PDEs, Special Functions and Basic Fourier Analysis

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Dieses Buch widme ich meiner Frau in Dankbarkeit

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Preface

The development of special functions and differential equations have over three centuries occurred in perfect harmony and with a view towards applications to physics. The course tries to keep this synergy in focus, to unfold these development and present them naturally keeping technicalities at bay.

Over the years surprising connections with many other branches of mathematics - particularly number theory, probability theory and combinatorics have been unearthed. The instructor hopes that students would be stimulated enough to take an occasional look at the many references to further education and explorations into interesting by-lanes of mathematical physics. A subject that has developed in the hands of giants such as Gauss, Euler, Riemann, Laplace and Legendre to name a few, cannot fail to whet the appetite of the mathematically inclined students.

Happy reading !!

Introducing the players - Equations of mathematical physics

In this course we shall focus on the fundamental partial differential equations that arise in mathematical physics and their ODE reductions notably the Legendre and Bessel's equations. The evolution of a physical system is governed by partial differential equations such as

1. The wave equation
2. The heat equation
3. Maxwell's equations of electromagnetism
4. The Schrödinger equation.

In the heart of each of these lies the Laplace operator. In the case of Maxwell's equations, each component of electric and magnetic field satisfies the wave equation. The steady state behavior is governed by the Laplace's equation.

Ubiquity of the Laplacian Why does nature favour the Laplacian? Well, if the physical system under consideration is homogeneous (translational symmetry), then the governing equations are constant coefficient equations. Now, if the solutions exhibit rotational symmetry (in the space variables) which would be so if the physical phenomenon under consideration is isotropic about the origin, then the governing equations would be such that the partial derivatives in space variables would always occur as polynomials in the Laplacian. This can be established mathematically [1].

Thus the Laplace's operator is the most important operator in mathematical physics and its ubiquity is a reflection of the symmetry of space with respect to the group of spacial rotations. One must take advantage of this symmetry and choose coordinate systems adapted to it.

The ODE reductions of the basic equations When the PDEs of mathematical physics are written in spherical polar coordinates and the variables are separated one immediately sees the appearance of the Legendre and associated Legendre equations:

1. Legendre's equation:
$$((1-x^2)y')' + p(p+1)y = 0.$$
2. The Associated Legendre's equation:
$$((1-x^2)y')' - m^2y/(1-x^2) + p(p+1)y = 0.$$

In problems such as *diffraction of light through a circular aperture* it is more expedient to use cylindrical coordinates and we see the emergence of the differential equation of Bessel which is

[1] See for example G. B. Folland, Introduction to Partial Differential Equations, Prentice Hall, New Delhi

3. Bessel's equation
$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

Thus it is no surprise that the radii of *Newton's rings* are expressible in terms of the zeros of Bessel functions. These ODEs are variable coefficient equations and as such their solutions are not expressible in terms of the elementary transcendental functions. Indeed they define higher transcendental functions.

Methods for representing solutions There are two basic methods of representing the solutions of these ODEs.

1. **Power series:** These are amenable to algebraic manipulations and are easy to obtain from the ODEs. But it is difficult to obtain information such as growth of solutions, zeros of solutions and estimates.

2. **Integral representations:** These are usually harder to arrive at but they are more useful to estimate the solutions, their derivatives and growth/decay properties when the parameters involved are small/large. One such method has already been seen in the earlier course namely the method of Laplace transforms. The ODEs studied were such that the solutions could be "read off" from their Laplace transforms but a systematic inversion of the Laplace transform would lead to an integral representation of solutions.

   Another useful technique to obtain integral representation is the Fourier transform. We shall see some of this later in the course.

0 Preparatory results on infinite series

We begin with some generalities on infinite series which are important inasmuch as power series and trigonometric series would appear in great profusion in these lectures.

Given an infinite sequence $a_0, a_1, a_2, \dots$ we wish to assign a meaning to

$$a_0 + a_1 + a_2 + \dots$$

The infinite sum above is called an *infinite series* and is defined to be

$$\lim_{n\to\infty} (a_0 + a_1 + a_2 + \cdots + a_n).$$

The limit (if it exists) is denoted by $\sum_{n=0}^{\infty} a_n$ and we say the series CONVERGES. If the limit doesn't exits the series is said to be DIVERGENT. The finite sum $a_0 + a_1 + a_2 + \cdots + a_n$ is called the n-th partial sum of the infinite series.

The simplest example is that of a geometric series where it is possible to find a closed expression for the sum of the first $n+1$ terms:

$$c + cr + \cdots + cr^n = \frac{c(r^{n+1} - 1)}{r - 1}$$

Passing to the limit as $n \to \infty$ we conclude

$$c + cr + cr^2 + \cdots = c/(1-r), \quad |r| < 1$$

If $|r| \geq 1$ the limit doesnt exist except in the trivial case when $c = 0$. Some easy consequences of the definition

Theorem 0.1: If the series $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $S_n = a_0 + a_n + \cdots + a_n$ and l be the limit of S_n as $n \to \infty$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = l - l = 0.$$

The converse of this result is not true. It is not difficult to see that

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

diverges but the n-th term goes to zero as $n \to \infty$. In fact we have the following general result:

Theorem 0.2 (Improper integral test): Suppose $f(x)$ is a positive monotone decreasing function defined on $[1, \infty)$ then

$$f(1) + f(2) + f(3) + \dots$$

converges if and only if the improper integral

$$\int_1^\infty f(x)dx$$

converges.
¿From the improper integral test we see at once that $1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges.

Exercises:

1. Show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

   converges if $p > 1$ and diverges if $p \leq 1$.

A series $\sum a_n$ is said to *absolutely converge* if

$$|a_0| + |a_1| + |a_2| + \dots$$

converges.

Question : Suppose

$$|a_0| + |a_1| + |a_2| + \dots$$

converges, what can one say about

$$a_0 + a_1 + a_2 + \dots ?$$

Theorem 0.3: An absolutely convergent series is convergent.
The proof is not difficult but we shall not discuss it here.

What is the significance of the concept of absolute convergence?? Well, *in general rearranging the terms of an infinite series may alter the character of the series (a convergent one can turn into a divergent one) and even it the rearranged series remains convergent the sum may change.* Before investigating this we insert another notion. A series that converges but not absolutely is said to be *conditionally convergent*

Theorem 0.4 (The Alternating series test): Suppose $a_0, a_1, a_2, \dots$ is a *monotone decreasing sequence of positive reals* such that $a_n \to 0$ as $n \to \infty$ then the series

$$a_0 - a_1 + a_2 - a_3 + a_4 - \dots$$

converges.

¿From this we infer immediately that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges but not absolutely, that is to say it is a conditionally convergent series.

Example forbidding rearrangements: Let us consider the two series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

and

$$T = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \dots$$

Well, let us examine T closely:

$$T = (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \dots$$

The last one equals

$$\frac{1}{2}\Big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\Big) = S/2.$$

Which means $T \neq S$. It turns out that $S = \ln 2$.

Theorem 0.5 (Riemann's rearrangement theorem): A conditionally convergent series may through a suitable rearrangement made to converge to ANY PREASSIGNED real value.

In the light of this the following result is remarkable.

Theorem 0.6 (Dirichlet's theorem): Rearranging the terms of an absolutely convergent series has no effect on its behaviour - that is to say it remains an absolutely convergent series. Moreover, the sum is also unaffected.

Riemann's theorem appeared in his Habilitationschrift written in 1854 and published posthumously in 1867 by Richard Dedekind, a year after Riemann's death. This work, originating from the study of Fourier series, also lays the foundations of the theory of integration.

The Cauchy product of two series: Given two convergent series

$$A = a_0 + a_1 + a_2 + \dots, \quad B = b_0 + b_1 + b_2 + \dots,$$

Their Cauchy product is defined to be the series

$$C = a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$$

The general term of the Cauchy product is the *finite sum*

$$a_0b_n + a_1b_{n-1} + \dots + a_nb_0.$$

Exercises:

2. Compute the Cauchy product of a geometric series with itself.

Question: Is C a convergent series and if so is it the case that $C = AB$?

Theorem 0.7 (The product theorem of Cauchy): Suppose A and B displayed above are *both absolutely convergent* then their Cauchy product C is also absolutely convergent and $C = AB$.
We shall have occasion to use this result frequently.

Remark: It is not necessary that both series be absolutely convergent. Mertens proved the following result:

Theorem 0.8 (Mertens): If the two series converge and at least one of them converges absolutely then the Cauchy product converges and $C = AB$.

Tests for absolute convergence We now discuss two useful criteria for absolute convergence of series.

Theorem 0.9 (D'Alembert's Ratio Test): Let $\sum a_n$ be a series of non zero numbers and

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

The series $\sum a_n$ converges absolutely if $L < 1$. The series diverges if $L > 1$ or $L = +\infty$. If $L = 1$ the test is inconclusive.

Theorem 0.10 (Cauchy's Root Test): Let $\sum a_n$ be a series and

$$L = \lim_{n\to\infty} |a_n|^{1/n}$$

The series $\sum a_n$ converges absolutely if $L < 1$. The series diverges if $L > 1$ or $L = +\infty$. If $L = 1$ the test is inconclusive.

Power Series: The tests of D'Alembert and Cauchy are particularly useful while dealing with series of the form

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

These series are called *power series.*

As an example let us take up

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Here the general term is $a_n = x^n/n!$. We apply D'Alembert's Ratio test

$$|\frac{a_{n+1}}{a_n}| = |x|/(n+1)$$

Proceeding to the limit as $n \to \infty$ we see that

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0$$

which is strictly less than one and so the series converges absolutely for any value of x real or complex.

The Exponential Series: Let $E(x)$ denote the sum of the above series namely,

$$E(x) = 1 + x + \frac{x^2}{2!} + \ldots.$$

The function $E(x)$ is defined for all complex values of x.

Sine and Cosine series: Show using the ratio test that both the series

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Converge absolutely for all values of x real or complex.

Exercises:

3. Show that the series $\sum_{n=1}^{\infty} x^n/n^2$ converges absolutely for $|x| < 1$ and diverges when $|x| > 1$. What happens when $|x| = 1$?

4. Discuss the series $\sum_{n=1}^{\infty} x^n/\sqrt{n}$.

5. Discuss the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} x^n$.

6. Discuss the series $\sum_{n=1}^{\infty} n^n x^n$

Differentiation theorem: Given a power series $\sum a_n x^n$ there exists in general an $R \geq 0$ such that the series converges absolutely for $|x| < R$ and diverges for $|x| > R$. Along the circle $|x| = R$ nothing can be said in general. It may happen that $R = +\infty$. This R is called the *radius of convergence* of the power series.

Theorem 0.11: The sum of the power series is differentiable infinitely often in the disc $\{x : |x| < R\}$. Furthermore if $f(x)$ denotes the sum of the power series then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

In other words the power series may be differentiated term by term within the disc of convergence.

Examples on Differentiation theorem:

Exercises:

7. Prove that

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots, \quad |x| < 1.$$

8. Prove that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, \quad |x| < 1.$$

Question: Are we justified in setting $x = 1$ in order to derive

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$
$$\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \dots ?$$

As such it is not justified but to legitimize it we need to invoke *Abel's limit theorem.*

Exponential Addition Theorem: As an illustration on the use of the Cauchy Product theorem let us compute the Cauchy product of the series for $E(x)$ and $E(y)$ with general terms $x^m/m!$ and $y^l/l!$ respectively. The n-th term of the Cauchy product is

$$\begin{aligned}\sum_{m+l=n} \frac{x^m}{m!}\frac{y^l}{l!} &= \sum_{m=0}^{n} \frac{x^m}{m!}\frac{y^{n-m}}{(n-m)!} \\ &= \frac{1}{n!}\sum_{m=0}^{n} \frac{n!}{m!(n-m)!} x^m y^{n-m} = \frac{(x+y)^n}{n!}\end{aligned}$$

Thus, the Cauchy product of $E(x)$ and $E(y)$ is

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y)$$

We have proved

$$E(x+y) = E(x)E(y).$$

We shall see an analogue of this result in chapter - II for the *Bessel's Functions.*

Power Series with Arbitrary Center The series

$$\sum_{n=0}^{\infty} c_n (x-p)^n$$

is said to be a power series with center p. The theory of this parallels the theory of power series with center at the origin and so need not be repeated here. In particular the power series has a radius of convergence R which may be zero, a positive real number or infinity. The disc of convergence is given by

$$|x - p| < R$$

The series converges absolutely in the disc of convergence and diverges on $|x - p| > R$.

Analytic Functions The sum of the power series $f(x)$ is differentiable infinitely often in the disc of convergence and the derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-p)^{n-1}$$

A function $f(x)$ defined on an open set is said to be *analytic* at a point p if the function admits a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-p)^n$$

valid in a disc $|x-p| < r$. Clearly the coefficients c_n would depend on the chosen point. For most familiar functions the Taylor series at a point gives the power series representation at the point. The function is said to be analytic throughout its domain it is so at each point of the domain.

Exercises:

9. Show that $x \cot x$ is analytic everywhere except at $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

10. Show that $-2x/(1-x^2)$ is analytic everywhere except at ± 1.

11. Examine whether $\sin \pi x / \log(1+x)$ is analytic at the origin. If it is analytic determine the first three terms of the power series expansion.
Hint: Suppose $1 + c_1 x + c_2 x^2 + \ldots$ is a power series with positive radius of convergence, then seek a power series for the reciprocal:

$$\frac{1}{1 + c_1 x + c_2 x^2 + \ldots} = d_0 + d_1 x + d_2 x^2 + \ldots$$

Multiply out, equate coefficients and check that $d_0, d_1, d_2, \ldots$ are uniquely determined. The series $d_0 + d_1 x + d_2 x^2 + \ldots$ has positive radius of convergence but we shall not prove this.

I - Power Series Solutions of ODEs - Legendre Polynomials

We shall solve second order differential equations

$$y'' + A(x)y' + B(x)y = C(x)$$

where $A(x), B(x)$ and $C(x)$ are analytic in a neighborhood of a point x_0 in the form of a power series centered at x_0. Thus we look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where coefficients a_n are determined through a recurrence relation.

The procedure is best illustrated through an example. Let us consider the *Legendre's equation:*

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0. \tag{1}$$

The coefficient $(1 - x^2)$ does not vanish at the origin and so origin is not a *singular point.* We seek a solution of (1) in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

Differentiating,

$$-2xy'(x) = \sum_{n=0}^{\infty} -2na_n x^n \tag{3}$$

and for the term $x^2y''(x)$ we have

$$x^2y''(x) = \sum_{n=0}^{\infty} n(n - 1)a_n x^n \tag{4}$$

The term y'' in the ODE is to be dealt with in the following manner:

$$y'' = \sum_{n=0}^{\infty} n(n - 1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2}.$$

It is important to arrange it so that the exponent of the general term in the series is n. Here it is $n-2$.

So we set $n - 2 = N$ in the series and we get

$$y'' = \sum_{N=0}^{\infty} (N + 2)(N + 1)a_{N+2}x^N.$$

Changing the dummy index N to n we see that

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n. \tag{5}$$

Consolidating we get

$$(1-x^2)y'' - 2xy' + p(p+1)y =$$
$$\sum_{n=0}^{\infty} x^n\Big((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n\Big) = 0$$

Equating the coefficient of x^n to zero we get the *recurrence relation:*

$$a_{n+2} = \frac{-a_n(p-n)(p+1+n)}{(n+2)(n+1)}, \quad n = 0, 1, 2, \dots$$

Setting $n = 0, 1, 2, \dots$ in succession we find the list of coefficients:

$$a_2 = -a_0 p(p+1)/2!,\ a_3 = -a_1(p-1)(p+2)/3!$$

$$a_4 = \frac{a_0 p(p-2)(p+1)(p+3)}{4!},\ a_5 = \frac{a_1(p-1)(p-3)(p+2)(p+4)}{5!}$$

The law of formation is clear:

$$a_{2n} = (-1)^n a_0 \frac{p(p-2)\dots(p-2n+2)(p+1)(p+3)\dots(p+2n-1)}{(2n)!}$$

and

$$a_{2n+1} =$$
$$(-1)^n a_1 \frac{(p-1)(p-3)\dots(p-2n+1)(p+2)(p+4)\dots(p+2n-1)}{(2n+1)!}$$

The general solution is thus given by

$$a_0\Big(1 - \frac{p(p+1)x^2}{2!} + \frac{p(p-2)(p+1)(p+3)x^4}{4!} - \dots\Big) +$$
$$a_1\Big(x - \frac{(p-1)(p+2)x^3}{2!} + \frac{(p-1)(p-3)(p+2)(p+4)x^5}{4!} - \dots\Big).$$

The coefficients a_0 and a_1 may be assigned arbitrary values and assigning the values $a_0 = 1, a_1 = 0$ we get one solution and another linearly independent one by setting $a_0 = 0, a_1 = 1$. Both are power series with unit radius of convergence *(Exercise).*

We see that if p is an integer then *exactly* one of the two series terminates and we have a polynomial solution of the Legendre equation. With a suitable normalization that we shall presently specify, these polynomials are called *Legendre Polynomials.*

The Legendre Polynomials Assume that $p = k$ is an integer and one of the two series described above terminates into a polynomial solution $f(x)$. One can show without much difficulty that $f(1) \neq 0$. *Try this out.* Suppose not. Set $x = 1$ in the Diff. Eq. to conclude that $f'(1) = 0$ as well. Assume $f^{(n)}(1) = 0$. Differentiate the differential equation n-times using Leibnitz rule and then put $x = 1$. Apply induction on n.

Now that $f(1) \neq 0$ we can normalize our solution $f(x)$ by dividing by $f(1)$ and consider the solution

$$\frac{f(x)}{f(1)}.$$

This special solution is called the $k-$th *Legendre Polynomial.* It is customary to denote this as $P_k(x)$.

We record here the three defining properties of $P_k(x)$. Note that $k = 0, 1, 2, \ldots$.

1. $P_k(x)$ is a polynomial of degree k
2. $P_k(x)$ satisfies the Legendre Equation:
$$(1 - x^2)P_k''(x) - 2xP_k'(x) + k(k + 1)P_k(x) = 0$$
3. $P_k(1) = 1$.

It is clear that $P_k(x)$ is an odd function if k is odd and an even function if k is even. Also

$$P_0(x) = 1$$

Orthogonality Properties of Legendre Polynomials:

Theorem 1.1: If $k \neq l$ then $P_k(x)$ and $P_l(x)$ are orthogonal in the following sense:

$$\int_{-1}^{1} P_k(x)P_l(x)dx = 0.$$

Proof: To prove this we begin with the differential equations

$$(1 - x^2)P_k'' - 2xP_k' + k(k + 1)P_k = 0 \tag{1}$$

and

$$(1 - x^2)P_l'' - 2xP_l' + l(l + 1)P_l = 0 \tag{2}$$

We shall write these equations can be written in a more following convenient form known as *self-adjoint form.*

$$\frac{d}{dx}\Big((1 - x^2)P_k'\Big) + k(k + 1)P_k = 0 \tag{3}$$

and

$$\frac{d}{dx}\Big((1 - x^2)P_l'\Big) + l(l + 1)P_l = 0 \tag{4}$$

Multiply (3) by P_l, (4) by P_k, subtract and integrate over $[-1, 1]$. Integration by parts would confirm that

$$(k(k + 1) - l(l + 1)) \int_{-1}^{1} P_k(x)P_l(x)dx = 0.$$

Since $k \neq l$ and are non-negative integers, the factor $k(k+1) - l(l+1) \neq 0$ and the proof is complete.

Exercise: Explain what happens if k and l are not non-negative integers.

Theorem 1.2 (Fundamental Orthogonality Lemma): Suppose V is a vector space endowed with inner product with respect to which $\{v_0, v_1, v_2, \dots\}$ and $\{w_0, w_1, w_2, \dots\}$ are two orthogonal systems of non-zero vectors. Further assume that

$$\text{span}(v_0, v_1, \dots, v_k) = \text{span}(w_0, w_1, \dots, w_k), \quad \text{for every } k = 0, 1, 2 \dots$$

Then, for certain scalars c_k ($k = 0, 1, 2 \dots$),

$$v_k = c_k w_k, \quad \text{for every } k = 0, 1, 2 \dots$$

Proof is an Exercise. First think of what happens in ordinary Euclidean spaces like $\mathbb{R}^n$. Geometrical considerations suggests the proof.

Exercise: Consider the sequence of polynomials

$$Q_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Show that $Q_n(x)$ has degree n for every n. Further show that the sequence is orthogonal with respect to weight 1 namely,

$$\int_{-1}^{1} Q_n(x) Q_m(x) dx = 0, \quad m \neq n.$$

¿From this infer the following result:

$$P_n(x) = c_n Q_n(x), \quad \text{for every } n = 0, 1, 2 \dots$$

for a certain sequence of constants $\{c_n\}$.

Rodrigues' Formula: Compute the constants c_n in the last slide by evaluating $Q_n(1)$. Deduce the following formula due to *Olinde Rodrigues.*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$

For more information on the work of Rodrigues see the Book Review by W. P. Johnson, in the American Math. Monthly, Volume 114, Oct 2007, 752-758[2]

Exercises:

1. Compute $\int_{-1}^{1} (P_n(x))^2 dx$
2. Show that $\int_{-1}^{1} (1 - x^2)(P_n'(x))^2 dx = 2n(n+1)/(2n+1)$. Hint: Multiply the Diff. Eqn by P_n and integrate by parts.
3. Use Rodrigues formula to prove that the Legendre polynomial of degree n has precisely n distinct roots in the open interval $(-1, 1)$. Use Rolle's theorem. Note: The roots were used by *Gauss in 1814 in his famous quadrature formula*[3].
4. Show that the Legendre polynomials satisfy the three term recursion formula

$$(n+1)P_{n+1} - x(2n+1)P_n + nP_{n-1} = 0.$$

[2] Available in the link: http://www.jstor.org/stable/27642326

[3] See the discussion on pp. 56-69 of S. Chandasekhar, Radiative transfer, Dover Publications, Inc., New York, 1960.

More Problems on Legendre Polynomials

5. Prove that $P_n'(1) = \frac{1}{2}n(n+1)$

6. Prove that $P_{n+1}' - xP_n' = (n+1)P_n$

7. Prove that $(x^2-1)P_n' - nxP_n + nP_{n-1} = 0$.

8. Prove that $P_{n+1}' - P_{n-1}' - (2n+1)P_n = 0$.

9. Prove that $xP_n' - P_{n-1}' = nP_n$.

10. Suppose $x^n = \sum_{j=0}^n c_j P_j(x)$ show that $c_n = 2^n(n!)^2/(2n)!$.

11. Use the method of series solutions to find the power series expansion of $(1+x)^a$ where a is any real number. Hint: Find an ODE satisfied by the function.

Generating function for the Legendre Polynomials: Given a sequence $\{a_n\}$ of real or complex numbers, their generating function is by definition the power series

$$\sum_{n=0}^{\infty} a_n t^n$$

Theorem 1.3:

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

For connections to potential theory[4]. There are several proofs of this important theorem and we select the one from Courant and Hilbert's monumental treatise[5]. Only a sketch of the proof is provided and the student can work out the details. According to the last exercise in the previous paragraph, the function $V(x,t) = (1-2xt+t^2)^{-1/2}$ can be expanded in an absolutely convergent series

$$V(x,t) = \sum_{n=0}^{\infty} R_n(x) t^n$$

where each $R_n(x)$ is a polynomial of degree exactly n (why?) and the series is valid for $x \in [-1,1]$ and for a certain interval $|t| < \rho$. Indeed $\rho = 0.4$ would suffice (why?).

In particular

$$\frac{1}{\sqrt{1-2xu+u^2}\sqrt{1-2xv+v^2}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R_j(x)R_k(x)u^j v^k$$

Integrating both sides with respect to x over the range $[-1,1]$ we get

$$\frac{1}{\sqrt{uv}} \ln\left(\frac{1+\sqrt{uv}}{1-\sqrt{uv}}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} u^j v^k \int_{-1}^{1} R_j(x)R_k(x)dx.$$

[4] see for instance, A. S. Ramsey, Newtonian Attraction, Camb. Univ. Press, p. 131 ff., or pp 121 - 134 of the more comprehensive and classic treatise of O. D. Kellog, Foundations of potential theory, Dover, New York, 1953.

[5] Methods of mathematical physics - I.

But the left hand side is (using the logarithmic series we have encountered earlier) given by

$$\frac{1}{\sqrt{uv}} \ln\left(\frac{1+\sqrt{uv}}{1-\sqrt{uv}}\right) = \sum_{n=0}^{\infty} \frac{2u^n v^n}{2n+1}$$

Comparing the coefficient of $u^j v^k$ in the last two expressions we get

$$\int_{-1}^{1} R_j(x)R_k(x)dx = 0, \quad \text{if } j \neq k.$$

Whereas if $j = k$ then

$$\int_{-1}^{1} (R_j(x))^2 dx = 2/(2j+1).$$

The fundamental orthogonality lemma now implies that $R_j(x) = P_j(x)$ for every $j = 0, 1, 2, \ldots$.

Remark: The student is not expected to remember this proof for the examinations.

The function $V(x,t)$ is the potential due to a point mass P placed at unit distance from the origin at a point X at distance t from the origin, where x is the cosine of the angle between OX and OP.

Bernoulli Polynomials and Bernoulli Numbers: This is just intended as an example. We inductively define the sequence of *Bernoulli polynomials* $\{B_n(x)\}$ as follows:

$$B_n'(x) = nB_{n-1}(x), \quad n \geq 1$$

and $B_0(x) = 1$. Further we demand

$$\int_0^1 B_n(x)dx = 0, \quad n \geq 1.$$

The student can easily determine the first few Bernoulli polynomials. The numbers $\{B_n(0)\}$ are called the *Bernoulli Numbers.*

Easy exercise: Show that

$$\sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} = C(t)e^{xt} \tag{1}$$

However it is not so easy to show $C(t) = t/(e^t - 1)$. Here is an interesting suggestion due to *Mr. Sanket Sanjay Barhate.* Integrate both sides of (1) with respect to x from 0 to 1. From the LHS we get 1 and on the RHS we get $C(t)(e^t - 1)/t$. At the formal level this computation is correct but the term by term integration needs justification. We need some decent estimate on $|B_n(x)|$ valid over $[0, 1]$. It may be noted that the Bernoulli numbers rapidly increase to infinity.

Thus

$$\sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} = \frac{te^{xt}}{e^t - 1}$$

So $te^{xt}/(e^t - 1)$ is the generating function for the sequence $\{B_n(x)/n!\}$.

It would not have been easy to *guess* the above formula. So the important thing was to obtain the closed form for the generating function $\sum_{n=0}^{\infty} B_n(x)t^n/n!$ (never mind it being formal). Let us now abandon the task of justifying the interchange of integration and infinite sums but verify the result obtained in an entirely different way. Let us define

$$\beta_n(x) = \left(\frac{d}{dt}\right)^n \frac{e^{tx}t}{e^t - 1}\Big|_{t=0}$$

Then $\beta_0(x) = 1$. Now,

$$\frac{d}{dx}(\beta_n(x)) = \left(\frac{d}{dt}\right)^n \frac{e^{tx}t^2}{e^t - 1}\Big|_{t=0}$$

Using Leibnitz rule we see immediately that $\beta_n(x) = n\beta_{n-1}(x)$. It is easy to justify (how?)

$$\int_0^1 \left(\frac{d}{dt}\right)^n \frac{e^{tx}t}{e^t - 1}dx = \left(\frac{d}{dt}\right)^n \int_0^1 \frac{e^{tx}t}{e^t - 1}dx.$$

Setting $t = 0$ in the last equation we get,

$$\int_0^1 \beta_n(x)dx = \left(\frac{d}{dt}\right)^n 1 = 0, \quad n = 1, 2, \dots.$$

Thus we conclude that $\beta_n(x) = B_n(x)$ and the proof is complete.

Some Classical Formulas

James Bernoulli - Ars Conjectandi (1713):

$$1^p + 2^p + \cdots + n^p = \frac{1}{n+1}\Big(B_{p+1}(n+1) - B_{p+1}(0)\Big), \quad p = 1, 2, 3, \dots$$

Euler:

$$1 + \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \cdots = \frac{(-1)^{p+1}(2\pi)^{2p}B_{2p}(0)}{2\cdot(2p)!}, \quad p = 1, 2, 3, \dots$$

James and John Bernoulli tried in vain to obtain the latter for $p = 1$. It was discovered by Euler in 1736. However James Bernoulli did not live to see the last displayed formula in which the numbers that bear his name feature so prominently.

Tchebychev's Differential Equation:

12. Discuss the series solutions of the Tchebychev's differential equation:

$$(1 - x^2)y'' - xy' + p^2y = 0$$

Show that if p is an integer, of the two linearly independent solutions *exactly* one of them terminates into a polynomial solution which after suitable renormalization is denoted by $T_n(x)$.

13. Rewrite the ODE in self-adjoint form and show that if $k \neq l$

$$\int_{-1}^1 T_k(x)T_l(x)(1 - x^2)^{-1/2}dx = 0.$$

In other words the Tchebychev's polynomials form an orthogonal system with respect to the weight function $(1 - x^2)^{-1/2}$.

14. Show that $\sin(p\sin^{-1}(x))$ and $\cos(p\cos^{-1}x)$ satisfy the Tchebychev's equation.

15. Show that $T_n(x) = \cos(n\cos^{-1}x)$. This means you need to prove first that the function on the right is a polynomial. Try this by induction. Then invoke uniqueness of T_n as a polynomial of degree n satisfying the ODE with appropriate normalization.

For more problems on Tchebychev's polynomials, the student is referred to pp 177-187 of L. Sirovich, Introduction to Applied Mathematics, Springer Verlag, 1988.

Hermite's Equation - Appears in Quant. Mech. See A. Beiser, Perspectives of Modern Physics, 1969, pp 180-187. The differential equation appears on p. 183.

16. Discuss the series solutions of Hermite's equation

$$y'' - 2xy' + 2\lambda y = 0$$

17. Write the equation in self-adjoint form. Hint: Multiply by $\exp(-x^2)$.

18. If n is a non-negative integer show that one of the series solutions terminates and we get a polynomial solution. After suitable normalization these are called the Hermite Polynomials $H_n(x)$.

19. Show that the Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$.

Orthogonal Polynomials in general The properties established for the Legendre Polynomials are rather typical of classical orthogonal systems of polynomials.

1. It is a general fact that if $\{f_n(x)\}$ is a sequence of orthogonal polynomials then the zeros of $f_n(x)$ are real distinct and lie in the interval of orthogonality.

2. The sequence $\{f_n\}$ satisfies a three term recursion formula.

3. The zeros of f_n and f_{n+1} interlace (we have not proved this for the Legendre polynomials).

4. There is an analogue for the Rodrigues formula for the system of Hermite, Tchebychev, Laguerre and other classical systems of polynomials. Note: The Laugerre polynomials arise when the sequence $1, x, x^2, \ldots$ is subjected to the Gram-Schmidt process with respect to the inner product on $[0, \infty)$ with weight function e^{-x}. These also arise from the Laguerre ODE we shall see later.

The book by Ian. Sneddon, Special functions of mathematical physics and chemistry, Longman Mathematical Texts, Longman, New York, 1980, contains very instructive list of problems on Legendre Polynomials (see pp. 96-105).

Additional Problems on Tchebychev's polynomials: These examples are all from chapter 6 of *L. Sirovich, Introduction to applied mathematics, Springer Verlag, 1988.*

20. *Recall that* $T_n(x) = \cos(n\cos^{-1}(x))$. *Use this to determine the three term recursion formula for the sequence* $\{T_n(x)\}$.

21. *Compute the integral*
$$\int_{-1}^{1} \frac{(T_n(x))^2 dx}{\sqrt{1-x^2}}$$

22. *As for the case of Legendre polynomials, show that the Tchebychev's polynomial* $T_n(x)$ *has* n *distinct roots in* $(-1,1)$. *Determine these roots.*

23. *Show that*
$$T_n(x) = \frac{1}{2}\Big\{(x - i\sqrt{x^2-1})^n + (x + i\sqrt{x^2-1})^n\Big\}$$
Hint: Write cosine in exponential form.

24. *Use the previous result to prove that the generating function for the sequence* $\{T_n(x)\}$ *is*
$$G(x,t) = \frac{1-tx}{1+t^2-2tx}.$$

25. *Use trigonometry to show that* $2T_m(x)T_n(x) = T_{m+n} + T_{m-n}$.

26. *Show that* $T_n(T_m(x)) = T_{mn}(x)$.

27. *Prove that*
$$\left(\frac{d}{d\cos\theta}\right)^{n-1} \sin^{2n-1}\theta = (-1)^{n-1}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n}\sin n\theta$$
This formula is due to C. G. J. Jacobi (1836). See p. 26 ff. of *G. N. Watson, Treatise on the theory of Bessel functions to understand its immense use in special functions.*
Hint: Put $t = \cos\theta$ and show that
$$f(t) = \left(\frac{d}{dt}\right)^{n-1}(1-t^2)^{n-\frac{1}{2}}$$
is a solution of Tchebychev's ODE whereby
$$f(t) = c_n \sin(n\cos^{-1} t).$$
To determine c_n divide both sides by $\sqrt{1-t}$ and let $t \to 1$.

II - Differential Equations with regular singular points

We shall discuss in detail the Bessel's functions of the first kind but as a preparation we shall have to begin with some basic notions on the gamma function. The gamma function was introduced into analysis by *L. Euler in 1729* in a letter to *Goldbach* seeking an analytic interpolation of the factorial function. Since then the function has been subjected to intense research and still interests mathematicians. For details on the historical developments and further references see the article by *G. K. Srinivasan, The gamma function - an eclectic tour, American Math. Monthly, Volume 114 (2007) 297-315.* A. M. Legendre introduced the notation $\Gamma(p)$ and since then the function is referred to as the gamma function

Definition: The gamma function is defined as an integral

$$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt, \quad p > 0.$$

We can also allow p to be a complex number with *positive real part.* In this case the factor t^{p-1} is defined as

$$t^{p-1} = \exp((p-1)\ln t),$$

where it must be recalled that the exponential function was defined earlier as an infinite series. The integral makes sense and defines an analytic function in the right half plane.

Basic Properties of the Gamma function: A simple integration by parts gives:

$$\Gamma(p+1) = p\Gamma(p), \quad \Gamma(1) = 1.$$

¿From this it immediately follows

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots.$$

It is easy to verify that

$$\Gamma(1/2) = \int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}.$$

The Stirling's Approximation Formula: It would be inappropriate not to mention at least briefly one of the most remarkable results of classical analysis obtained by *James Stirling* in his *Methodus Differentialis in 1730.*

Theorem 2.1 (Stirling's approximation formula) As $n \to \infty$,

$$n! \sim n^n e^{-n} \sqrt{2n\pi}$$

in the sense that the ratio tends to 1 as $n \to \infty$.

The proof may be found for instance in the appendix to the first chapter of *W. Feller: Introduction to the theory of probability, Volume - I, Wiley.*

The theorem can be extended to the gamma function. As $x \to \infty$,

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2x\pi}$$

in the sense that the ratio tends to 1 as $x \to \infty$.

We shall consider differential equations of the form

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

where $A(x), B(x)$ and $C(x)$ are analytic functions - except possibly for isolated singularities.

Definition (Singular point): In general any point where $B(x)/A(x)$ or $C(x)/A(x)$ fails to be analytic is said to be a singular point of the ODE. Suppose $A(x_0) = 0$ then x_0 may be a singular point of the differential equation since $B(x)/A(x)$ and $C(x)/A(x)$ may fail to be analytic at the point x_0. However there may be singularities other than the zeros of $A(x)$ as the following example indicates.

$$xy'' + (\cot x)y' - (\sec x)y = 0$$

has singular points at $0, \pm\pi, \ldots$ as well as $\pi/2 \pm \pi, \pi/2 \pm 2\pi, \ldots$.

We say that x_0 is a *regular singular point* if $(x - x_0)B(x)/A(x)$ and $(x - x_0)^2 C(x)/A(x)$ are both analytic at x_0. Otherwise the singular point is said to be *irregular.*

Definition After the substitution $x = 1/t$ if the origin is a ordinary/reg. sing./irr. sing. point then we say that the *point at infinity is an ordinary/reg. sing/irr. sing. point of the original equation.*

Check that the point at infinity is an irregular singular point of Airy's Equation $y'' - xy = 0$.

Examples of ODEs with regular singular points (except possibly ∞)

1. Legendre: $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$.
2. Tchebychev: $(1 - x^2)y'' - xy' + p^2 y = 0$.
3. Bessel: $x^2y'' + xy' + (x^2 - p^2)y = 0$.
4. Laguerre: $xy'' + (1 - x)y' + py = 0$.
5. Hypergeometric: $x(1 - x)y'' + (c - (a + b + 1)x)y' - aby = 0$.

Discuss the nature of the point at infinity for each of the above.

Theorem 2.2 (Fuchs-Frobenius): Suppose the ODE $A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$ has a *regular singular point* at $x = x_0$. Then the differential equation admits at-least one solution of the form

$$y(x) = (x - x_0)^\rho \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where the associated power series $\sum a_n(x - x_0)^n$ has *positive radius of convergence.*

The theorem was discovered by *Lazarus Fuchs in 1866 (Crelle's Journal) and in 1876 (Crelle's Journal) Frobenius described the algorithm for determining the complete system of solutions.*

We shall not prove the theorem here but as a motivation look at the case of Bessel's ODE rewritten as

$$x^2y'' + xy' - p^2y = -x^2y$$

So the Bessel's equation appears as a *perturbed Cauchy-Euler equation with "forcing function"* $-x^2y$. *For small values of* x *one hopes the solution would approximate the solution of the associated homogeneous equation*

$$x^2y'' + xy' - p^2y = 0$$

which has solutions

$$x^p \text{ and } x^{-p}.$$

This suggests the Ansatz

$$y(x) = x^{\pm p}\Big(a_0 + a_1x + a_2x^2 + \dots,\Big), \quad a_0 \neq 0. \tag{1}$$

Series of the form (1) are called Frobenius Series. The Fuchs-Frobenius theorem assures us of the correctness of the Ansatz (1). The reason for assuming $a_0 \neq 0$ is that if $a_0 = 0$ and $a_1 \neq 0$ we can factor out a x from the series and write

$$x^{1\pm p}\Big(a_1 + a_2x + a_3x^2 + \dots,\Big),$$

We can rename the exponent as well as the coefficients and get back to the Ansatz (1). Thus it is legitimate to assume that this has been done at the outset. Hereafter we shall always assume that $a_0 \neq 0$.

Determining the Frobenius series for an ODE This is done by substituting the Ansatz

$$x^\rho \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

into the ODE. First we must determine the exponent ρ known as the *Frobenius exponent* and then successive the coefficients $a_1, a_2, \dots$. Let us illustrate this for the Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{2}$$

Well,

$$x^2y'' = x^\rho \sum_{n=0}^{\infty}(n+\rho)(n+\rho-1)a_n x^n, \quad xy' = x^\rho \sum_{n=0}^{\infty}(n+\rho)a_n x^n$$

¿From $x^2y'' + xy' + (x^2 - p^2)y = 0$ we infer

$$x^\rho \sum_{n=0}^{\infty} a_n \Big((n+\rho)(n+\rho-1) + (n+\rho) - p^2\Big) x^n + \sum_{n=0}^{\infty} a_n x^{n+\rho+2} = 0.$$

The last series must be modified by the substitution $n + 2 = N$. The result is

$$x^\rho \sum_{n=0}^{\infty} a_n \Big((n+\rho)^2 - p^2\Big) x^n + \sum_{n=2}^{\infty} a_{n-2} x^{n+\rho} = 0.$$

Equating the $n = 0$ coefficient to zero we get (since $a_0 \neq 0$)

$$\rho^2 - p^2 = 0.$$

This quadratic which determines ρ is called the *indicial equation.* Thus

$$\rho = \pm p.$$

We shall assume $p \geq 0$ and work with the larger root p.

Equating the coefficient corresponding to $n = 1$ we get

$$a_1((p+1)^2 - p^2) = 0$$

which forces $a_1 = 0$.

Finally we have the recursion formula

$$n(n+2p)a_n = -a_{n-2}, \quad n = 2, 3, \dots$$

We see that all odd coefficients vanish and for the even coefficients which simplifies to

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(1+p)(2+p)\dots(n+p)}.$$

Exercise: Show that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+p)(2+p)\dots(n+p)}$$

has infinite radius of convergence. So we get the solution

$$a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n} n!(1+p)(2+p)\dots(n+p)}.$$

Bessel's functions of the first kind: In view of the properties of the Gamma function,

$$\Gamma(n+p+1) = (n+p)\Gamma(n+p) = \dots = (n+p)(n+p-1)\dots(1+p)\Gamma(1+p).$$

and the displayed expression in the last slide can be recast as

$$a_0 2^p \Gamma(1+p) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p} n!\Gamma(n+p+1)}$$

The *Bessel's function of the first kind* denoted by $J_p(x)$ is the solution with a specific value of

$$a_0 = \frac{1}{2^p\Gamma(1+p)}.$$

Thus

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\Gamma(n+p+1)}$$

It is clear that if $p = 0$ we can only get one solution by this method. Also note that if p is not an integer then we can repeat the above with the smaller root $-p$ as well and get a second solution. These two are linearly independent (how?).
Show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

What about $J_{3/2}(x)$? One can determine this directly from the definition but it would be easier to use the theorem that follows. *The Bessel's functions of half-integer order are expressible in terms of elementary functions.* We shall see later how this is reflected in the different behaviour of the Fourier transform of radial functions of two and three variables.

Basic Properties of Bessel's functions of the first kind For $k \in \mathbb{N}$, we DEFINE $J_{-k}(x) = (-1)^k J_k(x)$.

Theorem 2.3: The following relations hold:

$$\frac{d}{dx}\Big(x^p J_p(x)\Big) = x^p J_{p-1}(x), \quad \frac{d}{dx}\Big(x^{-p} J_p(x)\Big) = -x^{-p} J_{p+1}(x).$$

Proof: Differentiation theorem which permits term by differentiation of the series:

$$\frac{d}{dx}\Big(x^p J_p(x)\Big) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p} n!\Gamma(n+p+1)}$$

Assume $p > 0$ and we get

$$\frac{d}{dx}\Big(x^p J_p(x)\Big) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-1}}{2^{2n+p-1} n!\Gamma(n+p)}$$

and the RHS is precisely $x^p J_{p-1}(x)$.

In case $p = 0$ then after differentiation the series would begin with the $n = 1$ term. Well, with $p = 0$,

$$\frac{d}{dx}\Big(J_0(x)\Big) = \sum_{n=1}^{\infty} \frac{(-1)^n (x/2)^{2n-1}}{n!(n-1)!}$$

Put $n - 1 = N$ and we get

$$\frac{d}{dx}\Big(J_0(x)\Big) = -\sum_{N=0}^{\infty} \frac{(-1)^N (x/2)^{2N+1}}{N!(N+1)!} = -J_1(x) = J_{-1}(x).$$

as asserted. The other formula is left as an exercise.

We rewrite the formulas obtained in the previous theorem in a different form:

$$xJ_p' + pJ_p = xJ_{p-1}, \quad xJ_p' - pJ_p = -xJ_{p+1}$$

Hence

$$2pJ_p = x(J_{p-1} + J_{p+1}), \quad 2J_p' = J_{p-1} - J_{p+1}$$

Exercises:

1. Assume $p \geq 1$. Show that between two successive positive zeros of $J_p(x)$ there is a unique zero of $J_{p-1}(x)$ and a unique zero of $J_{p+1}(x)$.
2. Deduce an exact expression for $J_{\pm 3/2}(x)$ in terms of the sine/cosine functions.
3. Use induction to show that

$$J_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \Big(\frac{1}{x}\frac{d}{dx}\Big)^n \frac{\sin x}{x}$$

4. Prove that $|J_m(x)| \leq |x|^m e^{|x|}/m!$.

A bilateral series

$$\sum_{n=-\infty}^{\infty} a_n$$

is said to converge it each to the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ converges. The sum of the bilateral series is then the sum to the last two. Likewise one defines the notion of absolute convergence. Show that the bilateral series

$$\sum_{n=-\infty}^{\infty} J_n(x)$$

converges absolutely. What is the coefficient of x^t when each $J_n(x)$ is written as a power series and the resulting expression is rearranged? Ans: zero except when $t = 0$. Thus the sum of the bilateral series is 1. Hint: First prove that

$$\sum_{n=0}^{t} (-1)^n \binom{2t}{t-n} = \frac{1}{2}\binom{2t}{t}$$

Here is an alternate solution that avoids the tedious algebra involved in the previous slide. Suppose the sum is denoted by $C(x)$ and we assume term by term differentiation is valid. *In fact it is valid but the proof needs more background from analysis such as notions of uniform convergence.* Then

$$2C'(x) = \sum_{n=-\infty}^{\infty} (J_{n-1}(x) - J_{n+1}(x)) = 0.$$

Deduce that $C(x) = 1$.

The Schlömilch's Formula This is the formula giving explicitly the generating function for the bilateral sequence $\{J_n(x) : -\infty < n < \infty\}$.

Theorem 2.4 (Schlömilch):

$$\sum_{n=-\infty}^{\infty} J_n(x)t^n = \exp\left(\frac{tx}{2} - \frac{x}{2t}\right), \quad t \neq 0.$$

To prove this we denote the sum of the series on the left hand side by $F(x,t)$. *Show that the series converges for all values of* $t \neq 0$. The differentiation theorem gives

$$\frac{\partial F(x,t)}{\partial t} = \frac{1}{2}\sum_{n=-\infty}^{\infty} 2nJ_n(x)t^{n-1}$$

But $2nJ_n = x(J_{n-1} + J_{n+1})$. Hence

$$\frac{\partial F(x,t)}{\partial t} = \frac{x}{2}\sum_{n=-\infty}^{\infty} J_{n-1}(x)t^{n-1} + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_{n+1}(x)t^{n-1}$$

So finally we get the Diff. Eqn.

$$\frac{\partial F(x,t)}{\partial t} = \left(\frac{x}{2} + \frac{x}{2t^2}\right)F$$

Integrating we get

$$F(x,t) = \exp\left(\frac{xt}{2} - \frac{x}{2t}\right)C(x)$$

where we need to determine $C(x)$. Put $t = 1$ and we get

$$C(x) = \sum_{n=-\infty}^{\infty} J_n(x) = 1.$$

The proof is complete.

Integral Representation for the Bessel's function:

Theorem 2.5: For $m = 0, 1, 2, \ldots$ we have the integral representation:

$$J_m(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - m\theta)d\theta.$$

Proof: Put $t = \exp(i\theta)$ in Schlömilch's formula and we get

$$\sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta} = \exp(ix\sin\theta)$$

Multiply by $\exp(-im\theta)$ and integrating over $[-\pi, \pi]$ we get

$$2\pi J_m(x) = \int_{-\pi}^{\pi} e^{ix\sin\theta}e^{-im\theta}d\theta$$

Using $e^{iy} = \cos y + i \sin y$ we get

$$2\pi J_m(x) = \int_{-\pi}^{\pi} \cos(x \sin\theta - m\theta) d\theta + i \int_{-\pi}^{\pi} \sin(x \sin\theta - m\theta) d\theta$$

The second integral is zero (why?) and we get the desired result:

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x \sin\theta - m\theta) d\theta$$

This was the form in which Bessel originally presented in 1824 the functions that bear his name. Bessel was an astronomer in Königsberg. We shall later see an application to a famous problem in *celestial mechanics namely, inverting the famous Kepler Equation.*

Legendre Polynomials Again!! Laplace's Integral Representation An integral representation very similar to the one we have just obtained for Bessel's function was given by Pierre Simon Marquis de Laplace in his great work on Celestial Mechanics [6]

Theorem 2.6:

$$\begin{aligned} P_n(x) &= \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2 - 1} \cos\phi)^n d\phi. \\ P_n(x) &= \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2 - 1} \cos\phi)^{-n-1} d\phi. \end{aligned}$$

We immediately deduce that $|P_n(x)| \leq 1$ when $|x| \leq 1$.

Exercises: We prove the first formula. Call the integral $Q_n(x)$.

1. Why is $Q_n(x)$ a polynomial of degree n?
2. Verify that $Q_0(x) = 1 = P_0(x)$ and $Q_1(x) = x = P_1(x)$.

It suffices to show that the sequence $Q_n(x)$ satisfies the same three term recursion as $P_n(x)$ namely

$$(n + 1)Q_{n+1} - x(2n + 1)Q_n + nQ_{n-1} = 0.$$

Write A for $x + \sqrt{x^2 - 1} \cos\phi$ and

$$A^{n+1} = xA^n + A^n \sqrt{x^2 - 1} \frac{d}{d\phi} \sin\phi$$

Integrate by parts. We get

$$Q_{n+1} = xQ_n + \frac{n}{\pi} \int_0^{\pi} A^{n-1}(x^2 - 1) \sin^2\phi d\phi.$$

[6] Traité de méchanique célèste

Now write

$$\begin{aligned}(x^2-1)\sin^2\phi &= (x^2-1)-(\sqrt{x^2-1}\cos\phi)^2 \\ &= (x^2-1)-(A-x)^2 = 2Ax - A^2 - 1.\end{aligned}$$

Thus we get

$$Q_{n+1} = xQ_n - \frac{n}{\pi}\int_0^\pi (A^{n+1} + A^{n-1} - 2xA^n)d\phi$$

which readily translates to

$$(n+1)Q_{n+1} - x(2n+1)Q_n + nQ_{n-1} = 0.$$

For a different proof of this see Byerly, pp. 165-167. The second formula is easily deduced from the first as will be indicated in the exercises.

Lots of Exercises ! Have Fun!!

1. Show that

$$\frac{1}{2}\frac{d}{dx}[J_n^2 + J_{n+1}^2] = \frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2, \quad \frac{d}{dx}[xJ_nJ_{n+1}] = x(J_n^2 - J_{n+1}^2).$$

   and deduce from these the following

$$J_0(x)^2 + 2\sum_{n=1}^{\infty}(J_n(x))^2 = 1, \quad \sum_{n=0}^{\infty}(2n+1)J_nJ_{n+1} = x/2.$$

   Hint: Differentiate $x\sum(2n+1)J_nJ_{n+1}$.

2. Prove using Schlömilch's formula:

$$J_0(x)^2 + 2\sum_{n=1}^{\infty}(J_n(x))^2 = 1.$$

   Hint: Replace t by $-t$ is Schlömilch's formula.

3. Deduce that $|J_0(x)| \leq 1$ and $J_n(x)| \leq 1/\sqrt{2}$.

4. The last exercise shows in particular that the Laplace transform of $J_0(x)$ exists. Find the Laplace transform of $J_0(x)$. Do this by computing term by term Laplace transform from the series. Do this using the integral representation. Do this from the ODE - you will have some trouble determining the arbitrary constant.

5. Recall the definition of convolution from MA 108. Determine the convolution pf $J_0(x)u(x)$ with itself where $u(x)$ is the Heaviside unit step function. Hint: Use the convolution theorem.

6. Prove the Bessel addition theorem.

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y).$$

7. Show that $\sqrt{x}J_{\pm 1/3}\left(\frac{2}{3}x^{3/2}\right)$ satisfy the Airy's equation $y'' + xy = 0$.

8. Show that $xJ_0(x)$ is a solution of the ODE $y'' + y = -J_1(x)$. Deduce that

$$xJ_0(x) = \int_0^x \cos(x-t)J_0(t)dt.$$

9. Prove the following:

$$J_0(x) = \frac{1}{\pi}\int_{-1}^{1} e^{itx}(1-t^2)^{-1/2}dt$$

and more generally for non-negative integer values of k,

$$J_k(x) = \frac{x^p}{2^p\sqrt{\pi}\Gamma(p+\frac{1}{2})}\int_{-1}^{1} e^{itx}(1-t^2)^{k-1/2}dt$$

10. Starting from the series for $J_0(x)$ and the formula

$$\int_0^\pi \sin^{2n}\theta d\theta = \frac{\pi(2n)!}{2^n n!}$$

re-derive the integral representation for $J_0(x)$. By differentiating derive it for $J_1(x)$ as well.

11. Attempt a Frobenius series solution $y(x) = x^\rho \sum_{n=0}^{\infty} a_n x^n$ of the ODE $x^2y'' + (3x-1)y' + y = 0$. Why does the method break down?

12. Determine the indicial equation of the Hypergeometric ODEs. Show that it admits a series solution as well as a Frobenius series solution if c is not an integer.

13. Show that the Laguerre ODE has a power series solution. What is its indicial equation? Show that for a discrete set of parameter values it admits polynomial solutions and that these polynomials are orthogonal on $[0,\infty)$ with respect to the weight function e^{-x}.

14. Show that the sequence of polynomials

$$L_n(x) = e^x D^n(x^n e^{-x})$$

forms an orthogonal system with respect to the weight function e^{-x}. Determine the inner product of $L_n(x)$ with itself. Do the polynomials $L_n(x)$ satisfy the Laguerre ODE?

15. Show that the substitution $t = \frac{1}{2}(1-x)$ reduces the Legendre's equation to the Hypergeometric equation with $a = p+1, b = -p$ and $c = 1$.

Some points on Frobenius's algorithm

1. Suppose the indicial equation has roots ρ_1 and ρ_2 that DO NOT differ by an integer then each of them gives rise to a Frobenius series solution of the ODE and the two are linearly independent.

2. Suppose the roots ρ_1 and ρ_2 differ by an integer, the one with larger real part always leads to a Frobenius series solution. However if we attempt to use the root with smaller real part three possibilities occur as described below

A The procedure goes through and we do get a second linearly independent solution (such as in the case of Bessel's ODE of order $1/2, 3/2, \ldots$).

B The recurrence relation breaks down at some stage and its validity ceases.

C The recurrence relation does not break down but the resulting solution is a multiple of the solution obtained earlier. The trivial example being the case of equal roots.

In sub cases (B) and (C) the second solution has a logarithmic term. In the next slide we look at two examples that illustrate sub-cases (B) and (C).

Two illustrative examples - From the book of Earl. Rainville, Elem. Diff. Eqns pp. 338-341

16. Determine the indicial equation for the ODE

$$xy'' - (4 + x)y' = 2y = 0.$$

The roots differ by an integer and using either root one may proceed but the one obtained from the smaller root is a multiple of the solution obtained by using the larger root.

17. Consider the ODE

$$x^2y'' + x(1 - x)y' - (1 + 3x)y = 0.$$

Check that the indicial equation has roots ± 1.

We describe the procedure for finding the two solutions. Find the recurrence relation without putting in the value of c and call the resulting Frobenius series $y(x, c)$. Denote the LHS of the ODE by Ly. Then

$$L(y(x, c)) = a_0(c - 1)(c + 1)y(x, c).$$

Show that the choice $a_0 = c + 1$ leads to the equation

$$L(y(x, c)) = (c - 1)(c + 1)^2 y(x, c).$$

Differentiate and put $c = -1$ and we get the second solution. What happens if we put $c = -1$ in $y(x, c)$?

Second Solution of Bessel's Eqn. of Order Zero Find the second linearly independent solution of the Bessel's equation of order zero. Well, first substitute the Ansatz

$$y(x, \rho) = \sum_{n=0}^{\infty} a_n x^{n+\rho}$$

into the ODE and determine the successive coefficients without setting $\rho = 0$ so we get a function of x as well as ρ with an a_0 floating around. When substituted into the ODE we get

$$L(y(x, \rho)) = a_0 x^\rho \rho^2. \tag{1}$$

If we take $\rho = 0$ we get the good old $J_0(x)$. But we can differentiate both sides with respect to ρ and then put $\rho = 0$. We would get a second solution. Find this second solution which involves a logarithm. It may be useful to rethink about the Cauchy Euler equation with repeated roots and the cause for the appearance of logarithms,

18. Prove the mean value theorem for integrals: Suppose f, g are continuous on $[a, b]$ and $g > 0$ on (a, b) show that there is a $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Hint: First prove that if f is continuous

$$\int_a^b f(x)dx = f(c)(b-a), \qquad \text{for some } \ c \in (a, b).$$

Now use the integral of g over $[a, x]$ as a variable of integration.

19. Let $u(x) = \sqrt{kx}J_n(kx)$. Show that u satisfies the ODE

$$u'' = -\Big(k^2 - \Big(\frac{n^2 - \frac{1}{4}}{x^2}\Big)\Big)u$$

The last equation suggests that when x is very large $u(x)$ must behave like the sine function and $J_n(kx)$ must behave like $\sin kx/\sqrt{kx}$ and as such must have infinitely many zeros. We shall see that this is indeed so if $k > 1$. The last condition can be removed later.

20. Let $v(x) = \sin(x - a)$. Show that

$$\frac{d}{dx}(vu' - uv') = -uv\Big(k^2 - 1 - \Big(\frac{n^2 - \frac{1}{4}}{x^2}\Big)\Big)$$

21. Let a be so large that $k^2 - 1 - (n^2 - 1/4)/x^2 > 0$ on $[a, a+\pi]$. Integrate the equation obtained in the previous exercise over $[a, a+\pi]$ and use the MVT for integrals. So for some $c \in (a, a+\pi)$ we have

$$-(u(a+\pi) + u(a)) = u(c)\int_a^{a+\pi} v(x)\Big(k^2 - 1 - \Big(\frac{n^2 - \frac{1}{4}}{x^2}\Big)\Big)dx$$

Zeros of Bessel's Functions: Thus we see that $u(a), u(c)$ and $u(a+\pi)$ cannot all have the same sign. Thus u must have a zero in every interval $(a, a+\pi)$ for all $a >> 1$. We have proved,

Theorem 2.7: For $k > 1$, the function $J_n(kx)$ has infinitely many zeros for each $n \geq 0$.

Question: Explain why the condition $k > 1$ can be replaced by $k = 1$ or even $k > 0$? We shall see an application of this theorem to the theory of wave propagation. Another interesting proof via the integral representation is on pp. 76 - 78 of *D. Jackson, Fourier series and orthogonal polynomials, Dover, New York, 2004*. See also *G. N. Watson, Treatise on the theory of Bessel functions, p. 500 ff* for a discussion of the techniques used by L. Euler and Lord Rayleigh to compute the zeros of $J_p(x)$.

22. Prove Laplace's second integral representation:

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2 - 1}\cos\theta)^{-n-1}d\theta$$

Hint: A suitable change of variables in the first integral formula would do the job.

23. Use Laplace's integral representation to prove the following result of C. Neumann (1862)

$$\lim_{n\to\infty} P_n\Big(\cos\left(\frac{x}{n}\right)\Big) = J_0(x)$$

24. Writing the product of the integrals for $J_0(x)$ as a double integral show that (I. N. Sneddon, p. 145)

$$(J_0(x))^2 = \frac{1}{\pi}\int_0^{\pi} J_0(2x\sin\theta)d\theta.$$

Deduce the power series expansion of $(J_0(x))^2$.

Sneddon also gives a formula for $J_n(x)J_m(x)$.
Hint: The integral over the square $[0,\pi]\times[0,\pi]$ equals twice the integral over the triangle $0\le\theta+\phi\le\pi$ and $\theta\ge 0$, $\phi\ge 0$. Now use further symmetries and write the integral over this triangle as one-fourth of the integral along the square with vertices $(\pi,0)$, $(0,\pi)$, $(-\pi,0)$ and $(0,-\pi)$. Now change coordinates. See Watson, pp. 31-32.

III - Sturm-Liouville Problems and PDEs. The circular membrane:

As our first example of a boundary value problem, let us consider a circular membrane clamped along its rim and set into vibration. The mean position being along the $x - y$ plane and the origin at the center of the membrane. At time t let the displacement from the mean position be $z(x, y, t)$. It is well known (see Kreyszig, pp 616-618) that z satisfies the wave equation

$$c^2\Big(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\Big) = \frac{\partial^2 z}{\partial t^2} \tag{1}$$

where c denotes the wave speed. We seek a special solution of the form

$$z = (A\cos pt + B\sin pt)u(x, y). \tag{2}$$

More general solutions can then be determined by superposition. Substituting the Ansatz (2) in the PDE we get

$$c^2(A\cos pt + B\sin pt)\Delta u = -p^2(A\cos pt + B\sin pt)u$$

from which we conclude that u must satisfy the equation

Helmholtz Equation or the Reduced Wave Equation

$$\Delta u + k^2 u = 0,$$

where $k = p/c$. This equation is known as the *Helmholtz's equation* or the *reduced wave equation.*

Exercises:

1. Write the Laplace operator Δ in plane polar coordinates.

2. Write the Laplace operator in $\mathbb{R}^3$ in spherical polar coordinates. Computation gets very UGLY unless you use some cleverness.

The Helmholtz's equation in polar coordinates reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0.$$

We now separate the radial and angular variables by setting

$$u(r, \theta) = v(r)\cos n\theta,$$

where n must be an integer owing to 2π periodicity.

Exercise:

3. The function $v(r)$ satisfies

$$r^2v'' + rv' + (k^2r^2 - n^2)v = 0$$

Check that this is the Bessel's equation after a rescaling of the variable r.

The indicial for the equation is $\rho^2 - n^2$ and only the positive index gives a solution which is finite at the origin. Since $r = 0$ corresponds to the center of the membrane which always remains at finite distance, the only physically tenable solution is $J_n(kr)$.

Thus we see that our special solution is (recalling $p = ck$),

$$z(x, y, t) = J_n(kr)(A\cos ckt + B\sin ckt)\cos n\theta.$$

Since the membrane is clamped along the rim we see that the solution vanishes along $r = 1$ for all values of θ and t. Thus the following boundary condition must be satisfied:

$$J_n(k) = 0.$$

Thus the frequency k must be a zero of the Bessel's function J_n and we have seen that there is an infinite list of them. The frequencies therefore form a discrete set of values. The most general solutions are then obtained from superpositions whose coefficients are determined via initial conditions and Fourier Analysis. We illustrate this by means of an example where the oscillations are radial.

Radial vibrations of the circular membrane Suppose the initial conditions, the value of $z(x, y, 0)$ as well as $z_t(x, y, 0)$, are radial functions (that is depends only on $\sqrt{x^2 + y^2}$ then so would the solutions. Thus the term $\cos n\theta$ would disappear (that is $n = 0$) and we have a sequence of solutions

$$J_0(kr)(A\cos ckt + B\sin ckt),$$

where k runs through the discrete set of zeros of zeros of $J_0(x)$ say $\zeta_1, \zeta_2, \zeta_3, \ldots$. The most general solution is then

$$z(r, t) = \sum_{j=1}^{\infty} J_0(\zeta_j r)(A_j \cos c\zeta_j t + B_j \sin c\zeta_j t)$$

Setting $t = 0$ in $z(x, y, t)$ as well as $z_t(x, y, t)$ we get the following pair of equations for determining the coefficients A_j and B_j:

$$z(r, 0) = \sum_{j=1}^{\infty} A_j J_0(\zeta_j r)$$

$$z_t(r, 0) = \sum_{j=1}^{\infty} C_j J_0(\zeta_j r)$$

where $C_j = jB_j\zeta_j c$.

In order to proceed further we need a result from Analysis called the Bessel expansion theorem. We are obviously not equipped to prove this. The result is available for example in chapter 18 of the authoritative work *G. N. Watson, Treatise on the theory of Bessel functions, Second edition, Cambridge University Press, 1958.* See the historical introduction on pp. 577 - 579.

Theorem 3.1 (Bessel expansion theorem): Suppose $f(r)$ is a smooth function on $[0,1]$ then it admits a Fourier-Bessel expansion

$$f(r) = \sum_{j=1}^{\infty} A_j J_0(\zeta_j r)$$

The coefficients A_j are uniquely determined by the formula (due to *Lommel*)

$$A_j = \frac{2}{(J_1(\zeta_j))^2} \int_0^1 r f(r) J_0(\zeta_j r) dr.$$

So the coefficients A_j and B_j appearing in the solution of the vibration problem can be recovered from Lommel's formula applied to the initial conditions $z(r,0)$ and $z_t(r,0)$.

Orthogonality properties of Bessel's functions:

Exercises:

4. Write the Bessel's ODE in self adjoint form. Check that the operator $x\frac{d}{dx}$ is scale invariant.

5. Put $\phi_u(x) = J_p(xu)$ and check that

$$\Big(x\frac{d}{dx}\Big)\Big(x\frac{d}{dx}\Big)\phi_u(x) + (x^2u^2 - p^2)\phi_u(x) = 0.$$

6. Fix $p \geq 0$ and $\zeta_1, \zeta_2, \zeta_3, \ldots$ be the list of zeros of $J_p(x)$. Show that the family $\{J_p(\zeta_j x) \ : \ n = 1, 2, 3, \ldots\}$ is orthogonal over the interval $[0,1]$ with respect to the weight function x. *Warning: The cases $p = 0$ and $p > 0$ have to be dealt with separately.*

There remains the computation of

$$\int_0^1 x(J_p(\zeta_j x))^2 dx$$

7. Let ζ be a zero of $J_n(x)$. Multiply by $2x\zeta J_n'(\zeta x)$ the ODE satisfied by $J_n(\zeta x)$ and deduce that

$$2\int_0^1 (J_n(\zeta x))^2 x dx = (J_n'(\zeta))^2 = (J_{n+1}(\zeta))^2$$

8. Deduce the formula of Lommel. *We are not proving the Bessel expansion theorem. Only that if the expansion exists and the functions involved are smooth we are deriving the formula for the coefficients in a formal way.*

9. Determine the Bessel expansion for the constant function 1. Hint: Use $(x^p J_p(x))' = x^p J_{p-1}(x)$.

10. Show that $x^n = \displaystyle\sum_{j=0}^{\infty} \frac{2J_n(\zeta_j x)}{\zeta_j J_{n+1}(\zeta_j)}$.

A very interesting proof of the orthogonality property of the Bessel functions suggested by physical considerations is available on pp 324-325 of *Lord Rayleigh, Theory of Sound, Vol - I, Dover, 1945. We shall see yet another proof at the end of this chapter.*

Fourier-Legendre Series *This is a result similar in spirit to the Fourier Bessel expansion. Rather than state the theorem we give an example due to Lord Rayleigh, Theory of sound, Volume - II, p. 273.*

$$e^{itx} = \sum_{n=0}^{\infty}(2n+1)i^n\sqrt{\frac{\pi}{2t}}J_{n+\frac{1}{2}}(t)P_n(x).$$

Prove by induction

$$J_{n+\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi}n!}\Big(\frac{t}{2}\Big)^{\frac{1}{2}+n}\int_{-1}^{1}(1-x^2)^n\cos tx dx$$

Formally deduce the result of Lord Rayleigh. This expansion appears in connection with scattering of plane waves by a spherical obstacle.

Uniqueness of the solution obtained We shall now demonstrate that the solution to the problem of vibrating membrane is unique. Thus the solution we have obtained completely settles the matter. Suppose z_1 and z_2 are two solutions satisfying the same initial conditions and boundary conditions (namely vanishing along the boundary of the membrane $r = 1$). Then the difference $Z = z_1 - z_2$ also satisfies the PDE with zero initial and boundary conditions:

$$c^2\Delta Z - \frac{\partial^2 Z}{\partial t^2} = 0,$$
$$Z(x,y,0) = 0,\ Z_t(x,y,0) = 0,$$
$$Z(\cos\theta, \sin\theta, t) = 0.$$

The Energy Method The idea of proof is well-known under the name of *energy method.* The method is frequently employed in the analysis of PDEs. Multiply the differential equation by Z_t and integrate over the disc $D : x^2 + y^2 \leq 1$. Integration by parts gives (how?)

$$\frac{d}{dt}\iint_D Z_t^2 dxdy + c^2\iint_D (Z_{xt}Z_x + Z_{yt}Z_y)dxdy = 0.$$

¿From this we infer that the energy

$$E(t) = \iint_D (c^2Z_x^2 + c^2Z_y^2 + Z_t^2)dxdy = 0.$$

is constant in time. Since $E(0) = 0$ we see that Z is a constant and so is zero. The proof is complete.

11. Imitate the energy method to show that the twice continuously differentiable solution to the initial-boundary value problem

$$u_t = \Delta u,\ u(x,y,0) = f(x,y),\ u(\cos\theta,\sin\theta,t) = 0$$

for the heat equation is unique. Hint: Here the energy function is monotone decreasing in time.

12. Show that the twice continuously differentiable solution to the boundary value problem

$$\Delta u = 0, \quad \text{on } D$$
$$u(x,y,z) = f(x,y,z), \quad \text{on } \partial D$$

is unique where D is a region in $\mathbb{R}^3$ with a smooth boundary ∂D and f is a function prescribed along ∂D.

For more detailed discussion on these types of wave phenomena see

1. Courant and Hilbert, Methods of Mathematical Physics, Volume I. We have already cited this. For the discussion of vibration of a circular plate (which is more involved than the membrane) see pp. 307-308. These involve Bessel's functions of imaginary orders ip $(p > 0)$, known as the modified Bessel's functions.

2. Lord Rayleigh, Theory of Sound, Volume - I. This is a comprehensive account of the theory of vibrations. See particularly the long and detailed discussion on vibrating plate. This is still the best source on the *Physics of Vibrations.*

3. For other applications such as the *skin effect.* see F. Bowman, Introduction to Bessel's functions, Dover.

4. The Bessel functions also appear in optics. The radii of the successive interference fringes due to diffraction from a circular aperture are given in terms of the zeros of Bessel's function.

5. Besides the authoritative treatise of G. N. Watson mentioned earlier, the book of Byerly is particularly recommended especially the historical sketch at the end. The book is available on line.

The Vibrating String and regular Sturm Liouville Problems: Suppose we have a string of length l and density $\rho(x)$ stretched along the line segment $[0, l]$ and clamped at its two ends. The string is set into vibrations and the governing differential equation my be reduced to

$$y'' + \lambda\rho(x)y = 0 \tag{1}$$

where λ is a parameter. Since the string being clamped at its ends we have the boundary conditions (BC)

$$y(0) = y(l) = 0. \tag{2}$$

A λ for which a non-trivial solution exists is called the eigen-value of the B. V. Problem and the corresponding *non-trivial* solution the eigen-functions. The BC (2) is referred to as the Dirichlet boundary conditions. Other physical conditions lead to BC of the form

$$y'(0) = y'(l) = 0. \tag{3}$$

known as the Neumann BC. Later we shall see another BC that has some different features. We shall *assume that $\rho(x)$ is continuous and positive.*

As a simple example let us consider a uniform string of unit density and unit length. The problem to be solved is

$$y'' + \lambda y = 0, \quad y(0) = 0 = y(1).$$

Solving the ODE ($\lambda = 0$ doesnt give non-trivial solutions),

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

The condition $y(0) = 0$ immediately gives $A = 0$ and the condition $y(1) = 0$ gives

$$B \sin \sqrt{\lambda} = 0.$$

Since the solution in question is *non-trivial,* $B \neq 0$ which means

$$\sqrt{\lambda} = \pm\pi, \pm 2\pi, \ldots$$

Thus the solution to the boundary value problem exists only for a discrete set of values of the parameter and the eigen-values form a discrete set.

Theorem 3.2 (Orthogonality of Eigen Functions): For the boundary value problem on $[0, l]$

$$y'' + \lambda\rho(x)y = 0$$

with either Dirichlet or Neumann boundary conditions the eigen-functions corresponding to distinct eigen-values are orthogonal on $[0, 1]$ with respect to the weight function $\rho(x)$ namely,

$$\int_0^l u(x)v(x)\rho(x)dx = 0.$$

Proof: Suppose λ and μ are distinct eigen-values with eigen-functions u and v. Proof is similar to the orthogonality of Legendre Polynomials. We have the pair of equations:

$$\begin{aligned} u'' + \lambda\rho(x)u = 0, &\quad u(0) = 0 = u(l) \\ v'' + \mu\rho(x)v = 0, &\quad v(0) = 0 = v(1). \end{aligned}$$

Multiply the first by v, second by u, integrate over $[0, l]$ by parts and subtract. Details left to the student.

Exercise:

13. Discuss the orthogonality of eigen-functions for the Neumann BC.

Theorem 3.3 (Simplicity of the eigen-values): To each eigen-value of the BVP (1)-(2) with Dirichlet BC, there is only one eigen-function upto scalar multiples.

Proof Assume that u and v are two *linearly independent* eigen-functions with the same eigen value then the linear span of u and v is the set of all solutions of the homogeneous ODE

$$y'' + \lambda\rho(x)y = 0$$

and since both u and v vanishes at 0, it follows that every solution of the ODE vanishes at 0. This is plainly false since the solution of the initial value problem for this ODE with initial conditions

$$y(0) = 1, y'(0) = 0$$

which exists by *Picard's theorem,* does not vanish at 0. Contradiction.

Completeness of eigen-functions We now state a theorem on the completeness of the set of eigen-functions of the Sturm-Liouville problem

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(l).$$

We assume that the function $\rho(x)$ is positive and continuous on $[0, l]$.
The theorem is often stated under stronger hypothesis [7]. One can also discuss mean convergence but we shall not do so here.

Theorem 3.4: For the Sturm Liouville problem in question, there is an infinite sequence of eigen-values

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

tending to infinity. The set of eigen-functions $\{\phi_n \ : \ n = 1, 2, \dots\}$ is complete in the following sense. Each Lipschitz function $f(x)$ on $[0, l]$ can be expanded as a series

$$f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)$$

which converges for each $x \in (0, l)$. The coefficients are given by

$$c_j = \Big(\int_0^l f(x)\phi_j(x)\rho(x)dx \Big) \Big(\int_0^l \phi_j^2(x)\rho(x)dx \Big)^{-1}$$

Exercises:

14. Show that the eigen-values of
$$y'' + \lambda\rho y = 0$$
with boundary conditions $y(0) = 0 = y(1)$ are positive real numbers.

15. Determine the eigen-values and eigen-functions of the Sturm-Liouville problem
$$y'' - 2y' + (1 + \lambda)y = 0$$
with boundary condition $y(0) = 0, y(1) = 0$.

16. Determine the eigen-values and eigen functions of the Sturm-Liouville problem
$$x^2y'' + xy' + \lambda y = 0$$
on $[e, 1/e]$ with the periodic boundary conditions:
$$y(1/e) = y(e), \quad y'(1/e) = y'(e).$$

[7] R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume - I, p. 293. However E. C. Titchmarsh in his Eigen-function expansions associated with second order differential equations, Volume - I, Oxford, Clarendon Press, 1969, proves it with substantially weaker hypothesis (see page 12).

17. Show that the eigen-values of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \ y(1) + y'(1) = 0$$

are given by $\lambda = k^2$ where k satisfies $\tan k + k = 0$. Graphically show that there are infinitely many roots.

To see how this type of BC appears in physical problem, see the example on p. 117, §67 of Byerly's text.

18. A rigid body is rotated with uniform and fixed angular speed about an axis that is not specified. How would one choose the axis of rotation so as to maximize the rotational Kinetic Energy? Formulate the problem mathematically.

19. *Variational principles underlying eigen-values and eigen vectors/functions.* Suppose A is a $n \times n$ real symmetric matrix, show that the maximum and the minimum of the quadratic function

$$\langle Ax, x \rangle, \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

are both attained at eigen-vectors and the maximum and minimum values are the largest and smallest eigen-values of A.

Proof of the Min/Max properties of Eigen-values Let $Q(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$. This is a quadratic polynomial in n-variables and as such attains its minimum value at some point say $\mathbf{v}_1$ on the unit sphere in $\mathbb{R}^n$. We now *perturb* the vector $\mathbf{v}_1$ to say

$$\mathbf{w} = (\mathbf{v}_1 + \epsilon\mathbf{h})/\|\mathbf{v}_1 + \epsilon\mathbf{h}\|$$

and compare the values of the quadratic at the two points namely

$$Q(\mathbf{v}_1) \leq Q(\mathbf{w}), \quad \text{for all } \epsilon \quad \text{small enough.}$$

Thus, we get after cross multiplying by $\|\mathbf{v}_1 + \epsilon\mathbf{h}\|^2$, the following inequality valid for all small ϵ positive or negative !

$$Q(\mathbf{v}_1)\|\mathbf{v}_1 + \epsilon\mathbf{h}\|^2 \leq (\mathbf{v}_1 + \epsilon\mathbf{h})^T A(\mathbf{v}_1 + \epsilon\mathbf{h}).$$

Expanding and canceling off $Q(\mathbf{v}_1)$ we get

$$2\epsilon Q(\mathbf{v}_1)\mathbf{h}^T\mathbf{v}_1 \leq 2\epsilon\mathbf{h}^T A\mathbf{v}_1 + \epsilon^2(\mathbf{h}^T A\mathbf{h} - Q(\mathbf{v}_1)\|\mathbf{h}\|^2).$$

We now divide by ϵ and let $\epsilon \to 0$. Since ϵ may have either sign we get the pair of inequalities

$$\mathbf{h}^T(Q(\mathbf{v}_1)\mathbf{v}_1 - A\mathbf{v}_1) \leq 0, \quad \text{and } \mathbf{h}^T(Q(\mathbf{v}_1)\mathbf{v}_1 - A\mathbf{v}_1) \geq 0.$$

Thus we conclude $\mathbf{h}^T(A\mathbf{v}_1 - Q(\mathbf{v}_1)\mathbf{v}_1)$. Since $\mathbf{h}$ is arbitrary we see that

$$A\mathbf{v}_1 = Q(\mathbf{v}_1)\mathbf{v}_1$$

In other words the minimum is attained at an eigen-vector and the minimum value is the corresponding eigen-value.

Exercise: Examine carefully the computations and explain where have we used the fact that A is a symmetric matrix?

To proceed further, let S be the intersection of the unit sphere in $\mathbb{R}^n$ with the hyperplane

$$\mathbf{v} \cdot \mathbf{v}_1 = 0.$$

This is also a closed bounded set and the minimum of $Q(\mathbf{v})$ on S is attained at say $\mathbf{v}_2$ and the corresponding $Q(\mathbf{v}_2)$ is not less than $Q(\mathbf{v}_1)$ (why?).

Now we *perturb* $\mathbf{v}_2$ to

$$\mathbf{w} = (\mathbf{v}_2 + \epsilon\mathbf{h})/\|\mathbf{v}_2 + \epsilon\mathbf{h}\|$$

where h is chosen such that $\mathbf{h} \cdot \mathbf{v}_1 = 0$. Again $Q(\mathbf{v}_2) \leq Q(\mathbf{w})$. Multiplying the inequality through by $\|\mathbf{v}_1 + \epsilon\mathbf{h}\|^2$, expanding out and canceling $Q(\mathbf{v}_2)$ we get as before

$$2\epsilon\mathbf{h}^T(Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2) \leq \epsilon^2(\mathbf{h}^T A\mathbf{h} - Q(\mathbf{v}_2)\|\mathbf{h}\|^2).$$

Dividing by ϵ and letting $\epsilon \to 0$ we get the pair of inequalities from which we again deduce

$$\mathbf{h}^T(Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2) = 0.$$

This is now valid for all $\mathbf{h}$ such that $\mathbf{h} \cdot \mathbf{v}_1 = 0$. But this is also holds for $\mathbf{h}$ parallel to $\mathbf{v}_1$

Exercise: Verify the last statement.

Hence $Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2 = 0$.

Thus $Q(\mathbf{v})$ attains its minimum over S at an eigen-vector $\mathbf{v}_2$. The minimum value is the corresponding eigen-value. By construction $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$. Now minimize $Q(\mathbf{v})$ over the intersection of the unit sphere in $\mathbb{R}^n$ with the pair hyperplanes

$$\mathbf{v} \cdot \mathbf{v}_1 = 0, \quad \mathbf{v} \cdot \mathbf{v}_2 = 0$$

and the rest of the proof simply writes itself out. The process terminates after we have n orthogonal eigen-vectors of our matrix A. We have proved

Theorem 3.5 (Spectral Theorem): A real symmetric matrix has an orthonormal basis of eigen-vectors.

The analogue of the above result for self-adjoint Diff. Eqns with Dirichlet BC is a serious matter that has lead to a huge corpus of mathematical research - for these mathematical development against a historical back-drop see the introductory parts of *R. Courant: The Dirichlet's principle, conformal mappings and minimal surfaces, Dover Reprint, 2005. See the free preview of the first three pages of introduction on the Internet!*

Let us consider the problem of minimizing the "energy"

$$\int_0^1 (y'(t))^2 dt \tag{1}$$

subject to the condition

$$\int_0^1 y(t)^2 \rho(t) dt = 1. \tag{2}$$

where $y(t)$ ranges over continuous piecewise smooth functions with $y(0) = y(1) = 0$ and $\rho(x)$ is a positive continuous function on $[0, 1]$.

The Dirichlet Principle: The minimization problem (1)-(2) has a twice continuously differentiable solution which corresponds to the smallest eigen-value of the Sturm-Liouville problem

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(1). \tag{3}$$

The principal difficulty is in showing

(i) That the minimizer exists and

(ii) The minimizer is twice continuously differentiable.

These are rather deep waters. Riemann used these ideas motivated by potential theoretic considerations to prove his celebrated theorem in complex analysis known today as *The Riemann Mapping Theorem.*

20. Verify this formally. That is to say, if $y_0(x)$ is a twice continuously differentiable minimizer then $y_0(x)$ is an eigen-function of the Sturm-Liouville problem.

21. Try to give an intuitive (non-rigorous) geometric argument that the eigen-function corresponding to the smallest eigen value has no zeros in $(0,1)$. In physics books this is referred to as the *Fundamental Mode.*

22. Assuming that y_0 is the fundamental mode, we seek to minimize (1) subject to the condition (2) as well as

$$\int_0^1 y(t)y_0(t)\rho(t)dt = 0. \tag{2$'$}$$

then we get an eigen function with eigen value larger than the first one. Why is this so? Why is it the case that this eigen-function has at least one zero in $(0,1)$? Is it geometrically clear that there is precisely one zero in $(0,1)$?

23. How does one continue to construct the successive eigen functions?

Sturm's comparison theorem

24. Suppose ρ and σ are both continuous positive functions on $[0,1]$ and $\rho(x) > \sigma(x)$ for every $x \in [0,1]$. Let $y(x)$ and $z(x)$ be solutions of the pair of ODEs

$$y'' + \rho(x)y = 0, \quad z'' + \sigma(x)z = 0.$$

Prove that between two successive zeros of $z(x)$ there is at least one zero of $y(x)$.
Hint: Suppose a and b are successive zeros of $z(x)$ in $[0,1]$ and we may assume $z(x) > 0$ on (a,b). Suppose that $y(x)$ does not vanish on (a,b) and is positive there. Imitate the proof of orthogonality of eigen-functions but on $[a,b]$ and show that a sum of positive things adds up to zero.

See the commentary in Lord Rayleigh: Theory of Sound - I, pp. 217-222.

Returning to the Sturm-Liouville problem $y'' + \lambda\rho(x)y = 0$ with Dirichlet BC at 0 and 1, let us consider the solution $y(x,\lambda)$ of the ODE with initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

For small values of λ the solution $y(x,\lambda)$ has no zeros on $(0,1]$ and for a suitably large value of λ there is a unique zero in this interval. Use the Sturm's comparison theorem.

We must now select the value of λ such that the boundary condition at $x = 1$ is satisfied. In other words we must select λ such that

$$y(x, \lambda) = 0.$$

We need to show that the solution to this equation for x as a function of λ exists and is continuous. The proof of these assertions rely on two theorems of analysis namely the *existence uniqueness of IVP and the implicit function theorem.* Thus we see that there is a value λ_1 such that

$$y(1, \lambda_1) = 0$$

and we have the first eigen value and eigen function. Increasing the value of λ and invoking the same principles again we get a $\lambda_2 > \lambda_1$ such that $y(x, \lambda_2)$ is the second eigen function with eigen value λ_2 and this has exactly one zero in $(0, 1)$.

Synopsis of chapter III We started off by solving the wave equation on a disc of unit radius governing the vibrations of a circular membrane. The membrane is clamped along the rim which means the solution is zero on boundary. Separating off the time variable results in the Helmholtz' PDE. The circular symmetry makes it natural to employ polar coordinates and after separating radial and angular variables we see that the radial part satisfies the Bessel's equation of integer order. The fact that the membrane remains at finite distance singles out $J_n(kr)$ as the relevant solution. The other boundary condition forces the parameter k to run through a discrete set of values, the zeros of $J_n(x)$. We prove the infinitude of these zeros and we get a sequence of basic solutions from which one can obtain general solutions by superposition. Thus we have solved a singular Sturm-Liouville problem. For simplicity we assumed that the vibrations are radial which fixes $n = 0$.

We next take up the case of a vibrating string clamped at its ends. The wave equation in question is

$$\rho(x)u_{tt} - u_{xx} = 0$$

Substituting the Ansatz $u(x, t) = y(x)\phi(t)$ in the equation results in

$$\frac{\phi''(t)}{\phi(t)} = \frac{y''(x)}{\rho(x)y(x)}.$$

Since the left hand side is a function of t alone and the right hand side is a function of x alone both sides are constant say $-p^2$. Thus we get $\phi(t) = A\cos pt + B\sin pt$ whereas for $y(x)$ we get

$$y'' + p^2\rho(x)y = 0.$$

The physical condition that the string is clamped at the ends leads to the Dirichlet Boundary conditions $y(0) = 0 = y(l)$. This BVP is called a regular Sturm-Liouville problem. As in the case of a circular membrane, the set of values of the parameter p for which a non-trivial solution exists (known as eigen-functions) is discrete and these are called eigen-values.

We prove the simplicity of eigen-values and the orthogonality of eigen-functions corresponding to distinct eigen-values. The Sturm-Liouville problem is then given a variational formulation. The underlying principle is quite general and has wide applicability. The existence of the minimizer and its regularity is known as the Dirichlet principle. We also discuss an alternative approach to the existence of eigen-values and eigen-functions for the one-dimensional problem by using the Sturm's comparison theorem. The completeness of eigen-functions is a deep result in analysis and we can only state it here (both for the one-dimensional case of a vibrating string and the two dimensional case of a vibrating circular membrane).

I mention in passing that the case of an elliptical membrane has been considered by *Émile Léonard Mathieu* in 1868 and the resulting ODE known as the Mathieu equation:

$$y'' + (a + b\cos 2x)y = 0.$$

where b is given and a is an eigen-parameter. The equation has led to a long and rich chapter in the theory of analytic ODEs, generalized and studied by *G. W. Hill* in 1886 in his researches on Lunar motion. Unfortunately we are not in a position to say anything about these exciting theory in this elementary course!

Hope this thumb-nail sketch helps!!

IV - Fourier series and partial differential equations

The boundary value problem on $[-\pi, \pi]$ with periodic boundary conditions:

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi),\ y'(-\pi) = y'(\pi).$$

leads to the theory of *Fourier series.* The study began long before *Joseph Fourier* in connection with a physical problem by *Daniel Bernoulli in 1747* of a vibrating string. The method became popular after Fourier's work on heat conduction (circa 1807). Fourier Analysis has been the subject of intense research and occupies today a distinguished position in Mathematics.

Study of the paper by Roger Cooke, Uniqueness of trigonometric series and descriptive set theory 1870-1985, Archiv for the history of exact sciences **45** 281-334 (1993) would be an excellent project for any serious mathematically inclined student.

For the boundary value problem displayed in the previous paragraph, it is evident that the eigen-values are $0, 1, 4, \ldots$. Each eigen value other than 0 has two linearly independent eigen-functions namely

$$\sin nx, \quad \cos nx.$$

There is only one eigen-function with eigen-value 0 namely the constant function 1.

The Basic Issue: To develop a "fairly general function" $f(x)$ as a series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

We call the partial sum $a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ as a *trigonometric polynomial. If in addition* $|a_N| + |b_N| \neq 0$ *we say it is of degree* N.

Formula for Fourier coefficients Proceeding formally let us multiply equation (1) by $\cos mx$ and integrate term by term over $[-\pi, \pi]$ and we find

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx dx.$$

Likewise we find

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx dx.$$

For the case of a_0 we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx.$$

The Fourier series is by definition the series in (1) with the coefficients given by (2), (3) and (4). Question is now the validity of the equation (1).

Dirichlet's theorem The completeness theorem of the previous chapter has an analogue known as Dirichlet's theorem. Before stating the result we shall insert here a definition.

Definition A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be piecewise continuously differentiable if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that f is continuous on each $[t_j, t_{j+1}]$, differentiable on (t_j, t_{j+1}) and the derivatives have finite limits at the ends t_j and t_{j+1}, $j = 0, 1, 2, \ldots, n-1$.

The Heaviside unit step function $u(t)$ is piecewise continuously differentiable and so is

$$u(t)\cos t + u(1-t)\sin t.$$

Any Lipschitz function is piecewise continuously differentiable.

Theorem 4.1 (Dirichlet (1829)): Suppose $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is piecewise continuously differentiable then we have the development

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),\ x \neq \pm\pi$$

at points of continuity, where a_0, a_n and b_n are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

At a point of discontinuity x_0 or $x_0 = \pm\pi$, the series converges to

$$\frac{1}{2}(f(x_0+) + f(x_0-)).$$

The partial sums of the displayed series are trigonometric polynomials.

The series displayed in the last slide is called the *Fourier Series* of the function $f(x)$ and the sequences a_n, b_n the *Fourier coefficients* of $f(x)$. The proof of this theorem cannot be given here. Proof can be found on pp 71-73 of *Richard Courant and David Hilbert, Methods of Mathematical physics, Volume - I, Wiley (Indian Reprint), 2008.* Or p 56 ff. of the old but nevertheless excellent book *Byerly,* See particularly the brief commentary on §38 on page 62 and the detailed summary on pp. 267-274 and further references therein.

The issue with continuous functions Question: If $f(x)$ is *merely* assumed to be continuous does the above result hold ? This was believed to be so by several mathematicians including Dirichlet until *Paul Du Bois Reymond* after several abortive attempts at proving it, produced a counter example in 1875 ! Using ideas from set topology one can show that a *majority of continuous functions* display such *errant behaviour.*The simplified proof given by *Stephan Banach* is available in most texts.

The assumption of piecewise smoothness may be slightly weakened though. In applications one takes a periodic extension of $f(x)$ with period 2π:

$$f(x + 2\pi) = f(x), \quad x \in \mathbb{R}.$$

In case $f(\pi) \neq f(-\pi)$ then *we shall always redefine the value at these points as the arithmetic mean.* This convention w ill be followed throughout.

1. Compute the Fourier coefficients of the saw-tooth function $f(x) = x$, $-\pi < x < \pi$. When extended as a 2π periodic function you see a jump discontinuity of 2π at each of the points $\pm\pi, \pm 3\pi, \ldots$. Draw a graph of the function.

2. Determine the Fourier coefficients of the triangular wave train

$$f(x) = \pi - |x|, \quad -\pi \leq x \leq \pi$$

and sketch the graph of its 2π periodic extension. Use Dirichlet's theorem to deduce the value of

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

Deduce the value of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

3. Determine the Fourier coefficients of the square wave train which is given by $f(x) = \pi$ on the range $(0, \pi)$, $f(0) = 0$, extended as an odd function on $(-\pi, \pi)$ and then as a 2π periodic function on the whole real line.

Partial fraction expansion of cosecent and cotangent

4. Determine the Fourier coefficients of $f(x) = \cos ax$ where $a \notin \mathbb{Z}$ and deduce that

$$\pi \mathrm{cosec}(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a(-1)^n}{a^2 - n^2}$$

$$\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$$

5. Let $\zeta(2k) = \sum_{m=1}^{\infty} m^{-2k}$. Find the generating function $f(z)$ for the sequence $\{\zeta(2k)\}$ and write the result in terms of exponentials.

6. By comparing $2f(z) - 1 + i\pi z$ with the generating function for Bernoulli numbers derive the formula of Euler for

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \ldots$$

Remark: See pp. 277-281 of *Alain Robert, Advanced calculus for users, North Holland, 1989.*

The Riemann Lebesgue Lemma This important result states that the Fourier coefficients of a continuous function tends to zero as $n \to \infty$. The continuity assumption can easily be weakened but we shall not strive for the most general results.

7. Suppose $f(x) = x^k$ on the interval $[-\pi, \pi]$. Show that the Fourier coefficients a_n and b_n tend to zero as $n \to \infty$.

8. Deduce that for a polynomial $P(x)$, the Fourier coefficients tend to zero as $n \to \infty$.

9. Suppose $f(x)$ is continuous on $[-\pi, \pi]$ extend it continuously as a constant on $(-\infty, -\pi]$ and $[\pi, \infty)$. In the integral defining a_n perform the change of variables $x = y + \frac{\pi}{n}$ and prove that $a_n \longrightarrow 0$ as $n \to \infty$. The proof that $b_n \longrightarrow 0$ as $n \to \infty$ is similar.

Remark: This elegant argument goes back to Riemann himself. Modern proofs take a different approach via approximations as we shall see presently.

Question: Suppose we have sequences $\{a_n\}$ and $\{b_n\}$ such that they both tend to zero as $n \to \infty$ is it true that there is an integrable function whose Fourier coefficients are precisely these?

This question was answered relatively late in the development of Fourier series. An example appears in a letter dated 1910 by N. Luzin to Florenskii.

Example: The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log(n+1)}$$

is NOT the Fourier series of any integrable function on $[-\pi, \pi]$. See pp. 309-310 of Roger Cooke's article.

The Weierstrass's Approximation theorem We now state one of the most fundamental theorems in analysis whose proof may be found in Courant and Hilbert's book cited earlier (pp. 65-68). Since its publication in 1885, Several different proofs have been supplied for this result.

Theorem 4.2 (Weierstrass's Approximation theorem): Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then given any $\epsilon > 0$ there exists a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon, \quad x \in [a, b].$$

A charming commentary on the life of Weierstrass and the impact of this theorem on modern analysis is the article *A. Pinkus, Weierstrass and approximation theory, Journal of Approximation Theory* **107** *1-66 (2000)*

10. *Prove the Riemann-Lebesgue Lemma using the Weierstrass's approximation theorem.*

11. *Show that if $f(x)$ is continuously differentiable and the second derivative has only finitely many jump discontinuities then the Fourier coefficients decay to zero faster than $1/n^2$. One can then show (using analysis) that once term by term differentiation is valid.*

12. *Imitate the proof of Riemann Lebesgue Lemma to show that*

$$|J_0(x)| \leq c/\sqrt{x},$$

for some constant C. What about $J_n(x)$? Deduce that the Bessel's functions of integer orders decay like $1/\sqrt{x}$ for large x.

Mean Approximation *Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function then its L^2 norm is defined to be*

$$\|f\|_2 = \sqrt{\frac{1}{b-a} \int_a^b |f(t)|^2 dt}$$

Engineers are accustomed to calling it the Root Mean Square. We can define this for Riemann integrable functions as well but the word "norm" would be somewhat inappropriate in this context. The reason is that if we look at the function $g(x)$ which is zero at all but finitely many points then $\|g\|_2 = 0$ but g is not the zero function.

There is a way to get around this annoyance but the correct setting for this is the theory of *Lebesgue integrals.* Indeed *Lebesgue theory is the right backdrop for discussing Fourier Analysis* and whatever we discuss without its aid would necessarily be incomplete and at places would result in abuse of language and notation. With this disclaimer we now continue.

Definition: We say that a sequence f_n of square integrable functions *converges to f in mean* if

$$\|f_n - f\|_2 \longrightarrow 0, \qquad \text{as } n \to \infty.$$

Exercise: Show that if f, g and h are Riemann integrable then

$$\|f - g\|_2 \leq \|f - h\|_2 + \|h - g\|_2.$$

Theorem 4.3 (Least Square Approximation): Suppose $f(\theta)$ is a Riemann integrable function on $[-\pi, \pi]$ and $S_n(f)$ is the $n-$th partial sum of its Fourier series. For any arbitrary trigonometric polynomial $P(\theta)$ of the degree at most n,

$$\|f(\theta) - s_n(f)\|_2 \leq \|f(\theta) - P(\theta)\|_2$$

with equality if and only if $P(\theta) = s_n(\theta)$.

Exercises:

13. Prove the following theorem in Linear Algebra. Suppose V is a vector space endowed with an inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\mathbf{a}, \mathbf{b}$ are a pair of orthogonal vectors then we have the theorem of Pythagoras:

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2$$

14. Now show that $f - s_n$ and $s_n - P$ are orthogonal with respect to the inner-product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

Hint: First compute $\langle f - s_n(f), \cos mx \rangle$ for $m \leq n$. Use Pythagoras' theorem to prove the Least square approximation theorem.

Theorem 4.4 (Bessel's Inequality): Suppose $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a Riemann integrable function then

$$a_0^2 + \frac{1}{2}\sum_{j=1}^{n}(|a_j|^2 + |b_j|^2) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt, \quad n \in \mathbb{N}.$$

15. Prove the above theorem. Hint: We have proved $(f - s_n(f)) \perp (s_n(f) - P)$. Take $P = 0$.

Letting $n \to \infty$ we conclude

$$a_0^2 + \frac{1}{2}\sum_{j=1}^{\infty}(|a_j|^2 + |b_j|^2) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt.$$

Now we shall *invoke the Weierstrass' approximation theorem* to make the inequality into an equality knows as *The Parseval Formula.* The transition is highly non-trivial using analysis.

Some preliminaries for the main result

16. Suppose that $f(\theta)$ is an even continuous function on $[-\pi, \pi]$ show that given any $\epsilon > 0$, there is a trigonometric polynomial $P(\theta)$ such that
$$|f(\theta) - P(\theta)| < \epsilon, \quad -\pi \leq \theta \leq \pi.$$

17. Suppose $g(\theta)$ is an odd continuous function on $[-\pi, \pi]$ vanishing at $\pm\pi$ show that there is a trigonometric polynomial $Q(\theta)$ such that
$$|g(\theta) - Q(\theta)| < \epsilon, \quad -\pi \leq \theta \leq \pi.$$
Hint: Apply the previous to $g(\theta)\sin\theta$ and then we see that
$$|g(\theta)(1 - \cos^2\theta) - \sin\theta Q_1(\theta)| < \epsilon.$$
Now repeat the above with $g(\theta)\cos^2\theta$ and we see that
$$|g(\theta) - \cos^4\theta g(\theta) - Q_2(\theta)| < \epsilon.$$
Iterate sufficiently often.

Finally we have the ingredient we need:

18. Suppose $f(\theta)$ is a continuous function on $[-\pi, \pi]$ such that $f(\pi) = f(-\pi)$. Given any $\epsilon > 0$, there is a trigonometric polynomial $R(\theta)$ such that
$$|f(\theta) - R(\theta)| < \epsilon, \quad -\pi \leq \theta \leq \pi.$$

We are now ready to surge ahead and prove one of the fundamental results on Fourier Analysis.

Convergence in Mean and Parseval Formula

Theorem 4.5 (The Parseval Formula): Suppose $f, g : [-\pi, \pi] \longrightarrow \mathbb{C}$ are Riemann integrable functions and $s_n(f)$ is the $n-$th partial sum of its Fourier series for f then

1. $$\|f(\theta) - s_n(\theta)\|_2 \longrightarrow 0, \quad n \to \infty.$$

2. $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}d\theta = a_0'\overline{a_0''} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n'\overline{a_n''} + b_n'\overline{b_n''})$$
where a_n', b_n' and a_n'', b_n'' are the Fourier coeff. of f and g.

3. Taking $f = g$ in particular (and so $a_n' = a_n'' = a_n$ etc.,) we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

We shall prove this for continuous functions $f(\theta)$ such that $f(\pi) = f(-\pi)$. The transition to the general case is a fairly routine matter that shall be indicated as a list of three (optional) exercises:

1. The main point in this transition is that an arbitrary Riemann integrable function can always be approximated by continuous functions - in fact piecewise linear functions.

2. The continuous function can be altered at the two ends to ensure $f(\pi) = f(-\pi)$ such that the integrals (areas) are change by arbitrarily small amounts.

Proof of Parseval's formula: Let $\epsilon > 0$ be arbitrary. Choose a trigonometric polynomial $R(\theta)$ such that

$$\|f(\theta) - R(\theta)\|_2 < \epsilon.$$

Let n_0 be the degree of this trigonometric polynomial. Then by the least square approximation principle,

$$\|f(\theta) - s_n(f)\|_2 \le \|f(\theta) - R(\theta)\|_2 < \epsilon, \quad n \ge n_0.$$

This proves the first part of the theorem. To prove the third part note that $f(\theta) - s_n(\theta)$ is orthogonal to $s_n(\theta)$ and so by theorem of Pythagoras,

$$\|f\|_2^2 = \|f - s_n(f)\|_2^2 + \|s_n(f)\|_2^2$$

and letting $n \to \infty$ we conclude

$$\lim_{n\to\infty} \|s_n\|_2^2 \longrightarrow \|f\|_2^2$$

But $\|s_n\|_2^2 = |a_0|^2 + \frac{1}{2}\sum_{k=1}^{n}(|a_k|^2 + |b_k|^2)$ and the result is proved. The second part will be an exercise.

19. Prove the second part of Parseval's theorem applying the simple formula

$$4A\overline{B} = |A + B|^2 - |A - B|^2$$

to $A = f + g$ and $B = f - g$

20. Determine the Fourier series for the function $f(x) = x^2$ on $[-\pi, \pi]$. Use Parseval formula to show that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

21. Using $f(x) = \pi^2 x - x^3$, determine the sum of the series

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$$

22. Verify the results of the last two exercises with the Formula obtained by Euler in terms of the Bernoulli numbers.

Extending Parseval's theorem to Riemann integrable functions Here are three OPTIONAL exercises which shows how to get Parseval's theorem for Riemann integrable functions.

23. Show that if f is a Riemann integrable function on $[a, b]$ then given any $\epsilon > 0$ there is a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.

    Well, select a partition $P\ :\ a = t_0 < t_1 < \cdots < t_n = b$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

    Now let g be the continuous function whose graph on $[t_j, t_{j+1}]$ is obtained by joining the points $(t_j, f(t_j))$ and $(t_{j+1}, f(t_{j+1}))$.

24. Suppose $g : [-\pi, \pi] \longrightarrow \mathbb{R}$ is continuous, then given any $\epsilon > 0$, we can select a continuous function h on $[-\pi, \pi]$ such that $h(\pi) = h(-\pi)$ and $\|g - h\|_2 < \epsilon$

25. Use triangle inequality

$$\|f - s_n(f)\|_2 \le \|f-g\|_2 + \|g-h\|_2 + \|h - s_n(h)\|_2 + \|s_n(h) - s_n(g)\|_2 + \|s_n(g) - s_n(f)\|_2$$

and the Bessel's inequality to complete the proof of the mean convergence of Fourier series.

Now use the fact that $f - s_n(f)$ is orthogonal to $s_n(f)$ and the Pythagoras's theorem to deduce the Parseval's formula.

Descarte's isoperimetric problem This is one more important classic variational principle which goes back at least to René Descartes:

Theorem 4.6: Of all piecewise smooth closed curves with a *given perimeter* the circle encloses *maximum area.*

One can turn the problem around by fixing the area and minimizing the perimeter.

Theorem 4.7: Of all piecewise smooth closed curves enclosing a *given area* the circle has the *least perimeter.*

The theorem generalizes to higher dimensions in an obvious way.

Spacial isoperimetric theorem *"With a little knowledge of the physics of surface tension, we could learn the isoperimetric theorem from a soap bubble.*

Yet even if we are ignorant of serious physics, we can be led to the isoperimetric theorem by quite primitive considerations. We can learn it from a cat. I think you have seen what a cat does when he prepares himself for sleeping through a cold night: he pulls in his legs, curls up, and, in short, makes his body as spherical as possible. He does so obviously, to keep warm, to minimize the heat escaping through the surface of his body. The cat who has no intension of decreasing his volume, tries to decrease his surface. He solves the problem of a body with a given volume and minimum surface in making himself as spherical as possible. He seems to have some knowledge of the isoperimetric theorem." Quotation from p. 170 of
G. Polya, Mathematics and plausible reasoning, Princeton University Press, Princeton, 1954.

Hurwitz' Proof of isoperimetric theorem (1902): Let the piecewise smooth closed curve be parametrized by arc-length $(x(s), y(s))$, $0 \le s \le L$ and the curve is traced counter-clockwise. The area A is given by

$$A = \oint x dy = \int_0^L x \frac{dy}{ds} ds \qquad (How?)$$

Let $t = (2\pi s/L) - \pi$ so that t runs over the interval $[-\pi, \pi]$. Then

$$A = \int_{-\pi}^{\pi} x \frac{dy}{dt} dt \tag{1}$$

For the perimeter we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \frac{L^2}{4\pi^2}.$$

which is conveniently rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{\Big(\frac{dx}{dt}\Big)^2+\Big(\frac{dy}{dt}\Big)^2\Big\}dt=\frac{L^2}{4\pi^2}. \tag{2}$$

We now apply the Parseval's formula to (1) and (2). Let the $n-$th Fourier coefficients of $x(t)$ be a_n, b_n and those of $y(t)$ be c_n, d_n. For the area integral we get

$$A=\pi\sum_{n=1}^{\infty}n(a_nd_n-b_nc_n).$$

For second,

$$L^2=2\pi^2\sum_{n=1}^{\infty}n^2(a_n^2+b_n^2+c_n^2+d_n^2)$$

Thus we see

$$\begin{aligned}L^2-4\pi A &= 2\pi^2\sum_{n=1}^{\infty}\Big\{n^2(a_n^2+b_n^2+c_n^2+d_n^2)-2n(a_nd_n-b_nc_n)\Big\}\\ &= 2\pi^2\sum_{n=1}^{\infty}\Big\{(na_n-d_n)^2+(nb_n+c_n)^2+(n^2-1)(c_n^2+d_n^2)\Big\}\end{aligned}$$

Thus

$$L^2\geq 4\pi A$$

and the maximum value of the enclosed area equals $L^2/4\pi$. To determine the curve that achieves this, equality must hold which is so if and only if

$$na_n-d_n=nb_n+c_n=c_n=d_n=0,\quad n=2,3,\dots$$

and $a_1=d_1, b_1=-c_1$. Thus

$$x(t)=a_0+a_1\cos t+b_1\sin t,\ y(t)=c_0-b_1\cos t+a_1\sin t.$$

which represents a circle. The proof is complete.

Question: How do you know that the Fourier series of $x'(t)$ and $y'(t)$ are na_n, n_b and nc_n, nd_n respectively? Do you need to differentiate these Fourier series term by term? Is this justified? Can you prove the stated result without term by term differentiation??

Laplace's Equation on a disc We shall now apply the theory of Fourier series for solving some classical PDEs. We take up the Dirichlet problem for the Laplace equation on the unit disc $D=\{(x,y)\ :\ x^2+y^2\leq 1\}$. The problem seeks a twice continuously differentiable function u such that

$$\Delta u=0,\ \text{ on } D,\quad u(\cos\theta,\sin\theta)=f(\theta)$$

where we assume that f is Lipschitz and 2π periodic on $\mathbb{R}$. First we write the equation in polar coordinates:

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}=0.$$

Seeking special solutions in the form $u(x,y) = v(r)g(\theta)$ where $g(\theta)$ is 2π periodic,

$$(r^2 v'' + rv')/v = -g''/g$$

Since g is a function of θ alone and v is a function of r alone, either side must be a constant say k^2 and we get the pair of ODEs

$$r^2 v''(r) + rv'(r) - k^2 v(r) = 0, \quad g''(\theta) + k^2 g(\theta) = 0.$$

These have solutions

$$v(r) = Ar^k + Br^{-k}, \quad g(\theta) = C\cos k\theta + D\sin k\theta.$$

Now since $g(\theta)$ is 2π-periodic, we must have $k \in \mathbb{Z}$ (how?). Also since the solution is continuous at the origin, $k \geq 0$. Thus we get the solution in the form

$$u = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

To determine the coefficients of this Fourier expansion we must use the boundary condition. Setting $r = 1$ we get

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

from which we deduce the values of $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$.

Exercises:

26. Determine the solution of the Laplace's equation in the unit disc with the prescribed boundary value $|\sin\theta|$.

27. Show that if u is a harmonic function (that is $\Delta u = 0$) then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r\cos\theta, r\sin\theta) = u(0,0).$$

This is called the *mean value theorem* for harmonic functions.

The Poisson Kernel Let us continue with the formula obtained in the last slide:

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

But we know the formula for these Fourier coefficients and inserting these,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \Big(\cos n\theta \int_{-\pi}^{\pi} f(t)\cos nt dt + \sin n\theta \int_{-\pi}^{\pi} f(t)\sin nt dt \Big)$$

Since the integrals decay to zero by Riemann Lebesgue lemma it is easy to justify exchange of sum and integral (with $0 \leq r < 1$):

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(1 + 2 \sum_{n=1}^{\infty} r^n (\cos n\theta \cos nt + 2 \sin n\theta \sin nt) \Big) f(t) dt$$

The student is invited to sum the series

$$1 + \sum_{n=1}^{\infty} 2r^n \cos n(\theta - t)$$

and we get the result

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) f(t) dt}{1 + r^2 - 2r \cos(\theta - t)}$$

The expression $\Pi_r(\theta - t) = (1 - r^2)/(1 + r^2 - 2r\cos(\theta - t))$ is called the *Poisson Kernel.*

Problems on Poisson Kernel We still have to show that the solution obtained in the last slide does attain the value $f(\theta)$ on the boundary.

28. Show that the Poisson kernel is non-negative and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Pi_r(\theta - t) = 1.$$

29. Show that

$$\lim_{r \to 1-} |u(re^{i\theta}) - f(\theta)| = 0.$$

Hint: Write $f(\theta)$ as the integral w.r.t over $[-\pi, \pi]$ of $f(\theta)\Pi_r(\theta - t)/(2\pi)$. Now let $\epsilon > 0$ and I be an interval of length ϵ centered at θ. The integral over I is small for one reason and the integral over $[-\pi, \pi] - I$ is small for a different reason.

Poisson Formula for the Ball There is a corresponding result for the ball in $\mathbb{R}^3$ but to derive that we need to spend a little time with associated Legendre equations.

Theorem 4.8: Suppose given a continuous function $f(\mathbf{x})$ on the unit ball B centered at the origin in $\mathbb{R}^3$ then the solution of the boundary value problem

$$\Delta u = 0 \text{ on } B \quad u\Big|_{\partial B} = f$$

is given by

$$f(\mathbf{y}) = \int_{\partial B} \frac{(1 - \|\mathbf{x}\|^2) f(\mathbf{x}) dS(\mathbf{x})}{(1 + \|\mathbf{x}\|^2 - 2\|\mathbf{x}\| \cos\alpha)^{3/2}},$$

where α is the angle between $\mathbf{x}$ and $\mathbf{y}$.

For the proof see page 180 of *G. B. Folland, Fourier Analysis and its applications, Amer. Math. Soc, Indian Reprint by Universities Press, New Delhi, 2012.*

Problems on the Heat Equation

30. Let $u(x,t)$ be a smooth solution of the heat equation in the upper half-plane $t \geq 0$ such that $u(x,t) = u(x+2\pi, t)$ for all $x \in \mathbb{R},\ t \geq 0$. Show that the energy $\int_{-\pi}^{\pi} (u(x,t))^2 dx$ is a monotone decreasing function of time. Prove the same result if the integral is over any interval of length 2π. Deduce that a smooth $2\pi-$periodic solution of the heat equation with a given initial condition, if it exists, is unique.

31. Determine the periodic solution of the initial-boundary value for the heat equation $u_t - u_{xx} = 0$ on $-\pi \leq x \leq \pi$ and $0 \leq t \leq T$ subject to

$$u(-\pi,t) = u(\pi,t),\ u(x,0) = \pi^2 - x^2.$$

32. Solve the heat equation on the same rectangle above but with data $u(-\pi,t) = u(\pi,t) = 0,\ u(x,0) = \pi^2 - x^2$. This leads to a Sturm-Liouville problem with Dirichlet boundary conditions for which you would have to compute the eigen-values and eigen-functions.

33. Seek a periodic solution (with period 2π) of the heat equation $u_t = u_{xx}$ of the form $f(x)g(t)$ and write down the Ansatz for the most general solutions as superpositions of these special solutions. Given a smooth $2\pi-$periodic initial condition $u(x,0) = \phi(x)$, show that the solution (whose uniqueness has been established) is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\xi)\Big(\sum_{n=-\infty}^{\infty} e^{in(x-\xi)-n^2t}\Big)d\xi, \quad t > 0.$$

Call the sum of the series as $G(x-\xi,t)$ and we see that the solution is (a convolution?)

$$u(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\xi)G(x-\xi,t)d\xi.$$

34. Note that the series as such makes no sense when $t = 0$. How would you say that the solution displayed achieves the boundary value ϕ when $t \to 0$?

Wave equation

35. Solve the initial-boundary value problem for the wave equation

$$u_{tt} - u_{xx} = 0,$$

on the rectangle $-\pi \leq x \leq \pi$ and $0 \leq t \leq 1$ with initial conditions

$$u(x,0) = |\sin x|,\ u_t(x,0) = 0,$$

and periodic boundary condition $u(x,t) = u(x+2\pi,t)$.

36. Do the preceding for $2\pi-$periodic solutions but with data

$$u(x,0) = \pi^2 - x^2,\ u_t(x,0) = 0.$$

37. Solve the preceding two problems with the same initial conditions but with Dirichlet boundary conditions $u(-\pi,t) = 0 = u(\pi,t)$ instead of $2\pi-$ periodicity. Again you will run into a Sturm-Liouville problem as in the case of heat equation.

Additional Problems Determine the Fourier coefficients of $f(x) = (\pi^2 - x^2)^{-1/2}$. I. N. Sneddon, p. 141.

V - Fourier transforms and partial differential equations

Recall that in basic ODE theory where one studies equations with constant coefficients, special solutions were sought in the form

$$P(x)\exp(mx).$$

Here m is a root of the characteristic equation and $P(x)$ is a polynomial which would be non-constant when the characteristic equation has multiple roots.

Generalizing to the case of partial differential equations with constant coefficients (such as the fundamental equations arising in physics), it is natural to seek *plane wave* solutions

$$\exp i(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n) \tag{1}$$

and more general solutions can be obtained by superpositions.

In the case of partial differential equations, the characteristic equation would be a polynomial in several variables. For example taking the case of the wave equation

$$u_{tt} - u_{xx} = 0, \tag{2}$$

let us substitute the Ansatz (1) in the form $\exp i(at - bx)$ into (2) and we get

$$a^2 - b^2 = 0. \tag{3}$$

Equation (3) has infinitely many solutions and indeed two continuous families (λ, λ) and $(\lambda, -\lambda)$. We would now have to take a continuous superposition of the plane waves

$$\exp i\lambda(x+t), \quad \exp i\lambda(x-t)$$

which means we must consider the sum of two integrals

$$\int_{-\infty}^{\infty} f(\lambda)\exp i\lambda(x+t)d\lambda + \int_{-\infty}^{\infty} g(\lambda)\exp i\lambda(x-t)d\lambda. \tag{4}$$

Definition of the Fourier transform We are naturally led to the following

Definition: Suppose $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a function for which

$$\int_{-\infty}^{\infty} |f(t)|dt \tag{5}$$

is finite then the integral

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt$$

is called the Fourier transform of $f(t)$.

There are several conventions and we follow the one that is common in PDEs for example see p. 213 of *G. B. Folland, Fourier analysis and its applications.*

Exercises:

1. Let $f(t) = 1$ when $|t| \leq 1$ and $f(t) = 0$ otherwise. Compute the Fourier transform of f.
2. Compute the Fourier transform of the function $f(t)$ given by $f(t) = 1/\sqrt{1-t^2}$ if $|t| < 1$ and $f(t) = 0$ if $|t| \geq 1$.
3. Prove the Riemann Lebesgue lemma which states that if $f(t)$ is continuous and the integral (5) is finite then $|\widehat{f}(\xi)| \longrightarrow 0$ when $\xi \to \pm\infty$.
4. Compute the Fourier transform of $f(t) = 1/(a^2 + t^2)$ where a is a non-zero real number. Note that the cases $a > 0$ and $a < 0$ have to be dealt with separately.
   Hint: Let $I(\xi)$ be the integral. Find the Laplace transform of $I(\xi)$.
5. Compute the Fourier transform of $\exp(-a|t|)$ where $a > 0$.
6. Looking at the last two examples are you led to conjecture any general result?
7. Calculate the Fourier transform of $f(t) = \sin^2 t/t^2$ using the ideas of exercise (4) above.
8. One can also compute the Fourier transform of $f(t) = \sin t/t$ but a careful justification would have to wait. Why so? However proceed formally and try to arrive at the answer.
9. Try to calculate $f_a * f_b$ where $f_a(t) = a/(\pi(x^2 + a^2))$. Recall the definition of convolution. It is an integral from $-\infty$ to ∞. Don't be too surprised if the computation gets pretty ugly. This example comes up in Probability theory under the name of *Cauchy distribution.*

Fourier transform of the Gaussian: This is one of the most important examples in the theory of Fourier transforms and plays a crucial role in probability theory, number theory, quantum mechanics, theory of heat conduction and diffusive processes in general.

Theorem 5.1: Suppose $a > 0$. The Fourier transform of $\exp(-at^2)$ is the function

$$\sqrt{\frac{\pi}{a}}\exp(-\xi^2/4a).$$

Proof: We shall obtain a first order ODE for $I(\xi)$ given by

$$I(\xi) = \int_{-\infty}^{\infty} \exp(-at^2 - it\xi)dt = \int_{-\infty}^{\infty} \exp(-at^2)\cos(t\xi)dt.$$

Differentiate the integral with respect to ξ and we get

$$I'(\xi) = -\int_{-\infty}^{\infty} t\exp(-at^2)\sin(t\xi)dt.$$

which can be written as

$$I'(\xi) = \frac{1}{2a}\int_{-\infty}^{\infty} \frac{d}{dt}\Big(\exp(-at^2)\Big)\sin(t\xi)dt$$

Integrating by parts we get

$$I'(\xi) = -\frac{\xi}{2a}\int_{-\infty}^{\infty} \exp(-at^2)\cos(t\xi)dt$$

whereby we obtain the ODE $I' + \xi/(2a)I = 0$.

Exercises:

10. Integrating this linear ODE and complete the argument.

11. Compute the Fourier transform of $x^2 \exp(-ax^2)$ and more generally $x^{2k} \exp(-ax^2)$.

The Schwartz space $\mathcal{S}$ of rapidly decreasing functions This is a convenient class of functions introduced by Laurant Schwartz in his influential work *Theorié des distributions, Hermann, Paris.* This function space has the advantage that it is a vector space and it closed under differentiation as well as multiplication by polynomials. Besides it also contains enough functions so that any $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

can be approximated arbitrarily closely by functions in $\mathcal{S}$.

This space is particularly well-suited to the study of the Fourier transform. One proves all the results in the context of $\mathcal{S}$ where there are no technical obstructions to differentiation under integral sign or switching order of integrals. To pass on to the more general case one resorts to approximation techniques. The situation is reminiscent of the proof of Parseval formula for Riemann integrable functions.

Definition: The space $\mathcal{S}$ consists of all infinitely differentiable functions $f(t)$ such that for any $m, n \in \mathbb{N}$,

$$t^m \Big(\frac{d^n}{dt^n}\Big) f(t)$$

remains bounded.

We immediately see that for any polynomial $P(t)$ and any $a > 0$, the function $P(t) \exp(-at^2)$ lies in $\mathcal{S}$.

12. Show that $(\cosh at)^{-1}$ and $t(\sinh at)^{-1}$ lie in $\mathcal{S}$ for any $a > 0$.

13. One can show using the complex version of *Stirling's approximation formula* that

$$|\Gamma(a + it)|^2$$

lies in $\mathcal{S}$ for $a > 0$. This is an *Optional* exercise.

Properties of $\mathcal{S}$: The space $\mathcal{S}$ is obviously a vector space and it has the following further properties

1. Suppose $f(t)$ and $g(t)$ belong to $\mathcal{S}$ then so does $f(t)g(t)$.

2. It $f(t) \in \mathcal{S}$ then the derivative $f'(t)$ also lies in $\mathcal{S}$.

3. If $f(t) \in \mathcal{S}$ and $P(t)$ is a polynomial then $f(t)P(t) \in \mathcal{S}$.

4. For every $f(t) \in \mathcal{S}$ the integral of $|f(t)|$ over $\mathbb{R}$ exists and so for every $f(t) \in \mathcal{S}$ the Fourier transform is defined.

We shall now show that if $f(t) \in \mathcal{S}$ then $\widehat{f}(\xi)$ also lies in $\mathcal{S}$. Thus the Fourier transform maps $\mathcal{S}$ into itself as a linear transformation. Can you point out some of its eigen-values by looking at the list of Fourier transforms we have calculated?

Theorem 5.2: Suppose $f(t) \in \mathcal{S}$ then $\widehat{f}(\xi) \in \mathcal{S}$.

Proof: Differentiating under the integral is permissible since $f(t)$ decays rapidly. Thus

$$\Big(\frac{d}{d\xi}\Big)^n \widehat{f}(\xi) = (-i)^n \int_{-\infty}^{\infty} e^{-it\xi}(t^n f(t))dt$$

Rewrite the integrand as $\Big\{e^{-it\xi}t^n f(t)(1+t^2)\Big\}(1+t^2)^{-1}$. The bracketed term is bounded in absolute value. Multiplying the integral by $(-i\xi)^k$ we see

$$(-i\xi)^k\Big(\frac{d}{id\xi}\Big)^n \widehat{f}(\xi) = (-1)^n \int_{-\infty}^{\infty} \Big(\frac{d}{dt}\Big)^k e^{-it\xi}(t^n f(t))dt$$

Integrating by parts and using $D^k(t^n f(t)) \in \mathcal{S}$ we conclude $\xi^k D^n \widehat{f}(\xi)$ is bounded for every $n, k \in \mathbb{N}$. Since k is arbitrary, the result is proved.

Differentiation and multiplication

Theorem 5.3: Suppose $f(t) \in \mathcal{S}$ then

$$\begin{aligned} \widehat{\Big(\frac{d}{idt}\Big)(f(t))} &= \xi\widehat{f}(\xi) \\ \widehat{tf(t)}(\xi) &= \Big(-\frac{d}{id\xi}\Big)\Big(\widehat{f}(\xi)\Big) \end{aligned}$$

To prove the first part integrate by parts. To prove the second part, differentiate under the integral with respect to ξ:

$$\frac{d}{d\xi}\Big(\widehat{f}(\xi)\Big) = \frac{d}{d\xi}\int_{-\infty}^{\infty} e^{-it\xi}f(t)dt = \int_{-\infty}^{\infty} -ite^{-it\xi}f(t)dt$$

Divide by $-i$ and we get the second formula.

Hermite's ODE and Hermite polynomials again !!

14. Transform the Hermite's differential equation

$$y'' - 2xy' + 2\lambda y = 0$$

by the substitution $y\exp(-x^2/2) = u$. Ans: $u'' - x^2u + (2\lambda+1)u = 0$.

15. Show that is u is a solution for the transformed ODE in the previous exercise then $\widehat{u}$ is also a solution of the same differential equation. That is, the transformed equation is *invariant* under Fourier transform.

16. Show that at most one of the solutions of the transformed ODE lies in $\mathcal{S}$. If $\lambda \in \mathbb{N}$ then

$$H_n(x)\exp(-x^2/2)$$

where $H_n(x)$ is the $n-$th Hermite polynomial, lies in $\mathcal{S}$. Hint: The *Abel-Liouville formula.*

17. The Fourier transform is a linear transformation of $\mathcal{S}$ to itself. Show that $H_n(x)\exp(-x^2/2)$ are eigen-vectors of this linear map.

Inversion theorem

Theorem 5.4: Suppose $f(t)$ is a function in $\mathcal{S}$ then the function can be recovered from its Fourier transform via the formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi. \tag{6}$$

To prove this first try to substitute in the RHS of (6) the expression for $\widehat{f(\xi)}$ and invert the order of integrals. You will run into the following (hitherto meaningless) integral:

$$\int_{-\infty}^{\infty} \exp(i(t-x)\xi)d\xi$$

Proof of inversion theorem: The $\exp(-\epsilon x^2)$ **trick !** Equipped with this, we now write

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi-\epsilon\xi^2}d\xi$$

Now we put in the definition of $\widehat{f}(\xi)$ and we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(x)e^{i\xi(t-x)-\epsilon\xi^2}dx\Big)d\xi$$

We can now safely invert the order of integral and write

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{i\xi(t-x)-\epsilon\xi^2}d\xi\Big)f(x)dx$$

The inner integral is the Fourier transform of the Gaussian that we have computed !!

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty} \exp(-(x-t)^2/4\epsilon)f(x)dx$$

Putting $x = t + \sqrt{4\epsilon}s$ we obtain

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = 2\sqrt{\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t+\sqrt{4\epsilon}s)e^{-s^2}ds = 2\pi f(t)$$

The proof is complete.
Here is a bunch of exercises are collected all of which are amenable to the $\exp(-\epsilon t^2)$ trick.

Exercises:

18. Prove that
$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi = 2\pi f(0), \quad f \in \mathcal{S}.$$

19. Compute the Fourier transform of $(\sin at)/t$.

20. Compute the Fourier transform of $f(t) = (t^2+a^2)^{-1}$ by obtaining a second order ODE.

21. Use the $\exp(-\epsilon t^2)$ trick to obtain a second order ODE for the function

$$\frac{2}{\pi}\int_1^\infty \frac{(\sin tx)dt}{\sqrt{t^2-1}}, \quad x>0.$$

Can you recognize this as a familiar function? Also prove this by taking Laplace transform. This is from *H. Weber, Die Partiellen Differential-Gleichungen der mathematischen Physik, Vol - I, Braunschweig, 1900, based on Riemann's lectures. The formula appears on p. 175.* This representation is due to *Mehler (1872) and Sonine (1880). See p. 170 of G. N. Watson's treatise.*

Ramanujan's Formula: Here is an interesting formula due to Ramanujan that can be deduced from the $\exp(-\epsilon x^2)$ trick.

Theorem 5.5: Let $a>0$ then

$$\int_{-\infty}^\infty |\Gamma(a+it)|^2 e^{-it\xi}dt = \sqrt{\pi}\Gamma(a)\Gamma(a+\frac{1}{2})\text{sech}^{-2a}(\xi/2).$$

For the details of the proof see *D. Chakrabarty and G. K. Srinivasan, On a remarkable formula of Ramanujan, Archiv der Mathematik (Basel)* **99** *(2012) 125-135.*

The Jacobi theta function identity:

22. Show that if $f\in\mathcal{S}$ and $c>0$ then

$$\lim_{k\to\infty}\int_0^c \frac{\sin kt}{t}f(t)dt$$

exists. Hint: $(\sin kt)/t$ is the integral of $\cos ut$ over $[0,k]$. Is it necessary that $f\in\mathcal{S}$?

23. Show that if $0<c<\pi$ then

$$\lim_{k\to\infty}\int_0^c \frac{\sin kt}{\sin t}f(t)dt = \frac{\pi}{2}f(0), \quad f\in\mathcal{S}.$$

24. Modify the formula of the previous exercise if $0<c<2\pi$ and $\to\infty$ through odd-integer values. Hint: Break the integral into three integrals over $[0,\pi/2]$, $[\pi/2,\pi]$ and $[\pi/2,c]$. What if $c=\pi$? What if $c>2\pi$??

We shall see how this leads to the famous *theta function identity of Jacobi.*

25. With a little care one can continue from the last exercise and show *Oskar Schlömilch, Analytische Studien, Vol - II, Leipzig, 1847, p. 29,* that if $f\in\mathcal{S}$ and an even function,

$$\lim_{k\to\infty}\int_{-\infty}^\infty \frac{\sin kt}{\sin t}f(t)dt = \pi\Big\{f(0)+2f(\pi)+2f(2\pi)+2f(3\pi)+\ldots\Big\}$$

26. Recall the Fourier transform of the Gaussian:

$$\int_{-\infty}^{\infty} e^{-a^2t^2}\cos(2\xi t)dt = \frac{\sqrt{\pi}}{a}\exp(-\xi^2/a^2)$$

Set $\xi = 0, 1, \ldots, n$, add and deduce *Jacobi's theta function identity:*

$$\pi\Big\{1 + 2e^{-a^2\pi^2} + 2e^{-4a^2\pi^2} + \ldots\Big\} = \frac{\sqrt{\pi}}{a}\Big\{1 + 2e^{-1/a^2} + 2e^{-4/a^2} + \ldots\Big\}$$

Hint: $1 + \cos 2t + \cdots + \cos 2nt = \sin(2n+1)t/\sin t$.

One can now argue in general, without additional effort, to prove:

Theorem 5.6 (Poisson summation formula): Suppose $f \in \mathcal{S}$ then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \widehat{f}(2n\pi).$$

Optional: Try this! One can relax the requirement $f \in \mathcal{S}$. How about $f(x) = 1/(a^2 + x^2)$? Compute the sum of the series

$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \ldots$$

Compare this with the partial fraction decomposition of the cosecent obtained in the last chapter.

Theorem 5.7 (Parseval formula also known as Plancherel's theorem): Suppose $f(t)$ and $g(t)$ are in $\mathcal{S}$ then

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi \tag{7}$$

Exercises:

27. (Optional) Prove this using the $\exp(-\epsilon t^2)$ trick.

28. Apply this theorem to calculate the eigen-values of the Fourier transform which is a linear transformation from $\mathcal{S}$ to itself.

Theorem 5.8 (Convolution theorem): Suppose $f(t)$ and $g(t)$ are both in $\mathcal{S}$ then so is their convolution $(f * g)(t)$. Further

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{8}$$

We shall not prove that the convolution is in $\mathcal{S}$. Verification of (8) is an easy exercise. Observe the analogy with the corresponding result for Laplace transforms.

Heat equation again !! Let us now solve the initial value problem for the heat equation in the half-plane

$$u_t - u_{xx} = 0, \quad u(x,0) = f(x).$$

Let us assume to begin with $f(x) \in \mathcal{S}$ and compute the Fourier transform with respect to x:

$$\frac{d}{dt}(\widehat{u}(\xi,t)) + \xi^2\widehat{u} = 0, \quad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

This is an ODE in $\widehat{u}$ where ξ is regarded as a parameter.

$$\widehat{u}(\xi,t) = C\exp(-t\xi^2)$$

Putting in $t = 0$ we see that $C = \widehat{f}(\xi)$. Thus

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\exp(-t\xi^2)$$

The Heat Kernel Observe that $\exp(-t\xi^2)$ is the Fourier transform with respect to x, of the function

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/4t)$$

Appealing to the convolution theorem,

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\widehat{G}(\xi,t) = \widehat{G * f}(\xi,t)$$

Thus we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(s)\exp(-(x-s)^2/4t)\,ds \tag{9}$$

The function $G(x,t)$ called *the heat kernel,* plays a crucial role in *Probability theory.* The formula was derived assuming that the initial data $f(x)$ is in $\mathcal{S}$. However it makes perfect sense even if $f(x)$ is of exponential type !

Exercises:

29. Solve the heat equation $u_t - u_{xx} = 0$ with initial condition $u(x,0) = x^2$.

30. Solve the heat equation $u_t - u_{xx} = 0$ with initial condition $u(x,0) = \cos(ax)$. What about the solution with initial condition $\sin(ax)$.

31. Suppose the initial condition is a continuous function that is positive at say on $(-1,1)$ but zero outside $[-1,1]$ then the solution is positive at all points $u(x,t)$ no matter how large x is and how small $t > 0$ is. *Thus the effect of initial heat distribution in* $[-1,1]$ *is instantaneously propagated throughout space. Is this physically tenable?*

    Philosophical Question: How is it that the equation nevertheless is used to explain physical phenomena??

Jacobi theta function identity via heat equation Let us now assume that the initial data $f(s)$ in equation (9) is $2\pi-$ periodic. Then we write the RHS of (9) as

$$\frac{1}{\sqrt{4\pi t}}\sum_{-\infty}^{\infty}\int_{(2n-1)\pi}^{(2n+1)\pi} f(s)\exp(-(x-s)^2/4t)ds$$

which in turn can be recast as (how?)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\Big\{\sqrt{\frac{\pi}{t}}\sum_{n=-\infty}^{\infty} e^{-(x-s+2n\pi)^2/4t}\Big\}ds$$

Compare this with problem 33 of the previous chapter and re-derive the *Jacobi theta function identity. See Courant-Hilbert, Methods of mathematical physics, Volume - II, page 200.*

D'Alembert's solution of the wave equation At the beginning of the chapter we saw were led to looking for solutions of the wave equation

$$u_{tt} - u_{xx} = 0$$

in the form $u(x,t) = \phi(x+t) + \psi(x-t)$. Let us now supplement this with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Then

$$\phi(x) + \psi(x) = f(x), \quad \phi'(x) - \psi'(x) = g(x),$$

whereby

$$\phi'(s) = \frac{1}{2}(f'(s) + g(s)), \quad \psi'(s) = \frac{1}{2}(f'(s) - g(s))$$

We immediately get the formula

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds. \tag{10}$$

This is known as *D'Alembert's formula.*

Exercises:

32. Solve the wave equation $u_{tt} - u_{xx} = 0$ with initial conditions $u(x,0) = x$ and $u_t(x,0) = 1-x$.

33. Solve the wave equation $u_{tt} - u_{xx} = 0$ with initial conditions $u(x,0) = \sin x$ and $u_t(x,0) = 0$.

34. If v_1 is solution of the wave equation $u_{tt} - u_{xx} = 0$ and v_2 is the solution with the data of the first solution altered along $[2,3]$. Sketch the regions in the (x,t) plane where v_1 and v_2 definitely agree and the region where they could disagree.

35. A, B, C, D are the successive vertices of a rectangle whose sides have slopes ± 1. Show that if u is a solution of the wave equation,

$$u(A) + u(C) = u(B) + u(D).$$

Radial functions - the Bessel transform Let us now consider the multi-dimensional Fourier transform of a function $f(x_1, x_2, \ldots, x_n)$ which is assumed continuous and decaying rapidly enough so that the integrals appearing exist. The Fourier transform is defined as

$$\widehat{f}(\xi_1, \ldots, \xi_n) = \int_{\mathbb{R}^n} \exp(-i(x_1\xi_1 + \cdots + x_n\xi_n))f(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

Assume that the function f depends only on $\sqrt{x_1^2 + \cdots + x_n^2}$. Then the Fourier transform depends only on $\sqrt{\xi_1^2 + \cdots + \xi_n^2}$. To see this let us write

$$\langle \mathbf{x},\ \xi\rangle = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$$

Now let us choose a rotation matrix A such that $A\|\xi\|\mathbf{e}_n = \xi$. Then

$$\widehat{f}(\xi_1, \ldots, \xi_n) = \int_{\mathbb{R}^n} \exp(-i\|\xi\|\langle A\mathbf{e}_n, \mathbf{x}\rangle)f(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

But $\langle A\mathbf{e}_n, \mathbf{x}\rangle = \langle \mathbf{e}_n, A^T\mathbf{x}\rangle$. Now put $\mathbf{x} = A\mathbf{y}$. Since A is orthogonal, its determinant is one and applying the change of variable formula

$$\widehat{f}(\xi_1, \ldots, \xi_n) = \int_{\mathbb{R}^n} \exp(-i\|\xi\|y_n)f(y_1, \ldots, y_n)dy_1 \ldots dy_n.$$

We should now write it in polar coordinates. But we shall do this for $n = 2$ and $n = 3$ only. Write $f(x_1, x_2, \ldots, x_n) = F(r)$ where $r = \sqrt{x_1^2 + \cdots + x_n^2}$.

For $n = 3$, $y_3 = r\cos\phi$ and

$$\widehat{f}(\xi_1, \ldots, \xi_n) = \int_0^\infty F(r)r^2dr \int_0^\pi \exp(-ir\|\xi\|\cos\phi)\sin\phi d\phi \int_0^{2\pi} d\theta$$

Writing $\cos\phi = s$, we get

$$\widehat{f}(\xi_1, \xi_2, \xi_3) = 2\pi \int_0^\infty F(r)r^2dr \int_{-1}^1 e^{-irs\|\xi\|}ds = \frac{4\pi}{\|\xi\|}\int_0^\infty F(r)r\sin\|\xi\|rdr$$

For the case $n = 2$ we get

$$\widehat{f}(\xi_1, \xi_2) = \int_0^\infty rF(r)dr \int_{-\pi}^\pi \cos(r\|\xi\|\sin\theta)d\theta$$

Which is a *Bessel transform*

$$\widehat{f}(\xi_1, \xi_2) = 2\pi \int_0^\infty rF(r)J_0(r\|\xi\|)dr$$

More generally in even space dimensions it reduces to a Bessel transform and in odd dimension a "sine transform".

Principle of Equipartioning of Energy for the Wave Equation We derived the energy equation for $u_{tt} - u_{xx} = 0$.

36. Use Parseval's formula to compute the Fourier transform (with respect to $x-$variable) of the energy.

37. Calculate the Fourier transform with respect to the $x-$variable of the kinetic energy alone. Also compute the Fourier transform of the solution in terms of the initial conditions.

38. Calculate the limit as $t \to \pm\infty$ of the result obtained in the last exercise. Use the half angle formula and Riemann Lebesgue lemma.

39. Show that when time goes to infinity, the limiting value of kinetic energy is half the total energy.

See *R. Stricharz, A guide to distribution theory and Fourier transforms, CRC Press LLC, 1994.*

Airy's Function: *Airy studied the function that bears his name in the course of his investigations on the intensity of light in the neighborhood of a caustic (See G. N. Watson's treatise, p. 188). The work dates back to 1838. Before commencing on the discussion of Airy's function, here is a pointer to the interesting life of Sir George Biddell Airy:*
library.mat.uniroma1.it/appoggio/MOSTRA2006/airy
Another interesting account I had read long ago was by *Patrick Moore* but I am unable to locate it at the moment.
Airy's equation is the ODE

$$y''(x) - \frac{1}{3}xy = 0.$$

40. Formally take the Fourier transform of the Airy's differential equation $y'' - \frac{1}{3}xy = 0$. Use the inversion formula to obtain a representation

$$I(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\xi^3 + ix\xi)d\xi.$$

The integral converges only conditionally (how? maybe you should first discuss the integrals over $\mathbb{R}$ of $\sin(x^2)$ and $\cos(x^2)$). The steps leading to the integral are suspect and so one has to justify it in some other way.

41. Now replace ξ by $z = \xi + i\eta$ where the imaginary part η is positive and consider

$$I(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iz^3 + ixz)d\xi..$$

    The integral now along a line parallel to the real axis, converges absolutely and very rapidly. Differentiate under the integral sign and check that $I(x)$ satisfies the ODE.

42. The tricky business is to show that one can allow $\eta \to 0$ and get the integral of the last slide. How can one justify this when the integral converges only conditionally? Try to integrate by parts once and then see if you can allow $\eta \longrightarrow 0$. Those who have studied complex analysis can put it to good use. In fact the integral does not depend on η at all !! How would you verify this?

43. Determine the radius of convergence of the power series solution of Airy's equation. We have now both the series solution as well as the integral representation. Which one would you think is more useful?

Additional Problems

1. In the last chapter you computed the Fourier coefficients of $f(x) = (\pi^2 - x^2)^{-1/2}$. Assuming that the series converges on the open interval $(-\pi, \pi)$ deduce (I. N. Sneddon, p. 141)

$$J_0(x) = \frac{\sin x}{x}\left\{1 + 2x^2 \sum_{n=1}^{\infty} \frac{(-1)^n J_0(nx)}{x^2 - n^2\pi^2}.\right\}$$

VI - Application to Celestial Mechanics

In this last chapter of this course we discuss the original problem that led Bessel in 1824 to introduce the functions that bear his name. Wilhelm Bessel was an astronomer at Königsberg and his chief interest was the study of orbits of comets.

We must begin by recalling the three basic laws of planetary mechanics motion enunciated by Johannas Kepler. The discovery of these laws forms an interesting culmination of classical astronomy. Equipped with his calculus, Issac Newton, with his laws of dynamics able to explain:

1. *The motion of planets.*
2. *The precession of equinoxes.*
3. *The formation of tides.*

Astronomy hitherto an *empirical science* transformed into a *dynamical science.*

Kepler's laws of planetary motion The first two laws were enunciated in 1609 in *De Motibus Stellae Martis:*

Kepler's First Law: The planets move around the sun in elliptical orbits with the sun at one of the foci.

Kepler's Second Law: The radius vector joining the sun and the planet sweeps out equal areas in equal intervals of time. This is just a restatement of the law of conservation of angular momentum.

The third law (which is an approximate law neglecting the masses of planets) appeared much later in 1619 in his *Harmonices Mundi.*

Kepler's Third Law: The square of the period T is proportional to the cube of the semi-major axis a of the orbit.

The constant of proportionality was determined by Gauss in his famous book on Astronomy[8] and is known as Gaussian gravitational constant.

The Kepler problem Suppose the planet passes the perihelion Π at time $t = 0$. Placing the sun at the F, we measure all angles from the radius vector $F\Pi$. Let $X(t)$ be the position of the planet at time t and $\theta(t)$ be the angle between $F\Pi$ and FX. In astronomy one calls the function $\theta(t)$ the *True Anomaly.*

[8] Theoria Motus Corporum Coelestium

Kepler Problem: Find the function $\theta(t)$ as explicitly as possible.
PICTURE

A basic lemma: Let P and P' be corresponding points on the ellipse and the auxiliary circle. Then Area(AFP) refers to the area of the sector of the ellipse and Area(AFP') the corresponding sector of the auxiliary circle.

$$\frac{Area(AFP)}{Area(AFP')} = \frac{b}{a} = \frac{Area(Ellipse)}{Area(Circle)} \tag{1}$$

Exercise:

1. Prove this using integral calculus to compute the indicated areas.

We now use *Kepler's second law* and write

$$\frac{Area(AFP)}{Area(Ellipse)} = \frac{t}{T} = \frac{Area(AFP')}{Area(Circle)} \tag{2}$$

But if C denotes the center of the ellipse (and the circle) then

$$Area(AFP') = Area(ACP') - Area(\Delta FCP') \tag{3}$$

The last two areas are readily described and we get

$$Area(AFP') = \frac{1}{2}(a^2\phi - a^2\epsilon \sin\phi) \tag{4}$$

where ϕ is the eccentric angle of P and ϵ the eccentricity of the ellipse.

Substituting (4) in (3) we get

$$\frac{Area(AFP')}{Area(Circle)} = \frac{\phi - \epsilon\sin\phi}{2\pi}$$

or, using (2) we get

$$\phi - \epsilon\sin\phi = \frac{2\pi t}{T} \tag{5}$$

Equation (5) is the famous *Kepler Equation.* The number $2\pi t/T$ is called the *Mean Anomaly.*

Inverting the Kepler equation

Exercises:

2. Show that the function $\phi - \epsilon\sin\phi$ is strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. The inverse function $\phi(t)$ is a strictly increasing infinitely differentiable function.

3. Explain why the function $\phi(t)$ is an odd function. Use both geometrical reasoning as well as mathematical analysis.

4. Show that $\phi(0) = 0$ and $\phi(T/2) = \pi$.

The problem of inverting the Kepler equation has been studied by many eminent mathematicians such as *J. L. Lagrange, Memoirs of the Berlin Academie 1768-69. Also volume - II, p. 22 ff. of his Méchanique Analytique 1815.* In this connection Lagrange discovered the *inversion formula* that bears his name. The *Lagrange inversion formula* has important applications in quite un-related fields such as combinatorics.

Bringing in the periodicity Let us now look at $\phi(t+T)$. The Kepler equation gives

$$\phi(t+T) - \epsilon \sin \phi(t+T) = \frac{2\pi(t+T)}{T} = \frac{2\pi t}{T} + 2\pi.$$

which can be written as

$$\begin{aligned} \phi(t+T) - \epsilon \sin \phi(t+T) &= (\phi(t) - \epsilon \sin \phi(t)) + 2\pi \\ &= (\phi(t) + 2\pi) - \epsilon \sin(\phi(t) + 2\pi). \end{aligned}$$

By injectivity of the function $\lambda - \epsilon \sin \lambda$ we conclude

$$\phi(t+T) = \phi(t) + 2\pi. \tag{6}$$

Exercises:

5. Could you derive this directly from physical considerations?
6. Verify using (6) that the function $\psi(t) = \phi(t) - \frac{2\pi t}{T}$ is a periodic function with period T.

Clearly then $\psi\left(\frac{tT}{2\pi}\right)$ is periodic with period 2π. Let us write the Fourier series for $\psi\left(\frac{tT}{2\pi}\right)$:

$$\psi\left(\frac{tT}{2\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nt),$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \psi\left(\frac{tT}{2\pi}\right)\sin(nt)dt = \frac{2}{\pi}\int_0^{\pi}\left\{\phi\left(\frac{tT}{2\pi}\right) - t\right\}\sin(nt)dt.$$

Integrating by parts and recalling $\phi(T/2) = \pi$, $\phi(0) = 0$, we get

$$b_n = \frac{2}{n\pi}\int_0^{\pi} \frac{d}{dt}\left\{\phi\left(\frac{tT}{2\pi}\right) - t\right\}\cos(nt)dt.$$

This simplifies to

$$b_n = \frac{2}{n\pi}\int_0^{\pi} \frac{d}{dt}\phi\left(\frac{tT}{2\pi}\right)\cos(nt)dt = \frac{2}{n\pi}\int_0^{T/2} \phi'(s)\cos(2\pi ns/T)ds.$$

Using the Kepler equation again, the argument of the cosine can be re-written resulting in:

$$b_n = \frac{2}{n\pi}\int_0^{T/2} \phi'(s)\cos(n\phi(s) - n\epsilon\sin\phi(s))ds.$$

The change of variables $\phi(s) = \lambda$ now gives

$$b_n = \frac{2}{n\pi}\int_0^{\pi}\cos(n\lambda - n\epsilon\sin\lambda)d\lambda = \frac{2J_n(n\epsilon)}{n}.$$

The Fourier series now reads

$$\psi\left(\frac{tT}{2\pi}\right) = \phi\left(\frac{tT}{2\pi}\right) - t = \sum_{n=1}^{\infty}\frac{2J_n(n\epsilon)}{n}\sin nt.$$

So the eccentric angle $\phi(t)$ (also known in astronomy as the *Eccentric Anomaly*) can be written as a *Kaypten series:*

$$\phi(t) = \frac{2\pi t}{T} + \sum_{n=1}^{\infty}\frac{2J_n(n\epsilon)}{n}\sin(2\pi nt/T). \tag{7}$$

The true anomaly

7. Using elementary trigonometry, find a relation between the true anomaly and the eccentric anomaly.

For a short but quick historical survey see C. A. Ronan, Science, its history and development among world's culture, pp. 336-337.
For more on the Kepler problem and mathematical principles underlying celestial mechanics, the book by *J. M. Danby, Celestial Mechanics and dynamical astronomy, Kluwer Academic, 1991,* is HIGHLY recommended. This second edition contains computer experiments.
A more ambitious project would be to read the comprehensive two volumes
D. Boccaletti and G. Puccaco, Theory of Orbits, Vol- I, II, Springer Verlag, 2004.

One can try to expand the Bessel functions appearing in the Kaypten series and rearrange terms to get a power series in e. however it was known that the resulting series converges only when $e < 0.667$. The orbits of most comets exceed this number - Orbit of *Halley's comet* is 0.96 !!

It seems an investigation into why the series fails to converge beyond this threshold led Cauchy to develop the theory of functions of one complex variable. There is an imaginary singularity that prevents the power series from converging beyond the threshold value.

Further reading: The classic work of R. Courant and D. Hilbert has been cited several times. Besides this, there are several others. The serious student can scarcely do better than to begin a systematic study of the following three books which also have been referred in the slides.

1. G. B. Folland, Fourier Analysis and its applications,

2. A. Robert, Advanced calculus for users, North-Holland, 1989.

3. R. Strichartz, Guide to distribution theory and Fourier transforms.

VI - Acknowledgment

The instructor wishes to thank the many students who have pointed out errors as well as suggested alternate solutions to the problems. The response of students has been gratifying. Thanks also for patiently listening to the lectures and brooking the instructor's pestilence.

Solutions and hints to problems in chapter - I

1. Compute $\int_{-1}^{1}(P_n(x))^2dx$

Solution: We use Rodrigues's formula and write

$$\int_{-1}^{1}(P_n(x))^2dx = \frac{1}{n!^2 4^n}\int_{-1}^{1} D^n(x^2-1)^n D^n(x^2-1)^n dx.$$

Integration by parts gives, calling $1/(n!^2 4^n)$ as c_n,

$$-c_n\int_{-1}^{1} D^{n-1}(x^2-1)^n D^{n+1}(x^2-1)^n dx + c_n D^{n-1}(x^2-1)^n D^n(x^2-1)^n\Big|_{-1}^{1}.$$

The boundary terms vanish since ± 1 being $n-$fold roots of $(x^2-1)^n$,

$$D^k(x^2-1)^n\Big|_{\pm 1} = 0, \quad k = 0, 1, 2, \ldots, n-1.$$

Further integration by parts gives

$$\begin{aligned}\int_{-1}^{1}(P_n(x))^2dx &= (-1)^n c_n\int_{-1}^{1}(x^2-1)^n D^{2n}(x^2-1)^n dx\\ &= c_n(2n)!\int_{-1}^{1}(1-x^2)^n dx\\ &= c_n(2n)!\int_{-\pi/2}^{\pi/2}\cos^{2n+1}x dx\\ &= c_n(2n)!\frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}\int_{-\pi/2}^{\pi/2}\cos x dx.\\ &= \frac{2}{2n+1}\end{aligned}$$

as desired.

Use Rodrigues formula to prove that the Legendre polynomial of degree n has precisely n distinct roots in the open interval $(-1, 1)$.

Solution: Suppose $f(x)$ is a polynomial with a double root at a and b say $a < b$. Then by Rolle's theorem there is a $c \in (a, b)$ such that $f'(c) = 0$. But we know that

$$f'(a) = f'(b) = 0.$$

So we can apply Rolle's theorem to $f'(x)$ on each of the two intervals (a, c) and (c, b) to conclude that $f''(x)$ has a root in each of the intervals (a, c) and (c, b) and thus $f''(x)$ has at least two distinct roots in (a, b). Repeating the same argument one shows that if $f(x)$ has a triple root at a and b then $f'''(x)$ has at least three distinct roots in (a, b). The general situation is quite clear. Now the polynomial $(x^2 - 1)^n$ has an $n-$fold root at ± 1 and so its $n-$th derivative $D^n(x^2 - 1)^n$ has at least n *distinct roots* in $(-1, 1)$. But being a polynomial of degree n it cannot have more than n roots and so has exactly n distinct roots in $(-1, 1)$.

Show that the sequence of Legendre polynomials $P_n(x)$ satisfies the three term recursion formula

$$(n + 1)P_{n+1} - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0.$$

Solution: Coefficient of x^{n+1} in $(n + 1)P_{n+1} - (2n + 1)xP_n(x)$ equals:

$$\frac{(n + 1)(2n + 2)!}{2^{n+1}((n + 1)!)^2} - \frac{(2n + 1)(2n)!}{2^n(n!)^2} = 0.$$

So $(n + 1)P_{n+1} - (2n + 1)xP_n(x)$ is a polynomial of degree at most n and so can be written as

$$(n + 1)P_{n+1} - (2n + 1)xP_n(x) = \sum_{k=0}^{n} a_k P_k(x). \tag{1}$$

Since $P_{n+1}(x)$ and $xP_n(x)$ has the same parity and $P_n(x)$ has parity opposite to these, we see that $a_n = 0$. Let us now show $a_k = 0$ when $k = 0, 1, \ldots, n - 2$.

Using orthogonality of the Legendre polynomials we get

$$\int_{-1}^{1} ((n + 1)P_{n+1} - (2n + 1)xP_n(x))P_k(x)dx = a_k \int_{-1}^{1} (P_k(x))^2 dx.$$

Again by orthogonality the integral on the LHS is simply

$$-(2n + 1) \int_{-1}^{1} P_n(x)(xP_k(x))dx \tag{2}$$

Now $xP_k(x) = b_0P_0(x) + \cdots + b_{k+1}P_{k+1}(x)$ say. But $k + 1 \leq n$ and so by orthogonality again, the integral (2) is zero. Thus we see that $a_k = 0$ for $k = 0, 1, 2, \ldots, n - 2$. We are left with

$$(n + 1)P_{n+1} - (2n + 1)xP_n(x) = a_{n-1}P_{n-1}(x).$$

To compute a_{n-1} put $x = 1$ and recall that $P_k(1) = 1$ for all k.

Prove that $P_n'(1) = \frac{1}{2}n(n + 1)$.

Solution: Using Rodrigues' formula

$$P_n'(1) = \frac{1}{n!2^n} D^{n+1}(x-1)^n(x+1)^n \Big|_{x=1}.$$

Now using Leibnitz's rule for higher order derivatives of a product,

$$P_n'(1) = \frac{n+1}{n!2^n} n!n(2^{n-1}) = \frac{1}{2}n(n+1).$$

as desired.

Prove that $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$.

Solution: The polynomial $P_{n+1}'(x) - xP_n'(x)$ of degree n can be written as

$$P_{n+1}'(x) - xP_n'(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Multiply both sides by $P_k(x)$ and integrate over $[-1,1]$. Assume $k \leq n-1$. A single integration by parts gives for the LHS:

$$-\int_{-1}^{1} P_{n+1}(x)P_k'(x)dx + \int_{-1}^{1} P_n(x)(xP_k)'(x). \tag{1}$$

The boundary terms cancel out since both $P_{n+1}(x)P_k(x)$ and $(xP_k(x)P_n(x))$ equal 1 when $x = 1$. Thus

$$P_{n+1}(x)P_k(x) = (xP_k(x)P_n(x)) = (-1)^{n+k+1}, \quad \text{when } x = -1.$$

Since $P_k'(x)$ and $(xP_k(x))'$ are both polynomials of degree less than or equal to $n-1$, we have by orthogonality the integral (1) equals zero and so $a_k = 0$ when $k \leq n-1$.

Solution to problem 27: Proof of Jacobi's formula. Following the hint given, let $f(t) = D^{n-1}(1-t^2)^{(2n-1)/2}$. Then

$$\begin{aligned}
(1-t^2)f''(t) &= (1-t^2)D^{n+1}(1-t^2)^{n-\frac{1}{2}} \\
&= D^{n+1}(1-t^2)^{n+\frac{1}{2}} + 2(n+1)tD^n(1-t^2)^{n-\frac{1}{2}} \\
&+ (n^2+n)D^{n-1}(1-t^2)^{n-\frac{1}{2}}
\end{aligned}$$

Thus $(1-t^2)f'' - tf'(t) + n^2 f$ equals:

$$D^{n+1}(1-t^2)^{n+\frac{1}{2}} + t(2n+1)D^n(1-t^2)^{n-\frac{1}{2}} + (2n^2+n)D^{n-1}(1-t^2)^{n-\frac{1}{2}}.$$

The first term can be rewritten as

$$D^{n+1}(1-t^2)^{n+\frac{1}{2}} = -(2n+1)D^n(1-t^2)^{n-\frac{1}{2}}t.$$

Apply Leibnitz rule and things cancel out. The rest of the problem is clear.

Solutions for selected problems in chapter - II Prove Laplace's second integral representation

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{(x + \sqrt{x^2 - 1}\cos\theta)^{n+1}}$$

Solution: We assume $|x| \geq 1$ and use the first formula of Laplace

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1}\cos\theta)^n d\theta$$

Perform the change of variables

$$x + \sqrt{x^2 - 1}\cos\theta = 1/(x + \sqrt{x^2 - 1}\cos\phi)$$

Check that this change of variables is licit and complete the solution.

Use Laplace's integral representation to prove the following result of C. Neumann (1862)

$$\lim_{n\to\infty} P_n(\cos(x/n)) = J_0(x).$$

Solution: Substituting into Laplace's formula

$$P_n(\cos(x/n)) = (\cos(x/n))^n \frac{1}{\pi}\int_0^\pi (1 + i\tan(x/n)\cos\theta)^n d\theta \tag{1}$$

The factor $(\cos(x/n))^n$ tends to one as $n \to \infty$. Let us then focus on the integral. Note that if $a(\lambda) \to \alpha$ then by L'Hospital's rule,

$$(1 + \lambda a(\lambda))^{1/\lambda} \longrightarrow e^\alpha$$

Passing to the limit in (1) we get

$$\lim_{n\to\infty} P_n(\cos(x/n)) = \frac{1}{\pi}\int_0^\pi e^{ix\cos\theta} d\theta$$

Solutions and hints to problems of chapter - III

1. Show that the eigen-values of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

are roots of $k + \tan k = 0$, where $k^2 = \lambda$. Graphically demonstrate that there is an eigen value in every interval of length π.

Solution: If $\lambda = 0$ the solution of the ODE is $A + Bx$ and the boundary conditions force $A = 0 = B$. Thus 0 is not an eigen-value. If $\lambda \neq 0$, put $\lambda = k^2$ and the solution of the ODE is

$$y(x) = A\cos kx + B\sin kx$$

Setting $y(0) = 0$ we see $A = 0$. The other boundary condition gives

$$B\sin k + kB\cos k = 0,$$

which is $k + \tan k = 0$. The graphical argument is left to the student. The eigen functions are $\sin k_n x$, where $k_1, k_2, k_3, \ldots$ are the roots of $k + \tan k = 0$.

2. Suppose $y_0(x)$ is a minimizer for

$$Q[y] = \int_0^1 (y'(x))^2 dx$$

subject to the constraint

$$\int_0^1 (y(x))^2 \rho(x) dx = 1,$$

over the class S of all continuous piecewise once differentiable functions vanishing at 0 and 1. Assume that the minimizer $y_0(x)$ is twice differentiable then show that it satisfies the BVP

$$y'' + \lambda\rho(x)y(x) = 0, \quad y(0) = 0 = y(1).$$

where λ equals $Q[y_0]$.

Solution: Since the minimum is attained at $y_0(x)$ we have to begin with

$$\int_0^1 (y_0(x))^2 \rho(x) dx = 1. \tag{1}$$

Let us perturb $y_0(x)$ to $y_0(x) + \epsilon\phi(x)$ where $\phi(x)$ is an arbitrary smooth function vanishing at 0 and 1. We normalize by dividing by

$$N = \sqrt{\int_0^1 (y_0(x) + \epsilon\phi(x))^2 \rho(x) dx}$$

and consider the perturbation

$$w(x) = (y_0(x) + \epsilon\phi(x))/N$$

Then

$$Q[y_0] \leq Q[w] \tag{2}$$

for all small values of ϵ positive or negative.

Writing out (2) after multiplying through by N^2 and writing λ in place of $Q[y_0]$ we get

$$\lambda \int_0^1 (y_0(x) + \epsilon\phi(x))^2 \rho(x) dx \leq \int_0^1 (y_0'(x) + \epsilon\phi'(x))^2 dx.$$

Expanding out, using (1) and canceling out $Q[y_0]$ or λ from both sides we get

$$2\epsilon \int_0^1 (\lambda y_0(x)\rho(x) - y_0'(x)\phi'(x))dx + \epsilon^2(\text{Stuff}) \leq 0.$$

Dividing by ϵ and letting $\epsilon \to 0$, separating the cases when it tends to zero through positive and negative values, we get a pair of opposite inequalities that finally results in

$$\int_0^1 (\lambda y_0(x)\phi(x)\rho(x) - y_0'(x)\phi'(x))dx = 0.$$

Integrating by parts the second term we get

$$\int_0^1 (\lambda y_0(x)\phi(x)\rho(x) + y_0''(x))\phi(x) dx = 0.$$

Since ϕ is arbitrary, it follows that

$$\lambda y_0(x)\phi(x)\rho(x) + y_0''(x) = 0.$$

3. Explain why the the eigen-function corresponding to the first eigen-value of

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(1).$$

has no zeros on $(0, 1)$. Give an intuitive non-rigorous argument.

Solution: Suppose there is a zero $p \in (0,1)$. Assume the eigen-function $y_0(x)$ is positive on $(0,p)$ and negative on $(p,1)$. Consider the function $v(x)$ which equals $y_0(x)$ on $(0,p)$ and $-y_0(x)$ on $(p,1)$. The student is advised to draw pictures. Then

$$\lambda = \int_0^1 (y_0'(x))^2 dx, \quad \int_0^1 (y_0(x))^2 \rho(x) dx = 0.$$

Replacing $y_0(x)$ by $v(x)$ would not alter either of these. At this stage one can give a precise proof using mathematical analysis by appealing to the Dirichlet principle which asserts that $v(x)$ is twice differentiable and satisfies

$$v'' + \lambda\rho(x)v = 0.$$

It immediately follows (exercise) that $v'(p) = 0$ and so by Picard's theorem $v(x)$ is identically zero which is a contradiction. The geometrical argument runs as follows. One tweaks the function $v(x)$ making it flat and as a result the minimum value of the integral

$$\int_0^1 (y'(x))^2 dx$$

would get strictly below the smallest eigen-value leading to a contradiction.

1. Prove the Sturm's comparison theorem. Suppose $\rho(x)$ and $\sigma(x)$ are two continuous functions on an interval I such that $\rho(x) > \sigma(x)$ throughout I and $z(x)$, $w(x)$ are solutions of the ODEs

$$z''(x) + \rho(x)z(x) = 0, \quad w''(x) + \sigma(x)w(x) = 0.$$

Between two successive zeros of $w(x)$ there is a zero of $z(x)$.

Solution: Let a, b be successive zeros of $w(x)$ and assume $z(x)$ has no zeros in (a,b). We may assume that both $z(x)$ and $w(x)$ are positive in (a,b) (why?). Multiply the w equation by z, the z equation by w, integrate over $[a,b]$ and subtract. After a single integration by parts we get

$$w(x)z'(x)\Big|_a^b - z(x)w'(x)\Big|_a^b + \int_a^b (\rho(x) - \sigma(x))y(x)z(x)dx = 0.$$

Using the fact that $w(a) = w(b) = 0$, we are left with

$$z(a)w'(a) - z(b)w'(b) + \int_a^b (\rho(x) - \sigma(x))y(x)z(x)dx = 0.$$

First two terms are non-negative, the integral is strictly positive. Contradiction.

Solutions and hints to problems of chapter - IV Imitate the proof of Riemann-Lebesgue lemma to show that $|J_0(x)| \le c/\sqrt{x}$ for some constant c. We use the integral representation and look at the integral

$$I = \int_{-1}^1 \frac{e^{itx}}{\sqrt{1-t^2}}.$$

Performing the change of variables $s = t + \frac{\pi}{x}$,

$$I = -\int_{\frac{\pi}{x}-1}^{\frac{\pi}{x}+1} \frac{e^{itx}}{\sqrt{1-(t-\frac{\pi}{x})^2}}.$$

Adding we get

$$2I = \int_{\frac{\pi}{x}-1}^{1} e^{itx} A(x,t)dt + \int_{1}^{\frac{\pi}{x}+1} e^{itx} B(x,t)dt \tag{1}$$

The student is invited to write out the expression for $A(x,t)$ and $B(x,t)$. Further,

$$\Big| \int_{1}^{\frac{\pi}{x}+1} e^{itx} B(x,t)dt \Big| \leq \int_{1}^{\frac{\pi}{x}+1} |B(x,t)|dt.$$

The MVT for integrals is not directly applicable (why?) but the integral can be evaluated in terms of $\sin^{-1}$ and LMVT can be applied to get the $c/\sqrt{x}$ estimate. Equivalently the MVT for integrals can be applied after shrinking the intervals by arbitrarily small amount. We now have to deal with the first integral.

$$A(x,t) = \Big(\frac{1}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-(t-\frac{\pi}{x})^2}} \Big) \geq 0.$$

So estimating the integral we again evaluate it in terms of $\sin^{-1}$ and get the value

$$2\Big(\sin^{-1} 1 - \sin^{-1}(\frac{\pi}{x} - 1) \Big)$$

which behaves like c/x for large x. The proof is complete.

Euler's formula for $\zeta(2k)$.

1. We begin by computing the Fourier series of $\cos(ax)$ where $a \notin \mathbb{Z}$. Well, $b_n = 0$ for all $n \in \mathbb{N}$. Now,

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \cos(ax)dx = \frac{\sin a\pi}{a\pi}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos(ax)\cos(nx)dx = \frac{(-1)^{n-1} 2a \sin \pi a}{n^2 - a^2}$$

We apply Dirichlet's theorem and divide by $\sin a\pi$:

$$\frac{\pi \cos ax}{\sin a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^2 - a^2}, \quad x \in [-\pi, \pi]$$

Putting $x = 0$ and $x = \pi$ we get

$$\begin{aligned} \pi \mathrm{cosec}(a\pi) &= \frac{1}{a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2a}{n^2 - a^2}, \\ \pi \cot(a\pi) &= \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}, \end{aligned}$$

2. Let $\zeta(2k) = \sum_{m=1}^{\infty} m^{-2k}$. Find the generating function $f(z)$ for the sequence $\{\zeta(2k)\}$ and write the result in terms of exponentials.

Solution: We have
$$f(z)=\sum_{k=1}^{\infty}z^{2k}\sum_{m=1}^{\infty}\frac{1}{m^{2k}}.$$
It is easy to convince yourself that $\zeta(2k)$ has reasonable bound (for example $2^{-2k}+3^{-2k}<2\cdot 2^{-2k}$ etc.,) which makes it easy to interchange the order of summation and write
$$f(z)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{z^{2k}}{m^{2k}}.$$
The inner sum is a geometric series which can be readily summed:
$$f(z)=\sum_{m=1}^{\infty}\frac{z^2}{m^2-z^2}=\frac{z}{2}\sum_{m=1}^{\infty}\frac{2z}{m^2-z^2}=-\frac{z}{2}\Big(\pi\cot\pi z-\frac{1}{z}\Big)$$
Writing the result in exponential form we get
$$f(z)=\frac{1}{2}-\frac{i\pi z}{2}-\frac{i\pi z}{e^{2\pi iz}-1}$$
Replace z by iz and we get
$$\sum_{m=1}^{\infty}(-1)^k z^{2k}\zeta(2k)=\frac{1}{2}+\frac{\pi z}{2}-\frac{\pi z e^{2\pi z}}{e^{2\pi z}-1}=\frac{1}{2}-\frac{\pi z}{2}-\frac{\pi z}{e^{2\pi z}-1}.$$
We now compare it with the generating function for Bernoulli numbers
$$\sum_{m=0}^{\infty}\frac{B_m(0)z^m}{m!}=\frac{z}{e^z-1}.$$
Replacing z by $2\pi z$ and dividing by 2 and remembering that $B_0(0)=1, B_1(0)=-1/2$,
$$\frac{1}{2}-\frac{\pi z}{2}+\sum_{m=2}^{\infty}\frac{B_m(0)z^m(2\pi)^m}{m!}=\frac{z}{e^{2\pi z}-1}.$$
We conclude that $B_{2m+1}(0)=0$ and further
$$\sum_{m=1}^{\infty}(-1)^k z^{2k}\zeta(2k)=-\frac{1}{2}\sum_{m=1}^{\infty}\frac{B_{2m}(0)z^{2m}(2\pi)^{2m}}{(2m)!}.$$
Comparing the coefficient of z^{2k} on both sides we derive Euler's formula
$$\zeta(2k)=\frac{(-1)^{k-1}B_{2k}(0)(2\pi)^{2k}}{2\cdot(2k)!}.$$
Determine the Fourier series for the function $f(x)=x^2$ on $[-\pi,\pi]$. Use Parseval formula to show that
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}.$$

The function being an even function $b_n = 0$ for every n and

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{4(-1)^n}{n^2}$$

Appealing to Parseval's formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{\pi^4}{5} = |a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

Putting in the computed values and simplifying we get the stated result.

Find a harmonic function u in the unit disc in the plane satisfying the boundary condition

$$u(\cos\theta, \sin\theta) = |\sin\theta|.$$

Solution: Use the Poisson formula

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_0^{\pi} \frac{(1-r^2)\sin t}{1+r^2-2r\cos(t-\theta)}.$$

Put $t - \theta = s$ and compute the resulting integral. With a little patience you will get

$$\frac{(1-r^2)\cos\theta}{4\pi r}\log\left(\frac{1+r^2+2r\cos\theta}{1+r^2-2r\cos\theta}\right) - \frac{(1-r^2)\sin\theta}{4r} +$$
$$\frac{1+r^2}{2\pi r}\left\{\tan^{-1}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \tan^{-1}\left(\frac{1+r}{1-r}\tan(\theta/2)\right)\right\}$$

Solve the problem using Fourier series. This time the solution will be in the form of an infinite series.

Let u(x, t) be a smooth solution of the heat equation in the upper half-plane $t \geq 0$ such that $u(x,t) = u(x+2\pi, t)$ for all $x \in \mathbb{R}$ and for all $t \geq 0$. Show that the energy

$$E(t) = \int_{-\pi}^{\pi} (u(x,t))^2 dx$$

is a monotone decreasing function of time. Prove the same result if the integral is over any interval of length 2π. Deduce that a smooth $2\pi-$periodic solution of the heat equation with a given initial condition, if it exists, is unique.

Solution: Multiply the PDE by u and integrate over $[-\pi, \pi]$. We get

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} (u(x,t))^2 dx - \int_{-\pi}^{\pi} u u_{xx} dx = 0.$$

Integrating by parts once we get

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} (u(x,t))^2 dx + \int_{-\pi}^{\pi} (u_x(x,t))^2 dx + u u_x\Big|_{-\pi} - u u_x\Big|_{\pi} = 0. \tag{1}$$

Now $u(x+2\pi, t) = u(x,t)$ for all $x \in \mathbb{R}$ and for all $t \geq 0$ so differentiating with respect to x we see that u_x is also periodic with period 2π and so the two boundary terms in (1) cancel out and we conclude

$$E'(t) = \frac{d}{dt}\int_{-\pi}^{\pi} (u(x,t))^2 dx \leq 0,$$

proving that the energy is monotone decreasing. Now if $u_1(x,t)$ and $u_2(x,t)$ are two $2\pi-$periodic solutions then $v(x,t) = u_1(x,t) - u_2(x,t)$ is also a $2\pi-$periodic solution of the heat equation with zero initial condition. For this the energy function iz zero at time $t = 0$. Since the energy is non-negative and monotone decreasing, it is identically zero. Thus $v(x,t) = 0$ for all $t \geq 0$ which proves the uniqueness assertion.

Exercise Solve the initial-boundary value for the heat equation on the rectangle:

$$u_t = u_{xx},\ u(-\pi, t) = u(\pi, t) = 0,\ u(x, 0) = \pi^2 - x^2. \tag{1}$$

Solution: We seek special solutions of the form $f(x)g(t)$ and substituting this Ansatz into the PDE we get the pair of ODEs:

$$f''(x) + p^2 f(x) = 0, \quad g'(t) + p^2 g(t) = 0.$$

Thus the special solution is $\alpha x + \beta$ (corresponding to $p = 0$) or

$$e^{-p^2 t}(A\cos px + B\sin px),\ p \neq 0.$$

Putting in the boundary conditions $u(\pm\pi, t) = 0$ results in the Sturm Liouville problem

$$f''(x) + p^2 f(x) = 0, \quad p \neq 0.$$

with Dirichlet boundary conditions

$$A\cos p\pi + B\sin p\pi = 0,\ A\cos p\pi - B\sin p\pi = 0.$$

Note that $p = 0$ is not an eigen-value. Since we are looking for non trivial solutions A and B cannot be both zero so the determinant of this system must be zero which means $\cos p\pi \sin p\pi = 0$ which implies $2p \in \mathbb{Z}$. We have the list of eigen values

$$n, (2n-1)/2, \quad n = 1, 2, \dots$$

and the corresponding eigen functions

$$\sin nx, \cos(n - \frac{1}{2})x, \quad n = 1, 2, 3, \dots.$$

So the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t} B_n \sin nx + \sum_{n=1}^{\infty} e^{-(2n-1)^2 t/4} A_n \cos(n - \frac{1}{2})x$$

To determine the sequence of constants, put in the initial condition $t = 0$ and we get

$$\pi^2 - x^2 = \sum_{n=1}^{\infty} B_n \sin nx + \sum_{n=1}^{\infty} A_n \cos(n - \frac{1}{2})x$$

Using orthogonality of eigen-functions the student can readily compute the sequence of coefficients A_n and B_n.

Question: It is easy to see that $B_n =$ for every n (how?) and

$$\pi^2 - x^2 = \sum_{n=1}^{\infty} A_n \cos(n - \frac{1}{2})x$$

In the case of usual Fourier series of an even function, one expects a constant term. Isn't it strange that in this case there is no A_0 term in the series? Is there some contradiction of some sort??

Problem 33:

Solution: Following the standard procedure which should by now be clear to the students, the solution can be expressed as

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2 t}(A_n \cos nx + B_n \sin nx).$$

Putting in the initial condition

$$\phi(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx).$$

Using the formulas for the Fourier coefficients,

$$u(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(1 + 2\sum_{n=1}^{\infty} e^{-n^2 t}(\cos nx \cos ns + \sin nx \sin ns)\Big)\phi(s)ds$$

Solutions to selected problems in chapter - V Prove the Riemann Lebesgue Lemma for Fourier transforms. Suppose $f(t)$ is an absolutely integrable continuous function then its Fourier transform $\widehat{f}(\xi)$ tends to zero as $\xi \to \pm\infty$.

Solution: Let $\epsilon > 0$ be arbitrary. Then there is an $M > 0$ such that

$$\int_{|t|\geq M} |f(t)|dt < \epsilon/4$$

Let us make the change of variables $t = s + \frac{\pi}{\xi}$ in the integral defining the Fourier transform and add.

$$2\widehat{t}(\xi) = \int_{|t|\geq M} (f(t) - f(t+\frac{\pi}{\xi}))e^{-it\xi}dt + \int_{-M}^{M} (f(t) - f(t+\frac{\pi}{\xi}))e^{-it\xi}dt.$$

The first piece is less than $\epsilon/2$ in absolute value. To deal with the last piece note that the function is uniformly continuous on $[-M, M]$ and so there is a $\eta > 0$ such that

$$|f(t) - f(t+\frac{\pi}{\xi})| < \epsilon/4M, \quad |\frac{\pi}{\xi}| < \eta.$$

So the last integral in absolute value is less than $\epsilon/2$ as soon as $|\xi| > \pi/\eta$. The proof is complete.

Solution to problem 4: Let $I(\xi)$ denote the Fourier transform of $1/(a^2 + x^2)$ $(a > 0)$. so that

$$I(\xi) = \int_{-\infty}^{\infty} \frac{\cos \xi x dx}{x^2 + a^2}$$

The function $I(\xi)$ is even (why?) and so we may assume that $\xi > 0$. We take the Laplace transform of both sides with respect to ξ and we find

$$\mathcal{L}(I) = s\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + s^2)}$$

Using partial fractions we can complete the integration and we get $\pi/(a(s + a))$. Thus

$$I(\xi) = \frac{\pi}{a}e^{-a|\xi|}$$

Solution to problem 7: We shall see that in the course of the computation we arrive at the following integral that we shall determine first.

$$\phi(b) = \int_{-\infty}^{\infty} \frac{(1 - \cos(bt))dt}{t^2}$$

To compute $\phi(b)$ we may assume $b > 0$ and take Laplace transform with respect to b and we get

$$\mathcal{L}\phi = \int_{-\infty}^{\infty} \Big\{\frac{1}{st^2} - \frac{s}{t^2(s^2 + t^2)}\Big\}dt = \frac{1}{s}\int_{-\infty}^{\infty} \frac{dt}{s^2 + t^2} = \pi/s^2.$$

Thus the integral is $\pi|b|$. We are now ready to compute the Fourier transform of $\sin^2 t/t^2$.

Exercise: Compute the Fourier transform of $f(t) = (\sin^2 t)/t^2$.

Solution: Let us denote the Fourier transform by $I(\xi)$ and take the Laplace transform of $I(\xi)$ with respect to the ξ variable. Denoting by $L(s)$ the Laplace transform we get

$$L(s) = \frac{s}{2}\int_{-\infty}^{\infty} \frac{(1-\cos 2t)dt}{t^2(s^2+t^2)}$$

The integral can be decomposed using partial fractions into three integrals namely,

$$\int_{-\infty}^{\infty} \frac{\cos(2t)dt}{s^2+t^2}, \int_{\infty}^{\infty} \frac{(1-\cos 2t)dt}{t^2}, \int_{-\infty}^{\infty} \frac{dt}{s^2+t^2}.$$

The first two have been determined and the last is elementary. Thus

$$L(s) = \frac{\pi}{s} - \frac{\pi}{2s^2} + \frac{\pi}{2s^2}e^{-2s}.$$

Now using the shifting theorem for Laplace transforms we get $I(\xi) = 0$ if $\xi \geq 2$ and $I(\xi) = \pi - \pi\xi/2$ if $0 \leq \xi \leq 2$. Extend as an even function. The student is invited to draw a picture. Later we shall see another simpler way to get this result. Compute the convolution $f * f$ where $f(t) = 1$ if $|t| \leq 1$ and zero outside. Draw the graph of this convolution and compare it with the graph of the Fourier transform.

For the formal computation of the Fourier transform of $\sin t/t$. Denoting by $I(\xi)$ the Fourier transform we get assuming $a > 0$ and $\xi > 0$,

$$I(\xi) = \int_{-\infty}^{\infty} \frac{\sin at \cos \xi t dt}{t}.$$

The same method would work. Take the Laplace transform with respect to ξ and we get

$$s\int_{-\infty}^{\infty} \frac{\sin t dt}{t(s^2+t^2)}.$$

So we now have to calculate an integral of the form

$$\int_{-\infty}^{\infty} \frac{\sin xt dt}{t(t^2+s^2)}$$

The value of this integral is $\pi(1-e^{-s})/s^2$ (How?) and so the Fourier transform $\widehat{f}(\xi)$ equals $\pi\xi$ if $0 \leq \xi \leq 1$ and zero if $\xi \geq 1$. Extend as an even function.

Problems 14-16:

14 An easy calculation gives the following ODE for u:

$$u'' - x^2 u = -(1+2\lambda)u.$$

15 Now using theorem 41 we see at once that if u is a solution of this ODE that lies in $\mathcal{S}$ then $\widehat{u}$ also satisfies the same ODE. So the transformed ODE is invariant under Fourier transform.

16 If λ is an integer n we know that the original ODE has a polynomial solution $H_n(x)$ and so we have a solution $H_n(x)\exp(-x^2/2)$ for the transformed ODE and this solution lies in $\mathcal{S}$. The Abel-Liouville formula for the Wronskian of two linearly independent solutions is a non zero constant and so only one solution can lie in $\mathcal{S}$.

Well, otherwise the Wronskian would tend to zero as $x \longrightarrow \infty$ and being a constant would have to be identically zero. Now since $u(x) = H_n(x)\exp(-x^2/2)$ is a solution lying in $\mathcal{S}$ we see that $\widehat{u}$ is also a solution of the same equation in $\mathcal{S}$ and these two must be linearly dependent by the last exercise which means

$$\widehat{u} = cu$$

for some constant and u is an eigen-vector for the Fourier transform as a linear map from $\mathcal{S}$ to itself. One can show that there is an orthonormal basis of eigen-vectors but that uses more analysis.

Solution to problem 22: Following the hint provided,

$$\int_0^c \frac{\sin kt}{k} f(t)dt = \int_0^k du \int_0^c (\cos ut) f(t)dt = \int_0^k \frac{du}{u} \int_0^c (\sin ut)' f(t).$$

Hence,

$$\int_0^c \frac{\sin kt}{k} f(t)dt = \int_0^k \frac{du}{u} \Big\{ f(c)\sin uc - \int_0^c (\sin ut) f'(t)dt. \Big\}$$

The first piece on the RHS has limit $f(c)\pi/2$ as $k \to \infty$ (why?). The second piece can be written as

$$\int_0^c f'(t)dt \int_0^{kc} \frac{\sin v}{v} dv \longrightarrow \frac{\pi}{2} \int_0^c f'(t)dt \qquad (How?)$$

The rest is clear.

Solution to problem 23: Suppose $0 < c < \pi$. We simply write the integrand as

$$\left(\frac{\sin kt}{t}\right)\left(\frac{tf(t)}{\sin t}\right) = \frac{\sin kt}{t} g(t)$$

where $g(t) = tf(t)/\sin t$. We can apply the previous result with $g(t)$ in place of $f(t)$.

Solution to problem Assume $\pi < c < 2\pi$. Then as before

$$\lim_{k\to\infty} \int_0^{\pi/2} \frac{\sin kt}{\sin t} f(t)dt = \frac{\pi}{2} f(0).$$

We now take up the remaining two pieces. The substitution $\pi - t = s$ transforms the integral

$$\int_{\pi/2}^{\pi} \frac{\sin kt}{\sin t} f(t)dt$$

to (remembering that k is odd),

$$\int_0^{\pi/2} \frac{\sin ks}{\sin s} f(\pi - s)ds$$

which has limit $f(\pi)\pi/2$. Finally for the third piece with $s = t - \pi$ we get

$$\int_\pi^c \frac{\sin kt}{\sin t} f(t)dt = \int_0^{c-\pi} \frac{\sin ks}{\sin s} f(\pi + s)ds$$

which has limit $f(\pi)\pi/2$ as $k \to \infty$ through odd values. It is clear what happens when $c = \pi$.

To discuss the convergence of the integral in problem 40. Let us denote the integral without the $(2\pi)^{-1}$ factor as $I(x)$. Assume to begin with $x > 0$. We make the change of variables

$$\xi^3 + x\xi = t$$

and note that the derivative being positive makes this licit. Thus we get

$$I(x) = \int_{-\infty}^{\infty} \frac{\cos t dt}{(3\xi^2 + x)} = \int_{-\infty}^{\infty} \frac{1}{(3\xi^2 + x)} \frac{d \sin t}{dt} dt = -6 \int_{-\infty}^{\infty} \frac{\xi \sin t dt}{(3\xi^2 + x)^2}.$$

Now, the integral over $[-M, M]$ is anyway finite and we need merely focus on what happens when ξ is large. The integrand is easily estimated and convergence can be established. Explain why the condition $x > 0$ is not essential for the argument.