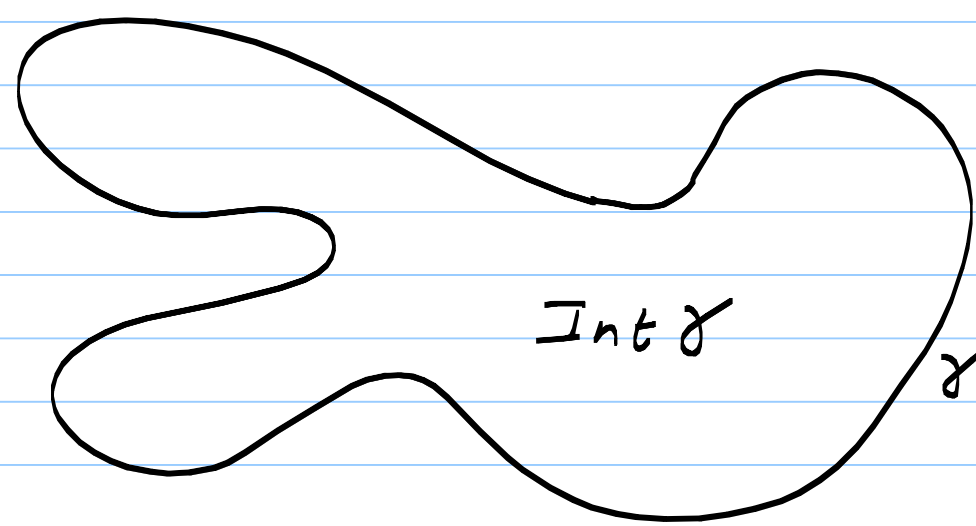
The Jordan Curve Theorem:

(i) A simple closed plane curve divides the plane into two open regions precisely one of which is unbounded.

(ii) The bounded component may be mapped onto a disc by a bijective continuous function with a continuous inverse.

<u>Def:</u> The bounded component is called the interior of γ

This is a deep theorem in Topology



Green's Theorem in the plane

Let $P(x,y)$, $Q(x,y)$ be continuously differentiable in an open set Ω and γ be a simple closed curve lying in Ω such that $\text{Int}\,\gamma \subset \Omega$.

Then

$$\oint_{\gamma} (P\,dx + Q\,dy) = \iint_{\text{Int}\,\gamma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

γ traced counter clockwise.

Example: Let $P = \dfrac{y}{x^2+y^2}$

$$Q = \frac{-x}{x^2+y^2}$$

Calculate $\oint P\,dx + Q\,dy$ along the circle $(\cos 2\pi t, \sin 2\pi t) ; 0 \le t \le 1$ and also compute $\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Does this Contradict Green's thm? Why?

Now suppose Ω is an open set and $f : \Omega \to \mathbb{C}$ is holomorphic (i.e the Cauchy Riemann Equations holds and the partial derivatives of $u = Re f$ and $v = Im f$ namely u_x, u_y, v_x, v_y are continuous)

$\int_\gamma f(z) dz$ denotes a Complex #

$$\int_\gamma (u+iv)(dx+idy)$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma (u\,dy + v\,dx)$$

Suppose that γ is a Simple closed curve lying in Ω such that $\mathrm{Int}\,\gamma \subset \Omega$ then we can apply Green's theorem to the individual pieces and get

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (u\,dy + v\,dx)$$

$$= \iint_{\mathrm{Int}\,\gamma} -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + \iint_{\mathrm{Int}\,\gamma} \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) dx\,dy$$

Which in view of the Cauchy Riemann Equations

Vanishes! Thus we have proved

Theorem (Cauchy's theorem):
If $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a simple closed curve in Ω such that $\operatorname{Int}\gamma \subset \Omega$ then

$$\oint_\gamma f(z)\,dz = 0$$

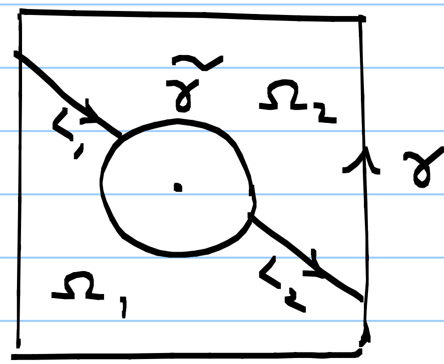
<u>Exercises</u>: Calculate $\displaystyle\oint_\gamma z^n\,dz$

where γ is the unit circle traced counterclockwise

Consider the cases $n \in \mathbb{N}$ and $n \in \mathbb{Z},\ n < 0$.

<u>Exercise</u>: Discuss how to compute $\displaystyle\oint_\gamma \frac{dz}{z}$ when γ is a square centered at the origin

Let us take up the Second example



Let $\tilde{\gamma}$ denote a circle $|z| = \rho$. Make two cross cuts L_1 and L_2 as shown above to get two regions Ω_1, Ω_2 between the Square and the circle.

Apply Cauchy's theorem to the boundaries of Ω_1 and Ω_2 and add.

We See that

The contribution from the cross-cuts cancel out leaving

$$\oint_{\gamma} \frac{dz}{z} = \oint_{\tilde{\gamma}} \frac{dz}{z}$$

The advantage is that the integral over a circle is easy to compute: put $z = \rho e^{it}$

$0 \le t \le 2\pi$

$$\oint_{\tilde{\gamma}} \frac{dz}{z} = \int_0^{2\pi} \frac{\rho i e^{it} dt}{\rho e^{it}}$$

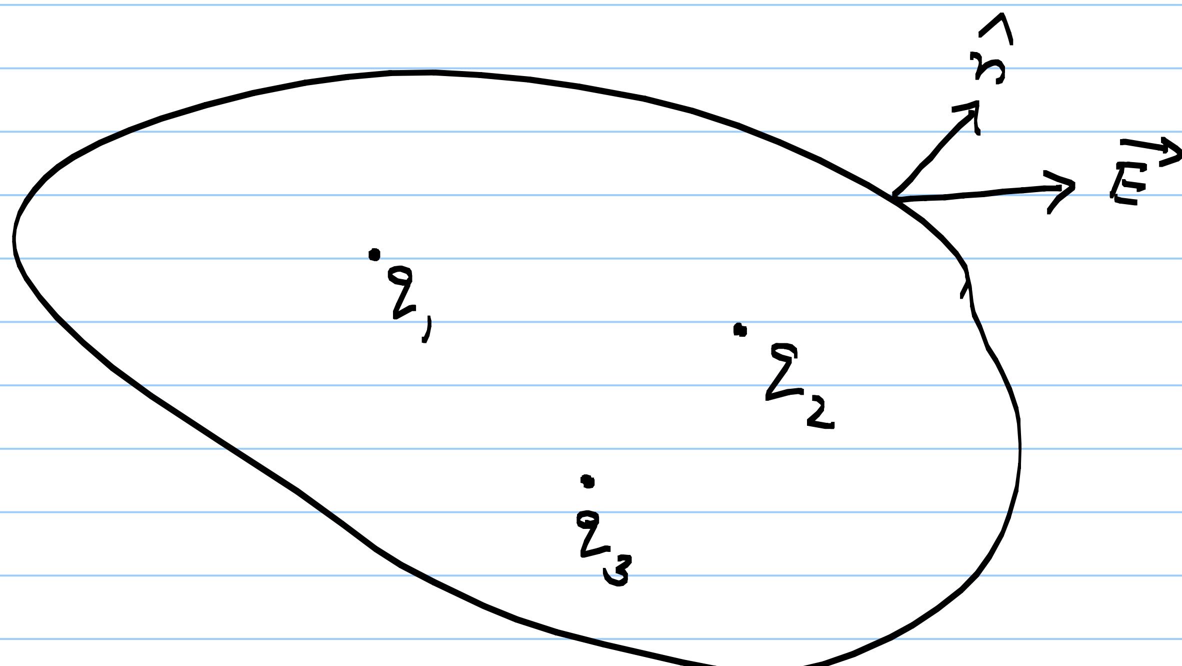
$$= i \int_0^{2\pi} dt = 2\pi i$$

This technique can be carried out in great generality

Recall the classical Gauss' divergence theorem

$$\int_S \vec{E} \cdot \hat{n} \, ds = \frac{1}{\epsilon_0} \sum_j q_j$$

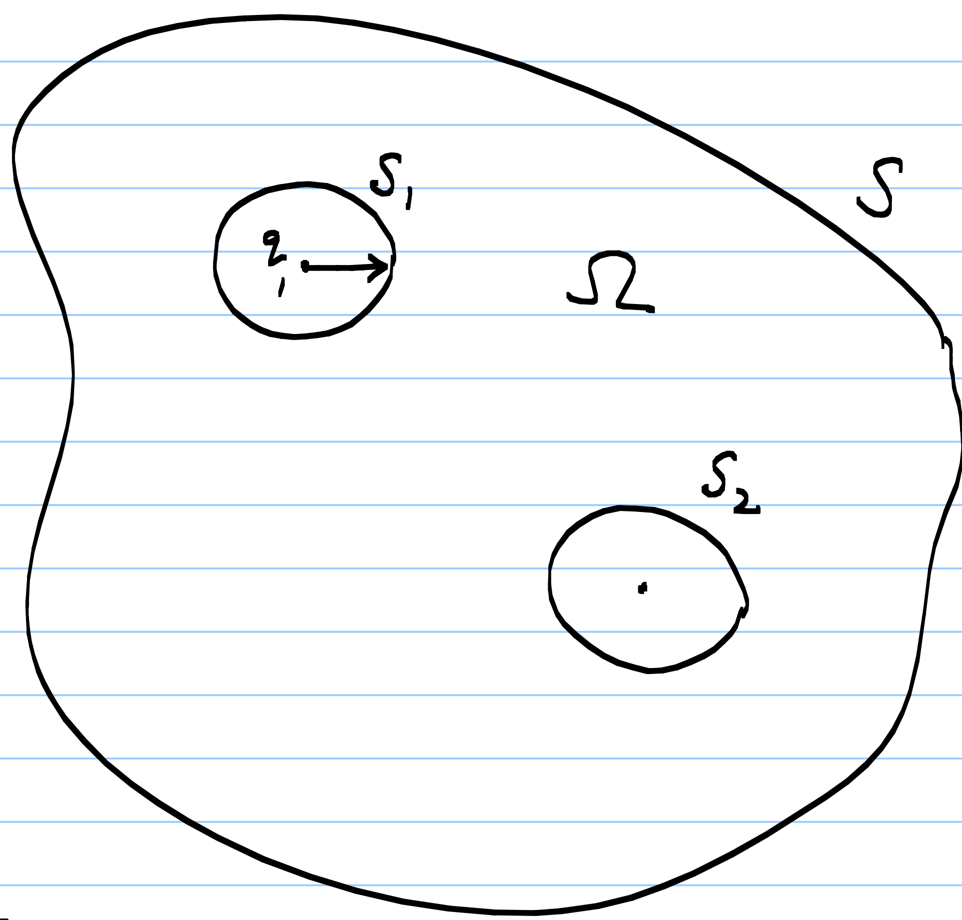
where the sum on the RHS is over the charges contained within the closed surface S.



$\hat{n}$ denotes the outer unit normal (as viewed from within the surface S)

Let us recall how one proves the Classical theorem of Gauss in Electrostatics.

At each of the Charges q_j Scoop out a small ball of radius ρ_j



Then $\operatorname{Div} \vec{E} = 0$ on Ω the region inside S and outside the Spheres $S_1, S_2, S_3 \ldots$

Then $$\int_{\partial\Omega} \vec{E}\cdot\hat{n}\, dS = \iiint_{\mathrm{Int}\,\Omega} \mathrm{Div}\,\vec{E}\, dx\,dy\,dz$$

(which is the 3D analogue of Green's theorem)

But $\partial\Omega$ Consists of $S, S_1, S_2, \ldots$

$$\int_{S} \vec{E}\cdot\hat{n}\, dS - \int_{S_1} \vec{E}\cdot\hat{n}\, dS - \int_{S_2} \vec{E}\cdot\hat{n}\, dS - \ldots = 0$$

(why the negative Sign ?)

Now let us Calculate

$$\int_{S_j} \vec{E}\cdot\hat{n}\, dS = \frac{q_j}{4\pi\epsilon_0}\int \frac{\vec{x}-\vec{a}_j}{|x-a_j|^3}\cdot\hat{n}\, dS$$

but $\hat{n} = \dfrac{\vec{x}-\vec{a}_j}{|x-a_j|}$ and so

$$\int_{S_j} \vec{E} \cdot \hat{n}\, ds = \frac{1}{4\pi\epsilon_0} \int_{S_j} \frac{q_j\, ds}{|\bar{x} - \bar{q}_j|^2}$$

$$= \frac{q_j}{\rho_j^2} \int_{S_j} ds \cdot \frac{1}{4\pi\epsilon_0}$$

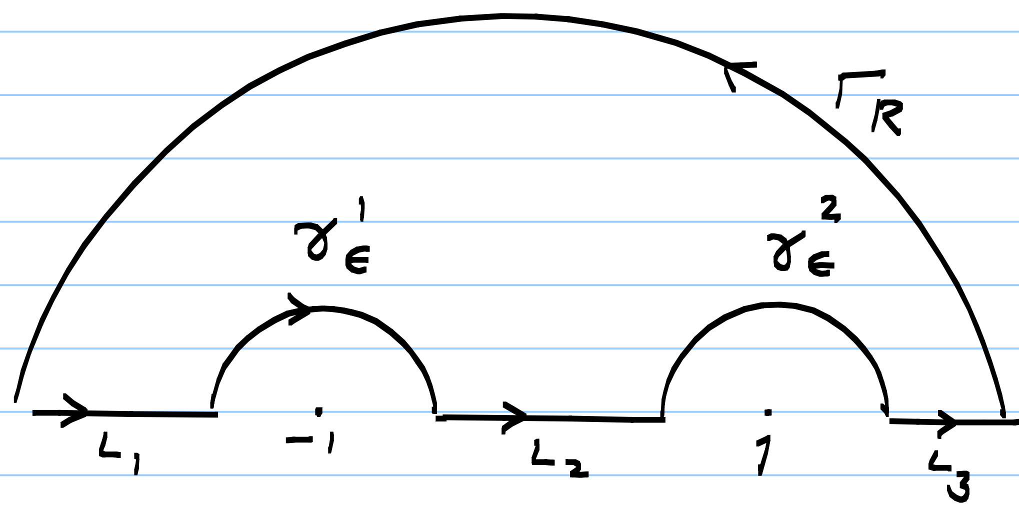
$$= \frac{q_j}{\rho_j^2} \cdot \frac{4\pi \rho_j^2}{4\pi\epsilon_0}$$

$$= q_j / \epsilon_0$$

In the Context of Complex analysis the integral of $\oint_\gamma \frac{dz}{z}$ where γ is a Simple Closed Curve Containing the Origin inside, is the Analogue of the integral of Gauss in Electrostatics

Exercise: Compute $\int_{-\infty}^{\infty} \frac{\cos \frac{\pi x}{2}}{1-x^2} dx$

Take $f(z) = \frac{\exp(i\pi z/2)}{1-z^2}$. Integrate $f(z)$ around a semi-circular contour with indentations at $z = -1$ and $z = -1$



Check: $\int_{\Gamma_R} f(z)dz \longrightarrow 0$ as $R \to \infty$

There are no poles within the contour

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ \int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_3} f(z)dz \right\}$$

$$= \int_{-\infty}^{\infty} \left(\cos \frac{\pi x}{2} dx \right) / (1-x^2)$$

The imaginary part cancels out.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{-e^{iz\pi/2}}{z+1}$$

$$= -\frac{1}{2} e^{i\pi/2} = -i/2$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{e^{iz\pi/2}}{1-z} = \frac{1}{2} e^{-i\pi/2}$$

$$= -i/2$$

Both arcs γ_ϵ^1, γ_ϵ^2 are traced clockwise

$$\lim_{\varepsilon\to 0} \left(\int_{\gamma_\epsilon^1} f(z) + \int_{\gamma_\epsilon^2} f(z)\,dz \right)$$

$$= -i\pi(-i) = -\pi$$

$$= -\pi$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi x}{2}\right) dx}{1-x^2} = -\pi$$

Exercise: (1) Compute

$$\int_{-\infty}^{\infty} \frac{\sin \pi x \, dx}{x(1-x^2)}$$

(2) Let us now try

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx \; ; \quad a, b \in \mathbb{R}, \quad a > b > 0$$

The obvious choice is

$$f(z) = \frac{e^{iaz} - e^{-ibz}}{z^2}$$

and the contour a semi-circle indented at the origin. The origin is a simple pole and fractional Residue Theorem applies.

Complete the task

(3) What if there is a double pole or a triple pole?

Frac. Res. Thm cannot be applied but a little cleverness helps.

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ and so

$$= \mathrm{Im}\left(\frac{1}{4}\left(\frac{3e^{ix} - e^{3ix}}{x^3}\right)\right)$$

It is tempting to take

$$f(z) = \frac{1}{4}\,\frac{3e^{iz} - e^{3iz}}{z^3}$$

and use a semi-circular contour indented at the origin. But the origin is a TRIPLE POLE.
Frac. Res. Thm. cannot be used!

Try modifying $f(z)$ suitably to make it work.

Well,

$$3e^{iz} - e^{3iz}$$

$$= 2 + 3z^2 + \dots$$

So that $$\frac{3e^{iz} - e^{3iz}}{z^3}$$

$$= \frac{2}{z^3} + \frac{3}{z} + \dots$$

If we modify $f(z)$ as

$$g(z) = \frac{1}{4}\left\{ \frac{3e^{iz} - e^{3iz} - 2}{z^3} \right\}$$

and work with $g(z)$ then along the real axis we get

$$\operatorname{Im} g(x) = \operatorname{Im} f(x)$$

$g(z)$ has a simple pole at 0

with residue $3/4$.

$$\int_{\Gamma_R} f(z)\,dz \xrightarrow[R\to\infty]{} 0$$

$$\int_{\gamma_\epsilon} f(z)\,dz \longrightarrow -i\pi \cdot \frac{3}{4} \quad \text{as } \epsilon \to 0$$

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz$$

$$\xrightarrow[\substack{R\to\infty \\ \epsilon\to 0}]{} i\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx \quad \text{and so}$$

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3}{4}\pi$$

Exercise:

Evaluate (1) $$\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\, dx$$

(2) You might wish to try computing

$$\int_0^{\infty} \frac{\sin^4 x\, dx}{x^3}$$ starting with

$$\frac{\sin^4 x}{x^3} = \frac{1}{8x^3}\left(\cos 4x - 4\cos 2x + 3\right)$$

which is the real part along $z = x$ of

$$\frac{1}{8z^3}\left(e^{4iz} - 4e^{2iz} + 3\right)$$

Origin is a double pole.

Modify it suitably.

But along which contour would you evaluate? Semi-circular contours are useless because $\frac{\sin^4 x}{x^3}$ is an odd function

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\,dz$$

$$= -i\pi \operatorname*{Res}_{z=i\pi} f(z)$$

So we get in the limit as $R \to \infty$
and $\epsilon \to 0$

$$I(1+\cosh\pi) - i\pi \operatorname*{Res}_{z=i\pi} f(z) = 0$$

$i\pi$ is a simple pole

So $$\operatorname*{Res}_{z=i\pi} f(z) = \lim_{z \to i\pi} \frac{(z-i\pi)}{\sinh z} \sin z$$

$$= \frac{(\sin i\pi)}{\cosh i\pi} = \frac{i \sinh \pi}{\cos \pi}$$

$$= -i \sinh \pi$$

$$\therefore I(1+\cosh\pi) - \pi \sinh \pi = 0$$

$$I = \frac{\pi \sinh \pi}{1+\cosh \pi}$$

To Calculate this

note that $\lim_{z \to a_1} \dfrac{z - a_1}{z^4 + 1}$

$$= \frac{1}{4a_1^3} \quad \text{(by L'Hospital's Rule)}$$

$$= \frac{a_1}{4a_1^4} = -\frac{a_1}{4}$$

On the other hand

$$\lim_{z \to a_1} \frac{z - a_1}{z^4 + 1} = \lim_{z \to a_1} \frac{z - a_1}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)}$$

$$= \lim_{z \to a_1} \frac{1}{(z - a_2)(z - a_3)(z - a_4)}$$

$$\therefore \frac{1}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} = \frac{-a_1}{4}$$

$$\therefore \oint_\gamma \frac{f(z)}{z - a_1} = -a_1 \cdot \frac{2\pi i}{4} = \frac{\pi i}{2}\left(\frac{-1 - i}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2\sqrt{2}}(1 - i)$$

$$\text{Now} \quad \oint_\gamma = \int_{L_1} + \int_{L_2} + \int_{\Gamma_R}$$

$$\int_{L_1} f(z)\,dz = \int_0^R \frac{dt}{1+t^4}$$

$$\int_{L_2} f(z)\,dz = -\int_0^R \frac{i\,dt}{1+t^4} \quad \text{(why neg sign ?)}$$

$$\int_{L_1} + \int_{L_2} = (1-i)\int_0^R \frac{dt}{1+t^4} \longrightarrow \int_0^\infty \frac{(1-i)\,dt}{1+t^4}$$

$$\int_{\Gamma_R} f(z)\,dz = \int_0^{\pi/2} \frac{iRe^{it}\,dt}{1+R^4e^{4it}} \longrightarrow 0 \quad \text{as } R \to \infty$$

Hence we get in the limit,

$$(1-i)\int_0^\infty \frac{dt}{1+t^4} = \frac{\pi}{2\sqrt{2}}(1-i) \quad \text{or}$$

$$\int_0^\infty \frac{dt}{1+t^4} = \frac{\pi}{2\sqrt{2}}$$

We shall later systematize this process of evaluating real integrals using complex analysis.

Theorem: Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Then given any $a \in \Omega$, we can find a $r > 0$ such that on the disc $|z-a| < r$ the function $f(z)$ is representable as a power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k \quad ; \quad |z-a| < r$$

The coefficients c_k are given by

$$c_k = \frac{f^{(k)}(a)}{k!}$$

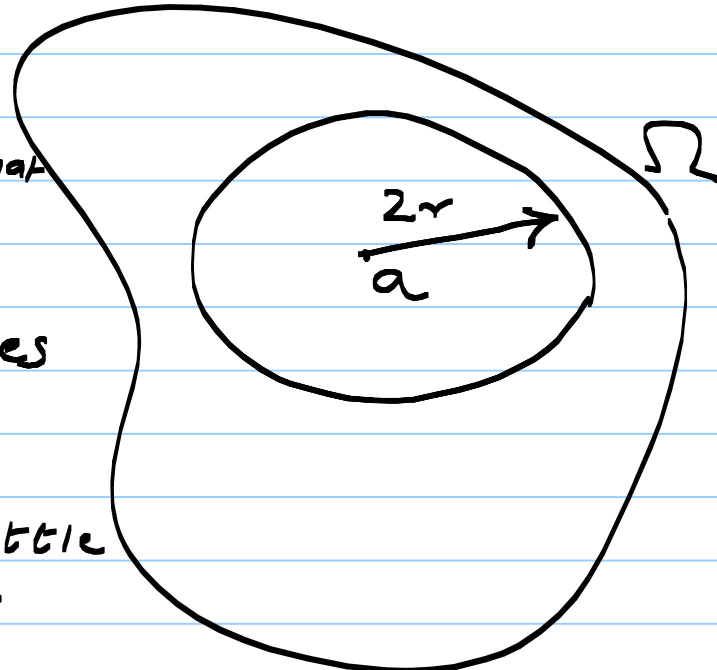
$$\boxed{c_k = \frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(\xi)\, d\xi}{(\xi - a)^{k+1}}}$$

Provided the closed disc of radius ρ centered at a lies in Ω

We shall have occasion to use this formula

Let $a \in \Omega$ and select r so that the closed disc

$\{|z-a| \leq 2r\}$ lies in Ω.



For z in the little disc $\{|z-a| \leq r\}$

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - a| = 2r} \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad \text{(traced Counter-Clockwise)}$$

$$= \frac{1}{2\pi i} \oint_{|\zeta - a| = 2r} \frac{f(\zeta)\,d\zeta}{\zeta - a - (z-a)}$$

$$= \frac{1}{2\pi i} \oint_{|\zeta - a| = 2r} \frac{f(\zeta)\,d\zeta}{(\zeta - a)\left[1 - \frac{z-a}{\zeta - a}\right]}$$

$$= \frac{1}{2\pi i} \oint_{|\zeta - a| = 2r} f(\zeta)\,d\zeta \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta-a)^{k+1}}\,d\zeta$$

Interchanging $\oint$ and $\sum$

we get

$$f(z) = \sum_{k=0}^{\infty} \frac{(z-a)^k}{2\pi i} \oint_{|\zeta-a|=2r} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{k+1}}$$

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k \qquad (*)$$

where,
$$c_k = \frac{1}{2\pi i} \oint_{|\zeta-a|=2r} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{k+1}}$$

as desired.

Since a power series may be diff. term by term, diff the series $(*)$ k-times and put $z=a$ and we get

$$c_k = \frac{f^{(k)}(a)}{k!}$$

More Integral Computations

Reduction to known integrals:

Recall:
$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt = \log\left(\frac{b}{a}\right)$$

$(0 < a < b)$

This integral is known as the Frullani integral. It is readily computed by evaluating in two ways the double integral

$$\int_0^\infty \int_a^b e^{-xt}\,dx\,dt$$

I do not know if it is possible to obtain this via complex analysis

You will need the Frullani integral to compute $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^3}\,dx$

Exercise: Calculate

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx \qquad 0 < a < b$$

by integrating a suitable function along the quadrant of a circle. and reducing it to the Frullani integral.

Exercise: By computing a suitable double integral in two ways Calculate

$$\int_0^\infty \frac{1}{x^2}\left(e^{-ax^2} - e^{-bx^2}\right) dx$$

Exercise: Compute $\displaystyle\int_0^\infty \left(\frac{\cos ax^2 - \cos bx^2}{x}\right) dx$

Exercise: Use real substitutions to Compute $\displaystyle\int_0^\infty \exp\left(-t - \frac{1}{t}\right)\frac{dt}{\sqrt{t}}$

Prove that

$$\int_0^\infty \cos\left(x^2 - \frac{1}{x^2}\right) dx$$

$$= \int_0^\infty \sin\left(x^2 - \frac{1}{x^2}\right) dx = \frac{\sqrt{\pi}}{2\sqrt{2}\, e^2}$$

Prove that

$$\int_0^\infty \left(\frac{\cos x^2 + \sin x^2 - 1}{x^2} \right) dx = 0$$

by integrating $\dfrac{e^{iz^2} - 1}{z^2}$

along the quadrant of a circle.

Next group of integrals:

Use of Rectangular Contours.

Use of Rectangular Contours.

Examples

(1) $$\int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}\, dx$$

(2) $$\int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{1+e^{x}} \quad ; \quad 0 < a < 1$$

(3) $$\int_{-\infty}^{\infty} \frac{e^{2\lambda x}\, dx}{\cosh x} = \frac{\pi}{\cos \lambda\pi}$$

$$-\frac{1}{2} < \lambda < \frac{1}{2}$$

The result is true for complex λ as well as long as $-\frac{1}{2} < \operatorname{Re}\lambda < \frac{1}{2}$

Thus, we get the Fourier transform of $\frac{1}{\cosh x}$.

(4) $$\int_{-\infty}^{\infty} \frac{\cosh cx}{\cosh \pi x}\, dx = \sec^2 \frac{c}{2} \; ; \; -\pi < c < \pi$$

(5) $$\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^{x}+e^{2x}} \; ; \; 0<a<2$$

$$= \frac{2\pi}{\sqrt{3}} \frac{\sin \frac{\pi}{3}(1-a)}{\sin a\pi}$$

In all these examples the denominator has infinitely many zeros located along the imaginary axis or in some vertical line.

A semi-circular contour (with radius R) would be unsuitable as we would get in the limit as $R \to \infty$

an infinite series

$$2\pi i \sum_{j=1}^{\infty} \operatorname{Res}_{z=p_j} f(z)$$

$p_1, p_2, p_3, \ldots$ list of poles in the upper half plane

Second, the denominators involve exponential functions that are periodic with ~~some~~ imaginary period

$$e^{2\pi i + z} = e^{2\pi i}$$

$$\cosh(z + 2\pi i) = \cosh z$$

If $\phi(z) = \sinh az$ then

$$\phi\left(z + \frac{2\pi i}{a}\right) = \phi(z) \quad \text{etc;}$$

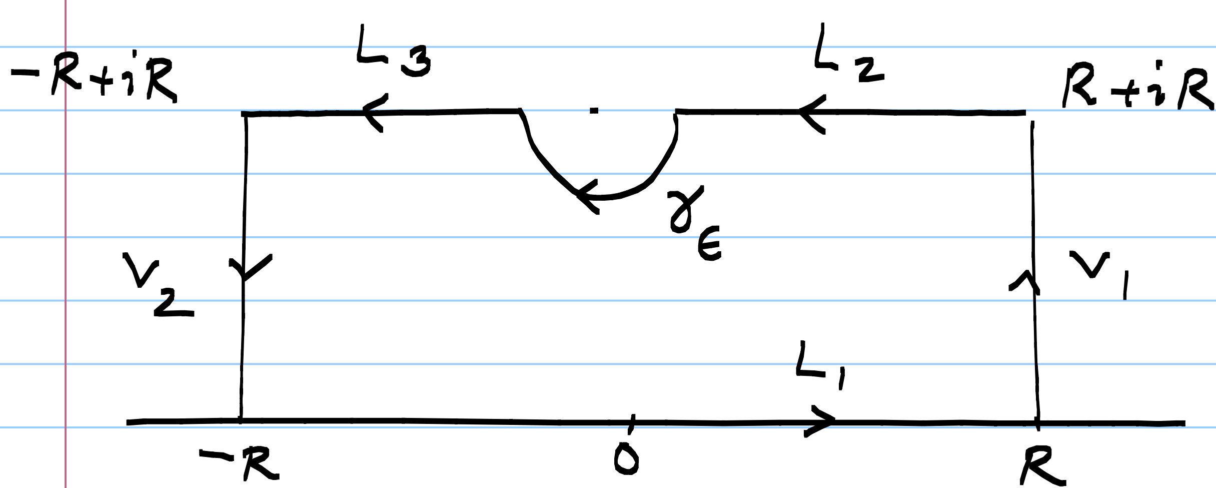
The Choice of a rectangular Contour. takes advantage of this periodicity.

Indeed the period decides the height of the rectangle as will become clear from examples.

(1) $\displaystyle I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}\, dx$

Let $f(z) = \dfrac{\sin z}{\sinh z}$ which is holomorphic in a neighborhood of the origin.

Take a rectangle with vertices $-R$, R, $R+i\pi$, $R-i\pi$ indented at $i\pi$ to avoid the pole.



$$\int_{L_1} f(z)\,dz \longrightarrow I \text{ (Desired integral) as } R \to \infty$$

Let us look at the contributions from V_1 and V_2:

Along V_1: $z = R + it$; $0 \le t \le \pi$
$dz = i\,dt$

$\sin z = \sin R \cosh t + i \cos R \sinh t$

$|\sin z|$ is seen to be bounded. by M say

$$\sinh z = -i \sin(iz)$$
$$= -i \sin(-t + iR)$$

$$|\sinh z|^2 = |\sin t \cosh R - i \cos t \sinh R)^2$$
$$= \sin^2 t \cosh^2 R + \cos^2 t \sinh^2 R$$
$$= \sin^2 t\,(1 + \sinh^2 R) + \cos^2 t\,\sinh^2 R$$
$$= \sin^2 t + \sinh^2 R > \sinh^2 R$$

So $$\left|\frac{\sin z}{\sinh z}\right| \le \frac{M}{\sinh R}$$

$$\left|\int_V f(z)\,dz\right| \le \int_0^{\pi} \frac{M}{\sinh R}\,dt = \frac{M\pi}{\sinh R} \xrightarrow{R \to \infty} 0$$

So $\lim\limits_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left(\int_{L_1} + \int_{L_2} \right)$

$$= 2i \int_0^\infty \frac{\sin t}{t}\, dt.$$

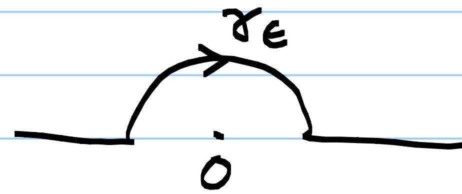
Finally,

$$\int_{\gamma_\epsilon} f(z)\, dz = \int_{\gamma_\epsilon} \frac{e^{iz}}{z}\, dz$$

$$= \int_{\gamma_\epsilon} \frac{e^{iz}-1}{z}\, dz + \int_{\gamma_\epsilon} \frac{dz}{z}$$

$$= \int_{\gamma_\epsilon} \frac{e^{iz}-1}{z}\, dz \quad - i\pi$$

Now $\dfrac{e^{iz}-1}{z}$ is bounded near 0 and so



Exercise: Suppose $g(z)$ is bounded along γ say $|g(z)| \leq M$

Then $\left| \int_\gamma g(z)\,dz \right| \leq M\,(\text{length } \gamma)$

Thus $\left| \int_{\gamma_\epsilon} \frac{e^{iz}-1}{z}\,dz \right| \leq M.\,\pi\epsilon \to 0$

as $\epsilon \to 0$

Now letting $\left.\begin{matrix} R \to \infty \\ \epsilon \to 0 \end{matrix}\right\}$ in the equation

$$\int_{L_1} + \int_{L_2} + \int_{\Gamma_R} + \int_{\gamma_\epsilon} = 0$$

We get

$$2i \int_0^\infty \frac{\sin t}{t} - \pi i = 0 \quad \text{or}$$

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Poles and essential Singularities

Residues.

Theorem (Riemann's Removable Singularities thm)

Suppose Ω is an open set, $p \in \Omega$ and $f : \Omega - \{p\} \to \mathbb{C}$ is analytic and bounded on $0 < |z-p| < r$

then $\lim_{z \to p} f(z)$ exists $= \ell$ say.

If we define $f(p) = \ell$ then the function f is analytic on Ω.

That is to ~~say~~ p is a removable Singularity

Que: Is 0 a removable Singularity of $\sqrt{z}$?

Note that 0 is a removable sing. of

$$\frac{\sin z}{z} \quad \text{and} \quad \frac{1-\cos z}{z^2}$$

Def: Suppose Ω is an open set, $p \in \Omega$ and $f : \Omega - \{p\} \to \mathbb{C}$ is analytic we say p is a pole of order k, where $k \in \mathbb{N}$

if $\lim_{z \to p} f(z)(z-p)^k$ exists and is non zero.

Ex: The origin is a pole of what order for $f(z) = \dfrac{1-\cos z}{z^5}$

What are the poles of

$z \cot^2 \pi z$? What are their orders?

Note: A pole of order one is called a simple pole.

A pole of order two is called a <u>double pole</u>.

A pole of order 3 is a <u>triple pole</u>.

Now, suppose p is a pole of order k ($k \in \mathbb{N}$) for $f(z)$. Then

$$\lim_{z \to p} (z-p)^k f(z) \text{ exists say } l \neq 0.$$

Thus

p is a removable singularity for the function $(z-p)^k f(z)$.

Assigning $(z-p)^k f(z)$ the value l at p we get an analytic function

Writing out its power series:

$$(z-p)^k f(z) = c_0 + c_1(z-p) + c_2(z-p)^2 + \cdots$$

But recall that $\lim_{z \to p} (z-p)^k f(z) \neq 0$

So $c_0 \neq 0$

Dividing by $(z-p)^k$ we get

$$f(z) = \frac{c_0}{(z-p)^k} + \frac{c_1}{(z-p)^{k-1}} + \cdots +$$

$$(**) \qquad \frac{\boxed{c_{p-1}}}{z-p} + c_p + c_{p+1}(z-p) + \cdots\cdots$$

Valid for $0 < |z-p| < r$

The coefficient of $\dfrac{1}{z-p}$ in the above expansion is called the <u>Residue of</u> <u>$f(z)$ at $z=p$.</u>

If we integrate $(**)$ over a circle $|z-p| = \rho$ we get

$$\oint_{|z-p|=\rho} f(z)\,dz = 2\pi i \; c_{p-1}$$

Or

$$\operatorname*{Residue}_{z=p} f = \frac{1}{2\pi i} \int_{|z-p|=\rho} f(z)\, dz$$

<u>Def</u>: If Ω is an open set in $\mathbb{C}$, $p \in \Omega$ and $f : \Omega - \{p\} \to \mathbb{C}$ is analytic we say p is an <u>essential singularity</u> of f if

$$\lim_{z \to p} (z-p)^k f(z) \text{ } \underline{\text{does not}}$$

<u>exist</u> for <u>any</u> k.

<u>Example</u>: Origin is an essential singularity of $e^{1/z}$.

Examine whether origin is a pole or an essential singularity of

$$\sin \frac{1}{z}$$

The Great Picard Theorem:

Suppose p is an essential singularity of $f(z)$ then for any $r>0$,

the function assumes every complex value with at most ONE exception

on the punctured disc

$$0 < |z-p| < r.$$

Computing Residues:

Suppose p is a simple pole of $f(z)$

then on $0<|z-p|<r$ we have

$$f(z) = \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + \dots$$

Multiplying by $(z-p)$ and letting $z \to p$

$$\operatorname*{Res}_{z=p} f(z) = \lim_{z\to p} (z-p)f(z)$$

(Simple poles only)

If p is a pole of order k then on $0 < |z-p| < r$

$$f(z) = \frac{c_{-k}}{(z-p)^k} + \frac{c_{-(k-1)}}{(z-p)^{k-1}} + \cdots + \frac{\boxed{c_{-1}}}{z-p}$$

$$+ c_0 + c_1(z-p) + \cdots\cdots$$

Multiply by $(z-p)^k$:

$$(z-p)^k f(z) = c_{-k} + \cdots + \boxed{c_{-1}}(z-p)^{k-1}$$

$$+ c_0 (z-p)^k + \cdots\cdots$$

Diff. $(k-1)$ times and let $z \to p$

$$\left(\frac{d}{dz}\right)^{k-1} (z-p)^k f(z) \Big|_{z=p} = (k-1)!\ c_{-1}$$

denotes $\lim\limits_{z \to p}$.

So,

$$\operatorname*{Res}_{z=p} f(z) = \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}(z-p)^k f(z)\Big|_{z=p}.$$

<u>Ex:</u> Determine the Residue at the origin of $f(z) = \dfrac{1-\cos z}{z^5}$.

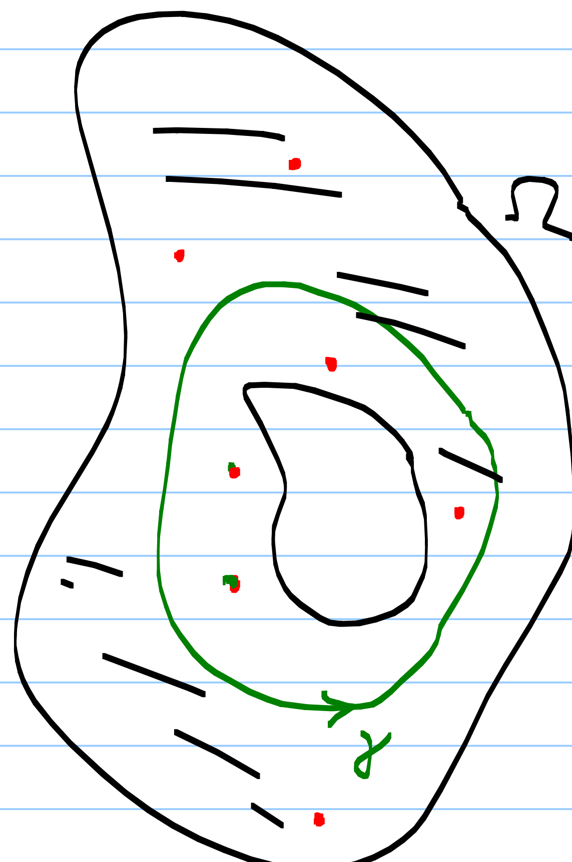
<u>Note.</u> Suppose $p_1, p_2, p_3, \ldots$ is a sequence of poles of a non constant holomorphic function and $p_n \to p$ then p is an essential Singularity of f. Thus the poles cannot accumulate a point in the domain of analyticity of a function.

The Residue formula:

(i) Suppose that $f : \Omega \to \mathbb{C}$ is analytic except for poles in Ω and γ is a simple closed curve in Ω not passing through poles of f

(ii) Assume further that f is analytic <u>inside</u> γ <u>as</u> <u>well</u> except at the poles that lie inside γ

Then
$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=p_j} \left(f(z)\right)$$



$p_1, \dots, p_k$ is the list of poles inside γ

Domain with a hole
γ encloses the hole

The hypothesis (ii) excludes the possibility of γ circuiting a hole in the domain.

The Cauchy Residue Formula is reminiscent of the Gauss' formula of Electrostatics and indeed is the planar version of Gauss' formula

Example. Let $f(z) = \dfrac{1}{1+z^4}$

Let a_1, a_2, a_3, a_4 be its roots in the first, second, third and fourth quadrants

$$\operatorname*{Res}_{z=a_j} f(z) = \lim_{z \to a_j} (z-a_j) f(z)$$

$$= \lim_{z \to a_j} \frac{z - a_j}{1 + z^4}$$

$$= \frac{1}{4 a_j^3} = \frac{a_j}{4 a_j^4}$$

but $a_j^4 + 1 = 0$ so

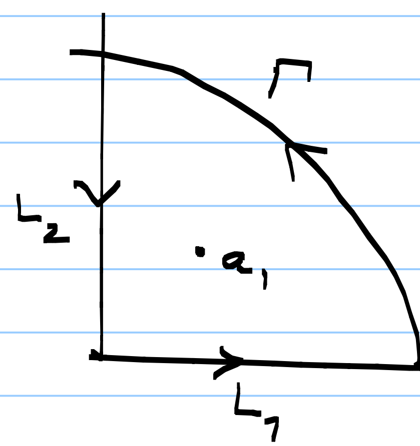
$$\operatorname{Res}_{z = a_j} f(z) = -\frac{1}{4} a_j$$

So

$$\oint_\gamma f(z)\,dz = 2\pi i \left(-\frac{a_1}{4}\right)$$

$$= -\frac{\pi i}{2} a_1$$

$$\int_{L_1} + \int_{L_2} + \int_{\Gamma_R} = \frac{-\pi i}{2} a_1$$



letting $R \to \infty$ we get

$$\int_0^\infty \frac{dt}{1 + t^4} - i \int_0^\infty \frac{dt}{1 + t^4} = \frac{-\pi i}{2} a_1$$

Since $\displaystyle\int_{\Gamma_R} f(z)\,dz \longrightarrow 0$ as $R \to \infty$

So $$\int_0^\infty \frac{dt}{1+t^4} = \frac{-i\pi}{2(1-i)} a_1$$

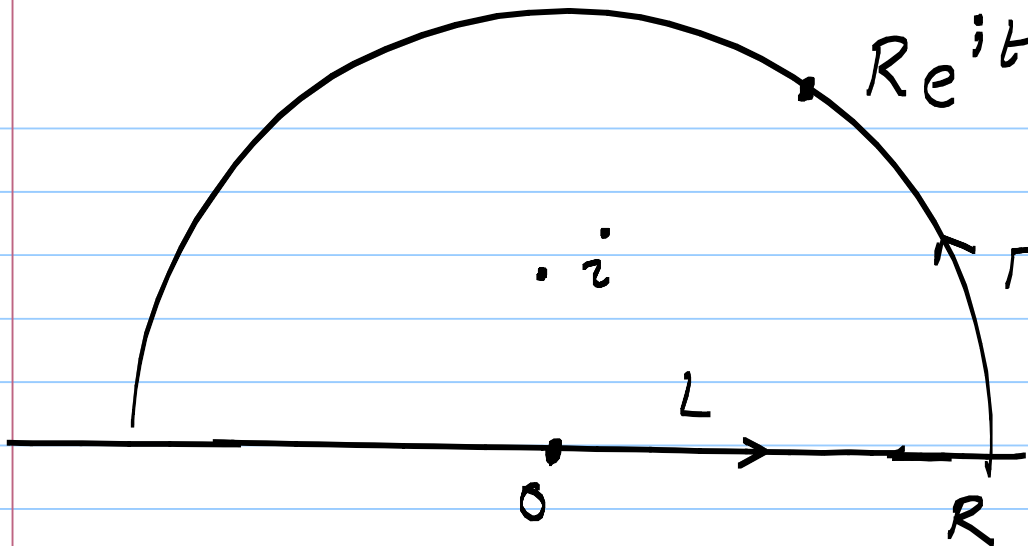
$$= \frac{\pi}{2\sqrt{2}}$$

Example: Let us Calculate

$$\int_{-\infty}^{\infty} \frac{(\cos at)\,dt}{1+t^2} \quad ; \quad a>0$$

Take $$f(z) = \frac{e^{iaz}}{1+z^2}$$

Compute $\underset{z=i}{\operatorname{Res}} f(z)$



$\gamma = L \cup \Gamma_R$ encloses the pole i

$$\int_L f(z)\,dz = \int_{-R}^{R} \frac{\cos at}{1+t^2}\,dt$$

$\longrightarrow I$ (Desired integral)

<u>Check:</u> $\int_{\Gamma_R} f(z)\,dz \longrightarrow 0$ as $R \to \infty$.

By Cauchy's Residue formula

$$\int_L f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

letting $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{(\cos at)\,dt}{1+t^2} = 2\pi i \operatorname*{Res}_{z=i}(f(z))$$

$$=$$

Exercise: Compute

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos t}{1+t^4}\,dt$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{1+t^6}$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{(1+t^4)^2}$

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{t^6+t^3+1}$

(v) $\displaystyle\int_{0}^{\infty} \frac{dt}{1+t^5}$

Integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

Where R is a rational function

Example: Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{a^2+1-2a\cos\theta} \, ; \qquad (a>1)$$

Sol: Idea is to transform it into an integral over the circle $|z|=1$

$$z = e^{i\theta} \, ; \quad 2\cos\theta = \left(z + \frac{1}{z}\right)$$

and $\dfrac{dz}{iz} = d\theta$ so that

$$I = \frac{1}{i} \oint_{|z|=1} \frac{1}{a^2+1-a\left(z+\frac{1}{z}\right)} \frac{dz}{z}$$

$$I = \frac{1}{i} \oint_{|z|=1} \frac{dz}{(a^2+1)z - a(z^2+1)}$$

$$= -\frac{1}{a\,i} \oint_{|z|=1} \frac{dz}{(z^2+1) - z\left(a+\frac{1}{a}\right)}$$

$$= \frac{-1}{a\,i} \oint_{|z|=1} \frac{dz}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \frac{-1}{ia} \oint_{|z|=1} \frac{f(z)\,dz}{\left(z-\frac{1}{a}\right)} \quad ; \quad f(z) = \frac{1}{z-a}$$

Since $a>1$, $0<\frac{1}{a}<1$ and so

$$I = \frac{-1}{ia} \cdot 2\pi i \, f\left(\frac{1}{a}\right)$$

$$= -\frac{2\pi}{a} \frac{1}{\frac{1}{a}-a} = \frac{2\pi}{a^2-1}$$

Prove that

$$\int_0^{2\pi} \frac{d\theta}{1-ae^{i\theta}} = \begin{cases} 0 & \text{if } |a|<1 \\ 2\pi & \text{if } |a|>1 \end{cases}$$

Prove that

$$I = \int_0^{\pi/2} (\sin 2nx) \cot x \, dx$$

$$= \frac{1}{4} \int_0^{2\pi} (\sin 2nx) \cot x \, dx$$

and hence evaluate the integral.

Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$

by transforming it into an integral over $|z|=1$.

This will illustrate the fact that a judicious choice of contour of integration can greatly simplify the problem.

Prove that

$$\int_0^{\pi} \left(\frac{\sin nx}{\sin x} \right)^2 dx = n\pi$$

$$\int_0^{\pi} \frac{\sin(nx + \frac{x}{2})}{\sin \frac{x}{2}} dx = \pi$$

Use of Sectoral Contours.

Example: Let us Calculate

$$I = \int_0^{\infty} \frac{dx}{1+x^5}$$

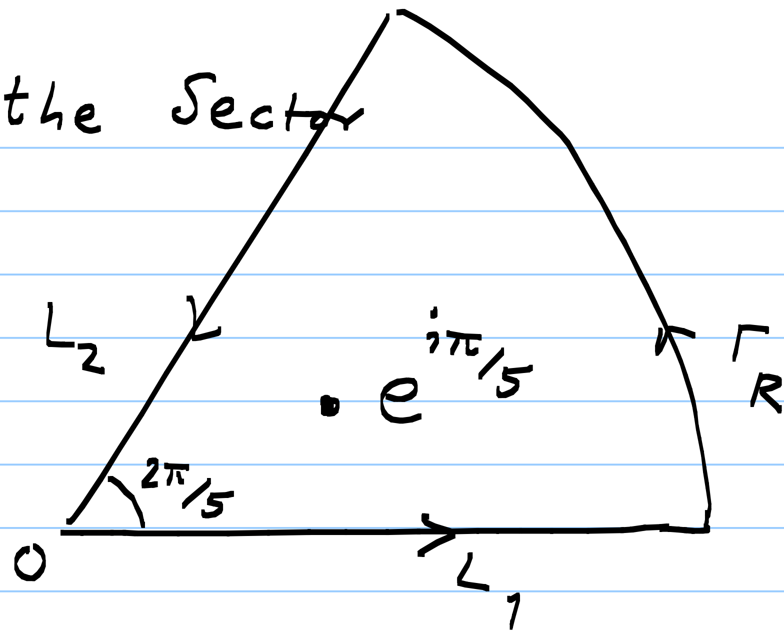
Take $f(z) = \dfrac{1}{1+z^5}$. The function has

simple poles at the roots of

$$z^5 + 1 = 0$$

One of these is $e^{i\pi/5}$

Consider the Sector



with opening angle $\frac{2\pi}{5}$

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz$$

$$= 2\pi i \operatorname*{Res}_{z=a} f(z) \;; \quad a = e^{i\pi/5}$$

Now $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$

$$\int_{L_1} f(z)\,dz = \int_0^R \frac{dt}{1+t^5} \longrightarrow I$$

as $R \to \infty$

Now $-L_2$ denotes the ray L_2 with reversed orientation

$$-L_2: \quad z = t e^{2\pi i/5} \ ; \ 0 \le t \le R$$

$$dz = e^{2\pi i/5} \, dt$$

$$\int_{L_2} f(z)\,dz = -\int_0^R \frac{e^{2\pi i/5}\,dt}{1+t^5}$$

$$= -e^{2\pi i/5} \int_0^R \frac{dt}{1+t^5}$$

$$\xrightarrow[\text{as } R \to \infty]{} -e^{2\pi i/5} \, \underline{I}$$

So letting $R \to \infty$ in the equation

$$\int_{L_1} + \int_{L_2} + \int_{\Gamma_R} = 2\pi i \operatorname*{Res}_{z=a} f(z)$$

We get

$$I\left(1 - e^{2\pi i/5}\right) = 2\pi i \operatorname*{Res}_{z=a} f(z)$$

$$I = \left(\frac{2\pi i}{1-e^{2\pi i/5}}\right) \operatorname*{Res}_{z=a} f(z)$$

Let us compute the residue
Since the pole is simple,

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} \frac{z-a}{1+z^5}$$

$$= \frac{1}{5a^4} = \frac{a}{5a^5}$$

$$= -\frac{a}{5} = -\frac{1}{5} e^{i\pi/5}$$

$$\therefore \quad I = -\frac{\pi}{5} \cdot \frac{2i\, e^{i\pi/5}}{1-e^{2\pi i/5}}$$

$$= \frac{\pi}{5} \frac{2i}{e^{i\pi/5} - e^{-i\pi/5}} = \frac{\pi}{5} \operatorname{Cosec} \frac{\pi}{5}$$

Exercise: Prove that if a is a rational number such that $0 < a < 1$ then

$$\int_0^\infty \frac{t^{a-1}\,dt}{1+t} = \frac{\pi}{\sin \pi a}$$

Does the result hold for all real numbers a in the range $0 < a < 1$?

Example:

Compute $\displaystyle\int_0^\infty \cos x^2\,dx$ and $\displaystyle\int_0^\infty \sin x^2\,dx$

Called Fresnel's Integrals

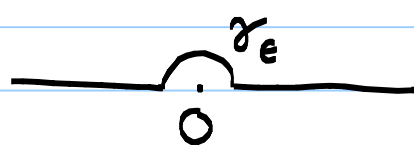
Compute e^{-z^2} along a sector with opening angle $\pi/4$

Fractional Residue Theorem

Recall that while computing $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

using $f(z) = \dfrac{e^{iz}}{z}$

we had to make a semi-circular indentation at the origin and had to



determine $\displaystyle\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz$. The

fractional residue theorem streamlines this process:

Theorem (Frac. Res. Thm): Suppose that $f(z)$ has a simple pole at p and

γ is a circular arc of radius ϵ subtending angle α at p traced counterclockwise

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\,dz = i\alpha \operatorname*{Res}_{z=p} f(z)$$

α of course is measured in radians

<u>proof</u>:

$$f(z) = \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + \cdots$$

$$= \frac{c_{-1}}{z-p} + g(z)$$ ~~Any~~

where $g(z)$ is analytic and hence bounded near p

$$\int_{\gamma_\epsilon} f(z)\,dz = c_{-1}\int_{\gamma_\epsilon} \frac{dz}{z-p} + \int_{\gamma_\epsilon} g(z)\,dz$$

$$= c_{-1}\, i\alpha + \int_{\gamma_\epsilon} g(z)\,dz$$

(Direct Calculation). Now,

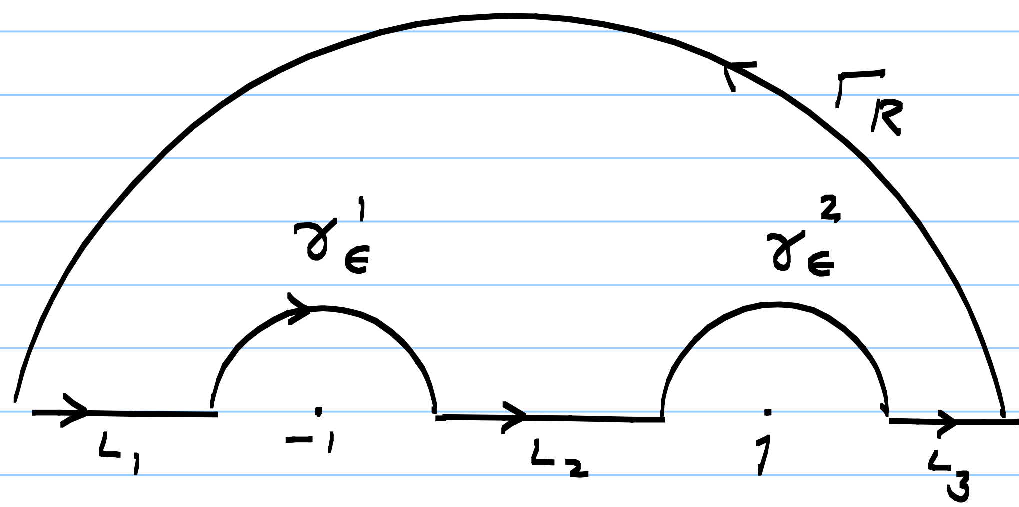
$$\left| \int_{\gamma} g(z)\,dz \right| \le (\text{Perimeter } \gamma_\epsilon)\left(\text{Bound on } |g|\right)$$

$$\longrightarrow 0 \text{ as } \epsilon \to 0$$

The proof is complete

Exercise: Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos \frac{\pi x}{2}}{1-x^2}\, dx$

Take $f(z) = \dfrac{\exp(i\pi z/2)}{1-z^2}$. Integrate $f(z)$ around a semi-circular contour with indentations at $z=-1$ and $z=-1$



Check: $\displaystyle\int_{\Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$

There are no poles within the contour

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz \right\}$$

$$= \int_{-\infty}^{\infty} \left(\cos \frac{\pi x}{2}\, dx\right) \Big/ (1-x^2)$$

The imaginary part cancels out.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{-e^{iz\pi/2}}{z+1}$$

$$= -\frac{1}{2} e^{i\pi/2} = -i/2$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \frac{e^{iz\pi/2}}{1-z} = \frac{1}{2} e^{-i\pi/2}$$

$$= -i/2$$

Both arcs γ_ϵ^1, γ_ϵ^2 are traced clockwise

$$\lim_{\varepsilon \to 0} \left(\int_{\gamma_\epsilon^1} f(z) + \int_{\gamma_\epsilon^2} f(z) dz \right)$$

$$= -i\pi(-i) = -\pi$$

$$= -\pi$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(\frac{\pi x}{2}) dx}{1-x^2} = -\pi$$

Exercise: (1) Compute

$$\int_{-\infty}^{\infty} \frac{\sin \pi x \, dx}{x(1-x^2)}$$

(2) Let us now try

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \, dx \quad ; \quad a, b \in \mathbb{R}, \quad a > b > 0$$

The obvious choice is

$$f(z) = \frac{e^{iaz} - e^{-ibz}}{z^2}$$

and the contour a semi-circle indented at the origin. The origin is a simple pole and fractional Residue Theorem applies.

Complete the task

(3) What if there is a double pole or a triple pole?

Frac. Res. Thm cannot be applied but a little cleverness helps.

Example: $$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx$$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ and so

$$= \operatorname{Im}\left(\frac{1}{4}\left(\frac{3e^{ix} - e^{3ix}}{x^3}\right)\right)$$

It is tempting to take

$$f(z) = \frac{1}{4}\,\frac{3e^{iz} - e^{3iz}}{z^3}$$

and use a semi-circular contour indented at the origin. But the origin is a TRIPLE POLE. Frac. Res. Thm. cannot be used!

Try modifying $f(z)$ suitably to make it work.

Well,

$$3e^{iz} - e^{3iz}$$

$$= 2 + 3z^2 + \dots$$

So that

$$\frac{3e^{iz} - e^{3iz}}{z^3} = \frac{2}{z^3} + \frac{3}{z} + \dots$$

If we modify $f(z)$ as

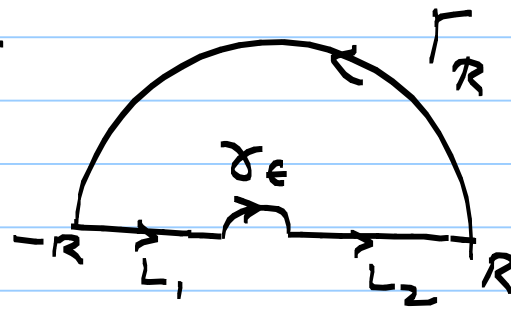
$$g(z) = \frac{1}{4}\left\{ \frac{3e^{iz} - e^{3iz} - 2}{z^3} \right\}$$

and work with $g(z)$ then along the real axis we get

$$\operatorname{Im} g(x) = \operatorname{Im} f(x)$$

$g(z)$ has a simple pole at 0

with residue $3/4$.



$$\int_{\Gamma_R} f(z)\,dz \xrightarrow[R \to \infty]{} 0$$

$$\int_{\gamma_\epsilon} f(z)\,dz \longrightarrow -i\pi \cdot \frac{3}{4}$$

as $\epsilon \to 0$

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz$$

$$\xrightarrow[\substack{R \to \infty \\ \epsilon \to 0}]{} i \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx$$ and so

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3}{4}\pi$$

Exercise:

Evaluate (1) $$\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\,dx$$

(2) You might wish to try computing

$$\int_0^{\infty} \frac{\sin^4 x\,dx}{x^3}$$ starting with

$$\frac{\sin^4 x}{x^3} = \frac{1}{8x^3}\left(\cos 4x - 4\cos 2x + 3\right)$$

which is the real part along $z = x$ of

$$\frac{1}{8z^3}\left(e^{4iz} - 4e^{2iz} + 3\right)$$

Origin is a double pole.

Modify it suitably.

But along which contour would you evaluate? Semi-circular contours are useless because $\frac{\sin^4 x}{x^3}$ is an odd function

Reduction to known integrals:

Recall:
$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log\left(\frac{b}{a}\right)$$

$(0<a<b)$

This integral is known as the Frullani integral. It is readily computed by evaluating in two ways the double integral
$$\int_0^\infty \int_a^b e^{-xt}\,dx\,dt$$

I do not know if it is possible to obtain this via complex analysis

You will need the Frullani integral to compute $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^3}\,dx$

Exercise: Calculate

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx \qquad 0 < a < b$$

by integrating a suitable function along the quadrant of a circle. and reducing it to the Frullani integral.

Exercise: By computing a suitable double integral in two ways

Calculate

$$\int_0^\infty \frac{1}{x^2}\left(e^{-ax^2} - e^{-bx^2}\right) dx$$

Exercise: Compute $\displaystyle\int_0^\infty \left(\frac{\cos ax^2 - \cos bx^2}{x}\right) dx$

Exercise: Use real substitutions to

Compute $\displaystyle\int_0^\infty \exp\left(-t - \frac{1}{t}\right) \frac{dt}{\sqrt{t}}$

Prove that

$$\int_0^\infty \cos\left(x^2 - \frac{1}{x^2}\right) dx = \int_0^\infty \sin\left(x^2 - \frac{1}{x^2}\right) dx = \frac{\sqrt{\pi}}{2\sqrt{2}\, e^2}$$

Prove that

$$\int_0^\infty \left(\frac{\cos x^2 + \sin x^2 - 1}{x^2} \right) dx = 0$$

by integrating $\dfrac{e^{iz^2} - 1}{z^2}$

along the quadrant of a circle.

Next group of integrals:

Use of Rectangular Contours.

Use of Rectangular Contours.

Examples

(1) $$\int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}\, dx$$

(2) $$\int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{1+e^{x}} \quad ; \quad 0 < a < 1$$

(3) $$\int_{-\infty}^{\infty} \frac{e^{2\lambda x}\, dx}{\cosh x} = \frac{\pi}{\cos \lambda\pi}$$

$$-\tfrac{1}{2} < \lambda < \tfrac{1}{2}$$

The result is true for complex λ as well as long as $-\frac{1}{2} < \operatorname{Re}\lambda < \frac{1}{2}$

Thus, we get the Fourier transform of

$$\frac{1}{\cosh x}.$$

(4) $$\int_{-\infty}^{\infty} \frac{\cosh cx}{\cosh \pi x}\, dx = \sec^2 \frac{c}{2} \, ; \quad -\pi < c < \pi$$

(5) $$\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^{x}+e^{2x}} \quad ; \quad 0<a<2$$

$$= \frac{2\pi}{\sqrt{3}} \; \frac{\sin \frac{\pi}{3}(1-a)}{\sin a\pi}$$

In all these examples the denominator has infinitely many zeros located along the imaginary axis or in some vertical line.

A semi-circular contour (with radius R) would be unsuitable as we would get in the limit as $R\to\infty$

an infinite series

$$2\pi i \sum_{j=1}^{\infty} \operatorname{Res}_{z=p_j} f(z)$$

$p_1, p_2, p_3, \ldots$ list of poles in the upper half plane

Second, the denominators involve exponential functions that are periodic with ~~some~~ some imaginary period

$$e^{2\pi i + z} = e^{2\pi i}$$

$$\cosh(z + 2\pi i) = \cosh z$$

If $\phi(z) = \sinh az$ then

$$\phi\left(z + \frac{2\pi i}{a}\right) = \phi(z) \quad \text{etc};$$

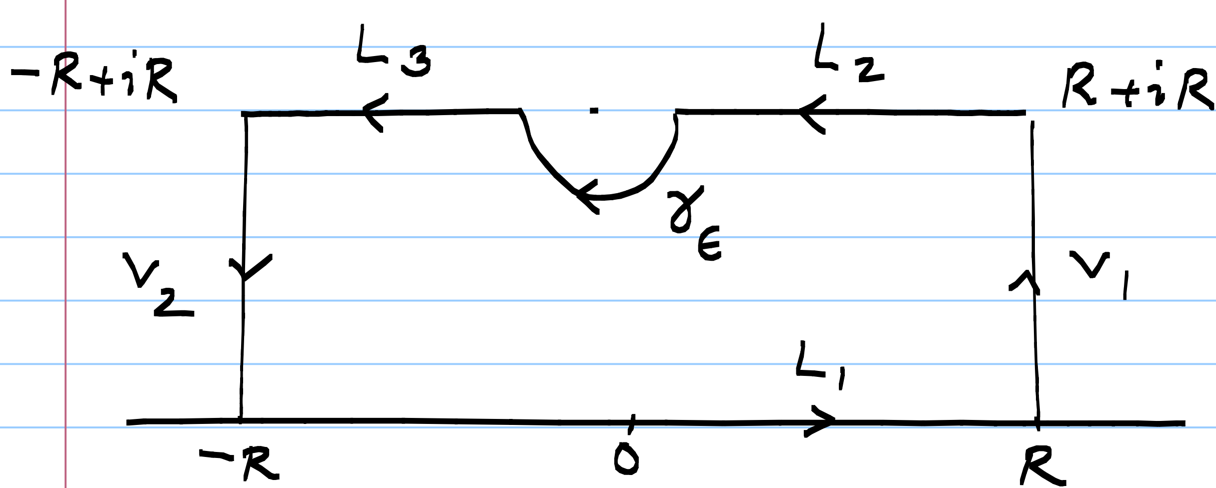
The choice of a rectangular contour, takes advantage of this periodicity.

Indeed the period decides the height of the rectangle as will become clear from examples.

(1) $\displaystyle I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}\, dx$

Let $\displaystyle f(z) = \frac{\sin z}{\sinh z}$ which is holomorphic in a neighborhood of the origin.

Take a rectangle with vertices $-R$, R, $R+i\pi$, $R-i\pi$ indented at $i\pi$ to avoid the pole.



$\displaystyle \int_{L_1} f(z)\,dz \longrightarrow I$ (Desired integral) as $R \to \infty$

Let us look at the contributions from V_1 and V_2:

Along V_1: $z = R + it$; $0 \le t \le \pi$
$dz = i\,dt$

$\sin z = \sin R \cosh t + i \cos R \sinh t$

$|\sin z|$ is seen to be bounded by M say

$$\sinh z = -i \sin(iz)$$
$$= -i \sin(-t + iR)$$

$$|\sinh z|^2 = |\sin t \cosh R - i \cos t \sinh R|^2$$
$$= \sin^2 t \cosh^2 R + \cos^2 t \sinh^2 R$$
$$= \sin^2 t\,(1 + \sinh^2 R) + \cos^2 t \sinh^2 R$$
$$= \sin^2 t + \sinh^2 R > \sinh^2 R$$

So $\left|\dfrac{\sin z}{\sinh z}\right| \le \dfrac{M}{\sinh R}$

$$\left|\int_V f(z)\,dz\right| \le \int_0^\pi \frac{M}{\sinh R}\,dt = \frac{M\pi}{\sinh R} \xrightarrow{R\to\infty} 0$$

Likewise $\int_{V_2} f(z)\,dz \longrightarrow 0$

$$\int_{L_2} f(z) + \int_{L_3} f(z)\,dz$$

$$= -\left\{ \int_{-R}^{-\epsilon} f(t+i\pi)\,dt + \int_{\epsilon}^{R} f(t+i\pi)\,dt \right\}$$

$$f(t+i\pi) = \frac{\sin(t+i\pi)}{\sinh(t+i\pi)}$$

$$= \frac{\sin t \cosh\pi + i \cos t \sinh\pi}{-\sinh t}$$

$$= (-\cosh\pi)\frac{\sin t}{\sinh t} - i\sinh\pi \cdot \frac{\cos t}{\sinh t}$$

$$\int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz$$

$$= \cosh\pi \left\{ \int_{-R}^{-\epsilon} \frac{\sin t}{\sinh t}\,dt + \int_{\epsilon}^{R} \frac{\sin t}{\sinh t}\,dt \right\} \longrightarrow (\cosh\pi)\underline{I}$$

$R \to \infty,\ \epsilon \to 0$

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\,dz$$

$$= -i\pi \operatorname{Res}_{z=i\pi} f(z)$$

So we get in the limit as $R \to \infty$ and $\epsilon \to 0$

$$I(1+\cosh\pi) - i\pi \operatorname{Res}_{z=i\pi} f(z) = 0$$

$i\pi$ is a simple pole

So $$\operatorname{Res}_{z=i\pi} f(z) = \lim_{z \to i\pi} \frac{(z-i\pi)}{\sinh z} \sin z$$

$$= \frac{(\sin i\pi)}{\cosh i\pi} = \frac{i \sinh \pi}{\cos \pi}$$

$$= -i \sinh \pi$$

$$\therefore I(1+\cosh\pi) - \pi \sinh \pi = 0$$

$$I = \frac{\pi \sinh \pi}{1+\cosh\pi}$$

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^{x}}$; $0<a<1$

Choose a rectangular contour of height 2π. This would enclose a simple pole at $i\pi$.

Ans: $\pi \operatorname{cosec} \pi a$

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sinh x \; dx}{1+e^{x}+e^{2x}}$

Compute the three integrals in the beginning of this discussion:

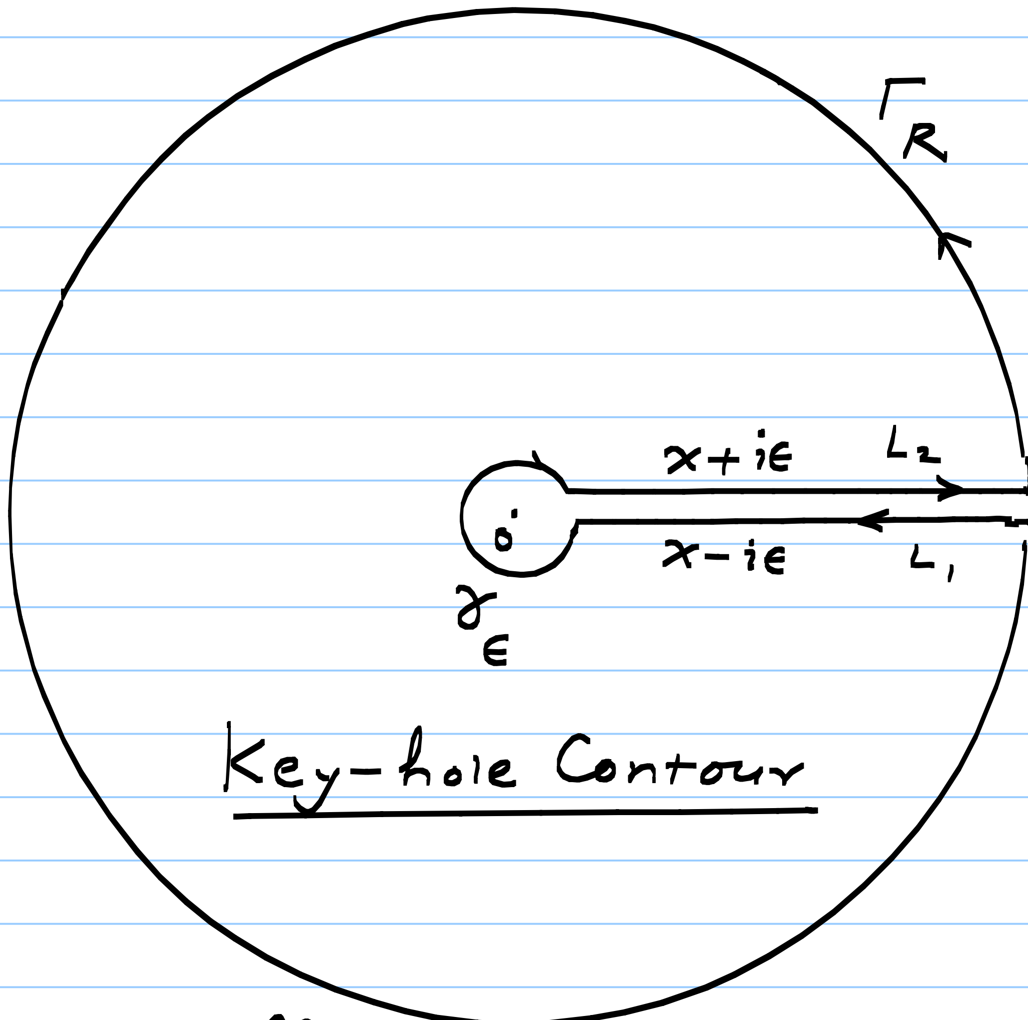
(i) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2\lambda x}}{\cosh x}\,dx$; $-\frac{1}{2} < \operatorname{Re}\lambda < 1$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\cosh cx}{\cosh \pi x}\,dx$; $|c|<\pi$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1+e^{x}+e^{2x}}$; $0<a<2$

Integrands Involving branch Points.

Use of the Keyhole Contour



Key-hole Contour

Compute $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x}\,dx$; $0<a<1$.

Let us take $f(z) = \dfrac{z^{a-1}}{1+z}$

$$= \frac{\exp (a-1) \log z}{1+z};$$

$\log z = \ln|z| + i \arg z$

defined on $\mathbb{C} - [0, \infty)$

$$0 < \arg z < 2\pi .$$

$f(z)$ has a simple pole at $z = -1$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \exp (a-1) \log z$$

$$= \exp[(a-1)(i \arg(-1))]$$

$$= \exp[(a-1) i\pi]$$

$$= -\exp a i\pi$$

By Cauchy's theorem

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{\gamma_\epsilon} f(z)\,dz + \int_{T_R} f(z)\,dz$$

$$= -2\pi i \exp i a\pi$$

$$\int_{\Gamma_R} f(z)\,dz = \int_{\epsilon}^{2\pi-\epsilon} \frac{R^{a-1} e^{(a-1)i\theta} \cdot Rie^{i\theta}\,d\theta}{1+Re^{i\theta}}$$

$$= i\int_{\epsilon}^{2\pi-\epsilon} \frac{R^a}{1+Re^{i\theta}}\, e^{ia\theta}\,d\theta$$

Since $0<a<1$, this integral $\to 0$ as $R\to\infty$

Show that $\displaystyle\int_{\gamma_\epsilon} f(z)\,dz \to 0$ as $\epsilon\to 0$.

Now $$\int_{L_2} f(z)\,dz = \int_{\epsilon}^{\infty} \frac{(x+i\epsilon)^{a-1}\,dx}{1+x+i\epsilon}$$

$$\to \int_0^{\infty} \frac{x^{a-1}\,dx}{1+x} = I$$

I is the desired integral

Now along L_1 the lower lip of the cut, $$f(z) = \frac{(x-i\epsilon)^{a-1}}{1+x-i\epsilon} \to \frac{|x|^{a-1}\cdot e^{2\pi i a}}{1+x}$$

So that

$$\int_{L_2} f(z)\,dz \to -\int_0^\infty \frac{|x|^{a-1} e^{2i\pi a}}{1+x}$$

$$= -e^{2\pi i a} \underline{I}$$

(why minus sign ??)

$$\therefore \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left(\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz \right)$$

$$= (1-e^{2\pi i a}) \underline{I}$$

So letting $R\to\infty$, $\epsilon\to 0$ in the equation

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{\gamma_\epsilon} f(z)\,dz +$$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}_{z=-1} f(z)$$ we get

$$(1-e^{2\pi i a}) I = -e^{i\pi a} \cdot 2\pi i$$

$$I = \frac{2\pi i}{e^{i\pi a} - e^{-i\pi a}}$$

$$I = \pi \operatorname{cosec} \pi a$$

<u>Exercise</u>: Use Keyhole Contour to Compute the following:

$$\int_0^\infty \frac{\sqrt{x}\ln x\,dx}{1+x^2}\ ;\quad \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}\ ;$$

$$\int_0^\infty \frac{(\ln x)\,dx}{1+x^2} \text{ and } \int_0^\infty \frac{(\ln x)^2\,dx}{(1+x^2)}$$

Use a Keyhole Contour to prove that

(i) $$\int_0^\infty \frac{(\ln x)^2\,dx}{1+x+x^2} = \frac{16\pi^3}{81\sqrt{3}}$$

(ii) $$\int_0^\infty \frac{\sqrt{x}\,dx}{x^2+x+1} = \frac{\pi}{3}$$

(iii) $$\int_0^\infty \frac{x^{1/3}\,dx}{x^2+x+1} = \frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right)$$

(iv) $$\int_0^\infty \frac{x^{1/3}\,dx}{x^2-x+1} = \frac{4\pi}{3}\sin\left(\frac{2\pi}{9}\right)$$

Compute using a Keyhole Contour

$$\int_0^\infty \frac{x^{-a}\,dx}{x^2+2x\cos\phi+1}; \quad 0<a<1, \quad 0<\phi<\pi$$

Evaluate $$\int_0^\infty \frac{\ln x\,dx}{(x+a)(x+b)}$$

By integrating $\dfrac{\log z}{1+z^3}$ around a Keyhole Contour, Compute the value of the integral $$\int_0^\infty \frac{dx}{1+x^3}$$

More examples of integrals involving $\log x$, general powers etc; but using other Contours:

1) Use a Semicircular Contour to show that

$$\int \frac{\ln(1+x^2)\,dx}{1+x^2} = \pi \ln 2$$

Note: $\frac{1}{2}\ln(1+x^2) = \operatorname{Re} f(x)$ for a suitable holomorphic function. $f(z)$

2) Deduce from (1) $\displaystyle\int_0^1 \frac{\ln(x+\frac{1}{x})\,dx}{1+x^2} = \frac{\pi}{2}\ln 2$

3) $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2-1} = \frac{\pi^2}{4}$ using Semi-circular Contour

4) By integrating the function

$$f(z) = \frac{\log(1-iz)}{z^2 + 2z\sin\alpha + 1}$$

$0 < \alpha < \frac{\pi}{2}$ along a semi-circle

Show that
$$\int_{-\infty}^{\infty} \frac{\tan^{-1} x \, dx}{x^2 - 2x\sin\alpha + 1} = \frac{\pi\alpha}{2\cos\alpha}$$

<u>End of the Course</u>

Thank You!