Introductio in Analysin Infinitorum

A heuristic introduction to the theory of convergence

A lecture by Prof. G. K. Srinivasan (IIT Bombay) at KVPY 2016 at IISER Mohali.

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While this was certainly a monumental achievement, calculus remained a tool in the service of the physical sciences. We now leave this phase and turn to the epoch when out of calculus crystalized a distinguished branch of mathematics whose founder was L. Euler.

Metamorphosis from Calculus to Analysis

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Euler's understanding of infinite processes (theory of convergence) was deep, his reasoning incisive and often daring and even audacious (for instance his derivation of the infinite product factorization of the sine function). His Introductio in Analysin Infinitorum paved way for a systematic doctorine of limits to be established 75 years later by A. L. Cauchy.
Let us begin with Euler's exponential function as set out in chapter 7 of his book...

Compound interest formula and the exponential function

Recall the formula for compound interest. Assume that the rate of interest per annum is r and the principal amount is one unit (say one rupee or one dollar). We assume that the one year period is divided into n sub-periods and the interest is compounded over each period. The amount at the end of a year is then

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Question: What happens as n increases and goes to infinity?
Exercise: Prove using the Binomial Theorem that if $m < n$ then

$$\left(1+\frac{r}{m}\right)^m < \left(1+\frac{r}{n}\right)^n \tag{2}$$

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Next, do the numbers (1) grow unboundedly as n goes to infinity or is there a finite threshold value that serves as an upper bound?

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For example if $r = 1$ then the factorials in the denominators may be replaced by $2, 2^2, 2^3, \dots$ and we get a geometric series with common ration $1/2$ dominating left hand side for any n however large.

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Exercise: How about a general r? For instance if $r = 10$, then look at what happens from the 12th term onwards. Replace the factorials in the denominators by successive powers of 11. Generalize this idea and show that for any fixed r, the sequence

$$\left(1+\frac{r}{n}\right)^n, \quad n = 1, 2, 3, \dots$$

steadily increases but does not grow unboundedly but remains always less that a certain threshold value.

The exponential function is born !

The sequence converges as n tends to infinity and the limit which would obviously depend on r is denoted by $\exp r$:

$$\exp r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n \tag{3}$$

Thus we have at our disposal a new function $\exp r$ at least defined for all positive real values of r - though in fact it makes sense for all values of r.

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Remark:The exponential function is one of the most important functions in Mathematics.
Clearly,

$$\exp r < \exp s, \quad \text{if} \;\; r < s \tag{4}$$

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In other words, the exponential function is a strictly increasing function.

Four Easy Exercises:

Use the idea of monotone increasing/decreasing sequences bounded above/below and establish

9. If $0 < c < 1$ then the sequence of powers $1, c, c^2, c^3, \ldots$ converges to zero.

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12. Show that if $0 < c < 1$ then the sequence $c^{1/n}$, $n = 1, 2, 3, \ldots$ again converges to one.

The exponential addition theorem

Theorem 1

For any two real numbers r and s

$$\exp(r+s) = \exp r \exp s$$

$$\left(1+\frac{r}{n}\right)^n \left(1+\frac{s}{n}\right)^n = \left(1+\frac{r+s}{n}\right)^n (1+A)^n \tag{5}$$

where A is given by

$$A = \frac{rs}{n^2}\left(1+\frac{r+s}{n}\right)^{-1} \sim \frac{rs}{n^2}, \qquad \text{for large } n$$

so that

$$(1+A)^n \sim \left(\left(1+\frac{rs}{n^2}\right)^{n^2}\right)^{1/n}, \qquad \text{which converges to one}$$

by virtue of the third exercise in the previous slide and the fact that the inner parenthesis tends to $\exp(rs)$. Now the result follows from (5).

The number e

By induction one deduces,

$$\exp(r_1 + r_2 + \cdots + r_n) = \exp r_1 \exp r_2 \ldots \exp r_n. \tag{6}$$

and taking $r_1 = r_2 = \cdots = r_n = 1$ we see that

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$$(\exp(p/q))^q = \exp p = e^p$$

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$$\exp x = e^x, \qquad \text{for all rational values } x.$$

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Remark: It is not difficult to show that e is irrational.

Graph of the exponential function

We know that the exponential function is monotone increasing namely $\exp r < \exp s$ if $r < s$ for real values of r and s.

Question: How does the graph of the function $\exp x$ look?

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Imagine the graphs of the functions

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The graph of f is convex (when viewed from below the x-axis and this is the standard convention) whereas the graph of $g(x)$ appears concave.
Definition: A function $f(x)$ defined on an open interval is convex if the chord joining two points always lies above the graph. For example in the case of a parabola $y = x^2$ this is so.
However this is not so in case of the inverted parabola - the graph of the function $x(1 - x)$.

The derivative of a function

Recall that the slope of the tangent to a graph $y = f(x)$ at a point $P = (p, f(p))$ on it is the limit (if it exists) of the slopes of chords joining P and a neighboring point Q taken as Q tends to P. Draw pictures and convince yourselves that this is a reasonable approach and indeed an ancient idea !

The limit (if it exists) is called the derivative of the function at p.

Exercise: Compute the derivatives of the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$. Show further that for the function $f(x) = x^2$ the derivative increases with x whereas for the function $g(x) = \sqrt{x}$ the derivative decreases as x increases. This suggests the following:

Convexity and Derivatives

Theorem 2

Suppose $f(x)$ is a function defined on the positive real line (or any open interval for that matter) then $f(x)$ is convex if its derviative is an increasing function.

It is not difficult to prove this theorem and in fact it is a simple consequence of the Lagrange's Mean Value Theorem. However it is more important and educative for the student to convince himself/herself of the plausibilty of this assertion by studying the graphs of the MODEL cases

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Convexity is one of the most important ideas in Mathematics

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Convexity is one of the most important ideas in Mathematics and the exponential function is one of the most important functions in Mathematics and so it is natural to ask:

What about the Convexity of the exponential function?

Let us try and calculate the derivative of the exponential function as a limit of the slope of the chords:

$$\frac{\exp(x+h) - \exp x}{h}, \quad \text{as } h \to 0.$$

By the exponential addition theorem this is the same as

$$(\exp x) \lim_{h\to 0} (\exp h - 1)/h = \exp x$$

if we can convince ourselves that

$$\lim_{h\to 0} (\exp h - 1)/h = 1$$

A heuristic argument to demonstrate that the limit is one

Well, let us approximate $\exp h$ using the definition and look at

$$\frac{1}{h}\left(\left(1+\frac{h}{n}\right)^{n}-1\right)$$

Factorizing we see the appearence of an arithmetic mean

$$\frac{1}{n}\left(\left(1+\frac{h}{n}\right)^{n-1}+\left(1+\frac{h}{n}\right)^{n-2}+\cdots+1\right)$$

which lies between the least 1 and the gretest $\left(1+\frac{h}{n}\right)^{n-1}$ and the latter is approximately $\exp h$ which in turn approaches 1 as $h \to 0$.

Remark: The argument is decidedly heuristic but it is not difficult to tighten it to make it rigorous.

Convexity of the exponential function

Theorem 3

The derivative of the exponential function at x is again $\exp x$. The exponential function is convex.

Let us reformulate the notion of convexity in analytic terms. Let us take two points $(x, f(x))$ and $(y, f(y))$ on the graph of a function $f(x)$. The section formula now says that a point on the chord is given by

$$(tx + (1-t)y, tf(x) + (1-t)f(y)), \quad 0 \leq t \leq 1$$

whereas the corresponding point in the graph is

$$(tx + (1-t)y, f(tx + (1-t)y)), \quad 0 \leq t \leq 1.$$

So convexity would require that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad 0 \leq t \leq 1. \tag{8}$$

Consequence of the convexity of the exponential function:

Take $t = 1/2$ in (8) and we get

$$\exp\left(\frac{x+y}{2}\right) \leq \frac{\exp x + \exp y}{2}$$

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Now replace y by $(y+z)/2$ and we get

$$\exp\left(\frac{x+y+z}{3}\right) \leq \frac{\exp x + \exp y + \exp z}{3}$$

Exercise: Prove that

$$\exp\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \leq \frac{1}{n}\Big(\exp x_1 + \exp x_2 + \cdots + \exp x_n\Big)$$

Cauchy's theorem on means:

Theorem 4

The Geometric mean of n positive real numbers does not exceed their Arithmetic mean.

To see this use the last displayed formula and set $a_1 = \exp x_1, a_2 = \exp x_2, \ldots, a_n = \exp x_n$. The exponential function is continuous and so assumes all positive real values.

Exercise: We have already indicated that $\exp x$ is differentiable and hence continuous. Now use the definition to show that $\exp x > 1 + x$ and so $\exp x$ assumes arbitrarily large positive values. Expontial addition theorem gives $(\exp x)(\exp(-x)) = \exp 0 = 1$ and so $\exp x$ assumes arbitrarily small positive real values as well. Invoke the intermediate value theorem.

Cauchy's theorem on means continued

The harmonic mean of $a_1, a_2, \ldots, a_n$ is by definition

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

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Exercise: Prove that the Harmonic Mean is less than or equal to the Geometric mean (the numbers are all assumed to be positive).

Cauchy's first limit theorem:

Theorem 5

Suppose $a_1, a_2, a_3, \ldots$ *is a sequence of real numbers such that*

$$\lim_{n\to\infty} a_n = l$$

Then, the sequence of arithemtic means also converges to l *namely,*

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Remark: A little contemplation on this would make the result look plausible - after all, averaging makes the distribution of data smoother and if the data hasa long tail close to a constant then the mean would be close to the same constant !

Let us continue with some ramifications of Cauchy's first limit theorem for a sequence of positive reals. Assume that $l = 0$. Then

$$0 < \left(a_1 a_2 \dots a_n\right)^{1/n} \leq \frac{1}{n}\left(a_1 + a_2 + \cdots + a_n\right)$$

The term on the extreme right tends to l by Cauchy's first limit theorem and $l = 0$ as we have assumed. Hence we conclude that the sequence of geometric means also converges to l.

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Exercise: What about the case $l \neq 0$? The sequence $1/a_n$ converges to $1/l$ and so applying Cauchy's first limit theorem to the sequence of reciprocals, we deduce that the sequence of harmonic means converges to l.
With the Arithmetic mean and Harmonic means both converging to l the Geometric mean, which is sandwiched between these two, must also converge to l. Let us give this the status of a theorem:

Theorem 6

If $a_1, a_2, a_3, \ldots$ is a sequence of positive real numbers converging to l, then

$$\lim_{n\to\infty} \left(a_1 a_2 \ldots a_n\right)^{1/n} = l$$

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All this may sound like un-interesting technicality but we shall presently see that this is not so. We first deduce a useful consequence of this result.

Cauchy's Second Limit Theorem:

Suppose $b_1, b_2, b_3, \dots$ is a sequence of positive real numbers such that

$$\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right) = l$$

Then,

$$\lim_{n\to\infty}\left(b_n\right)^{1/n} = l$$

Exercise: (i) Show that $n^{1/n}$ converges to 1 as n tends to infinity.
(ii) More generally show that if $P(n)$ is a polynomial with leading coefficient positve then $(P(n))^{1/n}$ converges to 1 as n tends to infinity.
(iii) Show that if $A(n) = \sqrt{2\pi n}$ then $A^{1/n}$ tends to one as n tends to infinity.

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Hint: For (i) take $b_n = n$ and apply Cauchy's second limit theorem. The other two are similar.

Theorem 7

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e} \tag{9}$$

The result for sure can be stated in the more informal style

$$(n!)^{1/n} \sim ne^{-1}, \quad \text{as} \;\; n \to \infty \tag{10}$$

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Equation (11) is WRONG. Recall that if $Q(n)$ is one of n, $P(n)$ (a polynomial with positive leading coefficient) or $\sqrt{2\pi n}$ then we have three more valid results:

$$(n!/Q(n))^{1/n} \sim ne^{-1}, \quad , \text{ as } \ n \to \infty \tag{12}$$

And now raising (12) or (10) to the nth power results in several distinct possibilities. What then is the correct scenario?

The Stirling's Approximation Formula:

This is one of the most remarkable results in classical analysis predating by two decades Euler's Introductio in Analysin Infintorum (1748).

Theorem 8

$$n! \sim n^n e^{-n}\sqrt{2\pi n}, \quad \text{as} \ \ n \to \infty.$$

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The proof may be found for instance in the appendix to the first chapter of W. Feller: Introduction to the theory of probability, Volume - I, Wiley.

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2. The early twentieth centure witnessed the birth of modern physics, statistical and quantum mechanics. Here ratios of factorials of large numbers appear (Order of Avagadros number) and it is of vital importance to understand the precise order of magnitude of such ratios. See for instance the book of Arthur Beiser, Perspectives of Modern Physics, McGraw Hill, 1969.

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Exercise: The binomial coefficient

$$\frac{(2n)!}{n!\ n!} \sim c n^b a^n; \quad \text{as} \ \ n \to \infty.$$

Find the constants a, b and c.

The Ubiquitous Euler's Constant

Let us consider the sequence whose nth term is given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1) \tag{13}$$

where the logarithm is to the base e. Often one sees $\log n$ insted of $\log(n+1)$ but this will have no effect on the limit of (13) as n tends to infinity.

Note that since

$$e^x > 1 + x, \qquad \text{for } x > 0.$$

we have

$$\log(1+x) < x, \qquad \text{for } x > 0.$$

Then,

$$a_{n+1} - a_n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n+1}\right) > 0 \tag{14}$$

which means that the sequence is monotone increasing. We now show that the sequence converges to a limit.

Exercises: Show that the sequence

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}$$

converges as n tends to infinity. Deduce as a consequence that the sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots \tag{15}$$

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Let us however return to (14), set $n=1,2,3,\dots$ and add. If we can show that the sum obtained on the right hand side converges then it follows that the telescoping sum

$$(a_2-a_1)+(a_3-a_2)+(a_4-a_3)+\cdots+(a_{n+1}-a_n)$$

converges which means a_n must converge as n tends to infinity.

Exercise: Prove that

$$0 < x - \log(1+x) < x^2, \quad \text{if } 0 < x < 1/2. \tag{16}$$

The result is trivial if x is large and we need it for small positive x namely when $x = 1/(n+1)$.

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Remark: You can of course use calculus. but if you are not yet familiar with calculus simply take it on faith. One can look for a somewhat algebraic proof but it would be somewhat tedious.

Now we are ready. We know

$$0 < a_{n+1} - a_n < \frac{1}{n+1} - \log\left(1 + \frac{1}{n+1}\right) < \frac{1}{(n+1)^2}$$

So if we set $n = 1, 2, 3 \ldots$ and sum we would get that

$$a_n - a_1 < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots, \quad \text{which is finite !}$$

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Definition: The limit of the convergent sequence

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1)$$

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It is not yet known whether γ is rational or irrational. There is an interesting history to this with the names of Mascheronni and Gauss attached to it

Euler and the sequence of prime numbers

The analytic study of the distribution of prime numbers has its roots in the work of Euler. The proof of the infinitude of primes goes back to Euclid more than 2000 years and is considered as a paradigm for elegance in mathematical reasoning.
However, the problem of asymptotic distribution of primes presents challenges and are quite beyond reach through elementary arguments such as Euclid's proof of the infinitude of primes. It was Euler who laid the foundations of an analytic theory of numbers.

We begin with a simple observation on the behaviour of the harmonic sums

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

We use the method of grouping to show that the sums H_n grow unboundedly as n goes to infinity. Well,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) > 1 + \frac{2}{2}.$$

Further,

$$H_8 = H_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{2}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) > 1 + \frac{3}{2}.$$

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Exercise: Show that

$$H_{2^n} > 1 + \frac{n}{2}, \quad H_{2^n - 1} < n$$

Estimate how many terms must be added to obtain a sum exceeding 11.

Infinitude of Primes

Let us begin with a simple observation:

$$\exp x > 1 + x > 1/(1 - \frac{x}{2}), \quad 0 < x < 1.$$

and hence

$$\exp(2x) > (1 - x)^{-1} = 1 + x + x^2 + \ldots, \quad 0 < x < 1/2.$$

The inequality also hold for $x = 1/2$ as is readily verified. Let us now put $x = 1/p_k$ and

$$\exp(2/p_k) > \left(1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \frac{1}{p_k^3} + \ldots\right)$$

Now set $k = 1, 2, 3, \ldots N$ and multiply out:

$$\exp\left(\frac{2}{p_1}+\frac{2}{p_2}+\cdots+\frac{2}{p_N}\right) > \prod_{k=1}^{N}\left(1+\frac{1}{p_k}+\frac{1}{p_k^2}+\frac{1}{p_k^3}+\dots\right) \tag{17}$$

If we expand the product on the right hand side we get the sum of those reciprocals of natural numbers j with the property that j is the product of primes from the list $p_1, p_2, \dots, p_N$.

Now if there are only finitely many primes take N to be the number of primes. The right hand side of the last displayed formula must contain reciprocals of all the numbers of the form

$$p_1{}^{j_1} p_2{}^{j_2} \dots p_N^{j_N}$$

and by the unique factorization theorem (fundamental theorem of arithmetic) these are ALL the natural numbers whereby

$$\exp\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_N}\right) > 1+\frac{1}{2}+\frac{1}{3}+\dots$$

which is false since the left hand side is finite.

Euler's argument actually proves more namely

Theorem 9

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots +$$

diverges.

Since the Nth prime is larger than N and any natural number between 2 and N must have prime factors only among the list $p_1, p_2, \ldots, p_N$. So the sum on the right hand side of (17) is larger than

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

Hence

$$\exp\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_N}\right)$$

cannot remain bounded which in turn implies

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_N}$$

cannot remain bounded when N tends to infinity.

The Riemann Zeta function

The method of grouping that was discussed earlier can be used to establish the fact that

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

converges if $s > 1$ and diverges if $s \leq 1$.

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$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\ldots \tag{18}$$

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Exercise: Prove this by first expanding each factor on the right hand side as a geometric series, multiplying out and using the unique factorization theorem.

Birth of Analytic Number Theory

The facotrization (18) is the entry point into the field of Analytic Number theory. The formula states that the information concerning the primes is in a sense encoded in the zeta function $\zeta(s)$. The study of the distribution of primes involves a deep study of the zeta function - a study that was initiated by Euler.

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Question: Why then is this function called the Riemann zeta function? Before answering a few remarks are in order.

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Question: Why then is this function called the Riemann zeta function? Before answering a few remarks are in order.

Notation: $\pi(x)$ denotes the number of primes in the interval $[1, x]$. The infinitude of primes is simply the assertion that $\pi(x) \to \infty$ as $x \to \infty$. A close look at Euclid's argument gives the bound

$$\pi(x) \geq \ln(\ln x)$$

but this is ridiculously off the mark. This is one of the principal reason why Euler's proof though less elementary is more insightful.

The Prime Number Theorem

Looking through tables of pimes, Gauss and Legendre observed that the relative population of primes $x/\pi(x)$ changed additively by 2.3 when x changed mulitplicatively by a factor of 10. since $2.3 \sim \ln 10$ and $\ln x$ changes by 2.3 when x changes multiplicatively by 10 they made the daring conjecture in the late 18th century that

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Assertion (19) is called the Prime Number Theorem
The result defied all attempts at proof for more than a century. Ultimately it was settled independently by J. Hadamard (1896) and De la Valle Poussin (1898).

Extending the zeta function à la Euler

Suppose that $0 < s < 1$ then the alternating series

$$h(s) = 1 - \frac{1}{2^s} + \frac{1}{3^3} - \frac{1}{4^s} + \dots$$

converges.

Exercise: First try this out for $s = 1/2$. by grouping pairs of terms and rationalizing. The general case can be taken on faith.

Suppose that $s > 1$ then,

$$h(s) = \zeta(s) - 2^{1-s}\zeta(s)$$

so that

$$\zeta(s) = h(s)/(1 - 2^{1-s})$$

Since the right hand side makes sense even when $0 < s < 1$ it can be taken as the definition of $\zeta(s)$ for these values !

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The Riemann Hypothesis

In 1859 there appeared Riemann's epoch making memoir on the zeta function where for the first time the zeta function was studied as a function of the complex variable s. The function makes sense everywhere except at $s = 1$ where it has a singularity and

$$\zeta(s) \sim \frac{1}{s-1}, \quad s \to 1.$$

Euler's proof of the infinitude of primes essentially is equivalent to the fact that the zeta function becomes infinite at $s = 1$.

Riemann made a careful study of the zeta function proving many interesting results, sketching some results and a famous unproved conjecture. Over the next thirty years there was a deafening silence followed by a flurry of papers by Mertens, Mangoldt and many others. H. M. Edwards in his famous book on the Riemann's zeta function remarks that "it appears as though it took thirty years for the world to digest Riemann's ideas". Riemann work ulimately made it made it possible to establish the Prime Number Theorem. In fact the Prime Number Theorem is completely equivalent to the fact that

$$\zeta(1+it) \neq 0, \quad t \in \mathbb{R}.$$

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Riemann conjectured that all the zeros of the zeta function lying in the right half plane actually lie on the vertical line $\mathrm{Re}\, s = 1/2$, known as the critical line. The conjecture remains unproved and is one of the most famous unsolved problems in mathematics.

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The speaker wishes good luck to those in audience whose desire happens to be whetted to solve this problem. Thank You !