

Free groups, covering spaces and Artin's theorem

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1 Introduction:

One of the charming experiences from a first encounter with algebraic topology, in particular the theory of covering spaces, is the topological proofs of several important theorems in group theory. Massey [4] devotes his last chapter to some of these as a dénouement of the theory of fundamental groups and covering spaces. The theorem alluded to in the title of this note is the following unpublished result of Emil Artin.

Theorem 1 (Artin:) The commutator subgroup of the free group on two generators is not finitely generated.

This is a result that surely springs a surprise and belongs to the family of results in combinatorial group theory for which topological proofs as well as algebraic proofs are available. The latter usually employs the method of Schreier transversals (see [7], p. 163). The book of John Stillwell [10] is an excellent account describing the close interactions between combinatorial group theory and topology. An authoritative historical account is the book of B. Chandler and W. Magnus [1] (see particularly pages 96-97). John Stillwell [10] attributes the result to Emil Artin and provides a topological proof on page 101 employing the infinite grid

$$G = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$$

as a covering space (the universal abelian cover) for the wedge of two circles $S^1 \vee S^1$. A proof is also available in [2] (p. 175). The present note provides yet another very short and crisp topological argument, also employing the infinite grid but takes a somewhat different route than the above mentioned proofs which the author believes would be of some value. As a further simplification we avoid in this note the use of infinite graphs and their spanning trees.

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2 Free groups and covering spaces:

For the benefit of the reader let us recapitulate some of the notions used in this note. If V is a vector space, a subset $S \subset V$ is a basis of V if the following two properties are satisfied:

- (i) For every vector space W , every map $f : S \rightarrow W$ extends as a linear transformation $F : V \rightarrow W$ namely, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & V \\ & \searrow f & \swarrow F \\ & & W \end{array}$$

where, $\iota : S \rightarrow V$ is the inclusion map.

- (ii) The linear extension F in (i) is unique.

It is easy to see that this definition is equivalent to the standard definition given in elementary linear algebra courses. Indeed if S fails to be linearly independent, the linear extension F in (i) may not exist while if S fails to generate V then the map F in (i) (if it exists) would not be unique. The above definition though more abstract, has the advantage that it readily generalizes yielding the notion of a free group and a basis for a free group.

Definition 1: A group G is said to be free with basis $S \subset G$ if

- (i) For every group H , every map $f : S \rightarrow H$ extends as a group homomorphism $F : G \rightarrow H$ namely, $F \circ \iota = f$. Again with $\iota : S \rightarrow G$ denoting the inclusion, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & G \\ & \searrow f & \swarrow F \\ & & H \end{array}$$

- (ii) The extension F in (i) is unique.

Remark: If we replace every occurrence of the word “group” by “abelian group” we get the notion of a free abelian group G with basis S . In the context of free abelian groups we have:

Theorem 2: Suppose G is a free abelian group then,

- (i) Any two bases of G have the same cardinality and the common value is known as the rank of the free abelian group.
- (ii) A subgroup of a free abelian group is again free abelian.
- (iii) If G and H are free abelian and $G \rightarrow H$ is a surjective group homomorphism then rank of H cannot exceed the rank of G .

Proof: See [7] (pp. 100-101) for details on (i) and (ii). We remark that the proof proceeds by considering $G \otimes \mathbb{Q}$ as a \mathbb{Q} vector space. The map $g \mapsto g \otimes 1$ is a monomorphism of G into $G \otimes \mathbb{Q}$ and if S is a basis of G then the set $S^* = \{s \otimes 1 : s \in S\}$ is a basis of the vector space $G \otimes \mathbb{Q}$. From this one easily deduces (iii) or use exercise 7(i) on p. 105 of [7]. \square

If G is any group denote by $[G, G]$ its commutator subgroup and by $A(G)$ its abelianization given by $A(G) = G/[G, G]$ and $\eta : G \rightarrow A(G)$ the quotient map. We now prove that $A(G)$ is a free abelian group if G is a free group.

Theorem 3: Suppose G is a free group with basis S , then

- (i) $A(G)$ is a free abelian group with basis $\eta(S)$.
- (ii) The restriction $\eta|_S$ is injective.
- (iii) Any two bases of G have the same cardinality. The common value is called the rank of the free group.

Proof: From (i) and (ii) we immediately get (iii) upon invoking theorem 2. One can also give a direct proof [7] (p. 50, exercise 7). Let us quickly dispose off (ii). If $\eta(s_1) = \eta(s_2)$ ($s_1 \neq s_2$) then $s_1^{-1}s_2$ lies in the commutator subgroup $[G, G]$. We take $H = \mathbb{Z}$ and declare $f(s_1) = 1$, $f(s) = 0$ if $s \neq s_1$ and $s \in S$. Then $f : S \rightarrow \mathbb{Z}$ extends as a group homomorphism $F : G \rightarrow \mathbb{Z}$ and the target group being abelian (regarded additively), F must map $[G, G]$ to the trivial element $\{0\}$. Hence $F(s_1^{-1}s_2) = 0$ or $F(s_1) = F(s_2)$, which is a contradiction since $F(s_1) = f(s_1) = 1$ and $F(s_2) = f(s_2) = 0$.

Turning now to the proof of (i), let H be an abelian group and $f : \eta(S) \rightarrow H$ be any map. To show that f extends to a group homomorphism $F : A(G) \rightarrow H$, consider the map $f \circ \eta : S \rightarrow H$ which must extend as a group homomorphism $\phi : G \rightarrow H$. Now since H is abelian, by the universal property of quotient there is a unique group homomorphism $F : A(G) \rightarrow H$ such that $F \circ \eta = \phi$. Evaluating at a typical element of S we get

$$F(\eta(s)) = \phi(s) = f(\eta(s))$$

showing that F indeed does the job. \square

From the theory of covering spaces we need theorem 4 below. It is well known that if $q : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering projection the induced map $q_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is a monomorphism and the index of the subgroup $q_*(\pi_1(\tilde{X}, \tilde{x}_0))$ equals the cardinality of the fiber $q^{-1}(x_0)$.

Theorem 4: Let $q : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering projection between connected, locally path connected spaces. Then the following are equivalent:

- (i) $q_*\pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$.
- (ii) The group Deck(\tilde{X}, X) of deck transformations acts transitively on the fibers.

(iii) If γ is a loop in X based at x_0 then either every lift of γ is closed or none of the lifts of γ is closed.

Further, when these conditions hold we have the following important relation:

$$\text{Deck}(\tilde{X}, X) = \pi_1(X, x_0)/q_*(\pi_1(\tilde{X}, \tilde{x}_0)) \quad (1)$$

Proof: See [4] (p. 153 and p. 167). □

Definition 2: A covering projection satisfying any of the above equivalent conditions is called a regular covering.

We shall now consider the covering map $p : G \longrightarrow S^1 \vee S^1$ is given by

$$p(x, y) = (\exp(2\pi ix), \exp(2\pi iy)). \quad (2)$$

This covering appears in [6] and [8] to illustrate that the fundamental group of $S^1 \vee S^1$ is non-abelian. It is clear that the deck transformations of (2) are the translations:

$$T_{(m,n)}(x, y) = (x + m, y + n), \quad m, n \in \mathbb{Z}$$

whereby the group of deck transformations is $\mathbb{Z} \oplus \mathbb{Z}$. Turning to Artin's theorem, we denote by F_k the free group on k generators and C_k its commutator subgroup.

Theorem 5: (i) The covering p given by (2) is regular and $p_*(\pi_1(G))$ is the commutator subgroup of F_2 , the fundamental group of $S^1 \vee S^1$.

(ii) $p_*(\pi_1(G))$ is not finitely generated.

Proof: (i) Since the group of deck transformations of the covering is $\mathbb{Z} \oplus \mathbb{Z}$ and acts transitively on the fibers, the covering is regular and

$$\pi_1(S^1 \vee S^1)/p_*(\pi_1(G)) = F_2/p_*(\pi_1(G)) = \mathbb{Z} \oplus \mathbb{Z}. \quad (3)$$

By virtue of (3) we conclude that $C_2 \subset p_*(\pi_1(G))$. However since F_2/C_2 is also $\mathbb{Z} \oplus \mathbb{Z}$, we get by the third isomorphism theorem, $p_*(\pi_1(G)) = C_2$. Indeed, since p_* is injective, it establishes an isomorphism between $\pi_1(G)$ and C_2 .

(ii) Let $G_n = G \cap ([0, n] \times [0, n])$, the truncated grid, whose fundamental group is F_{n^2} . There is an obvious retraction from G onto G_n and so a surjection $r_* : \pi_1(G) \longrightarrow \pi_1(G_n)$. If $\pi_1(G)$ were finitely generated with say k generators, we would have a surjection $s : F_k \longrightarrow \pi_1(G)$ thereby providing a surjection

$$r_* \circ s : F_k \longrightarrow F_{n^2}.$$

On abelianizing, we get an epimorphism $A(F_k) \longrightarrow A(F_{n^2})$ which, in view of theorem 2(iii), is contradiction since n is arbitrary. □

The first homology group $H_1(G)$ is the free abelian group with countably infinite rank and $H_1(S^1 \vee S^1)$ is the free abelian group of rank two. The induced map $H_1(p) : H_1(G) \longrightarrow H_1(S^1 \vee S^1)$ is trivial in stark contrast with p_* which is a monomorphism.

Corollary 6: The induced map in homology $H_1(p) : H_1(G) \longrightarrow H_1(S^1 \vee S^1)$ is trivial.

Proof: Recall that the Poincaré-Hurewicz map $\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X)$ is a surjective group homomorphism with kernel as the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$. Further, Π_X is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(G, x_0) & \xrightarrow{p_*} & \pi_1(S^1 \vee S^1, (1, 1)) \\ \Pi_G \downarrow & & \downarrow \Pi_{S^1 \vee S^1} \\ H_1(G) & \xrightarrow{H_1(p)} & H_1(S^1 \vee S^1) \end{array}$$

Since by theorem 5 the image of p_* is precisely the commutator subgroup, the composite $\Pi_{S^1 \vee S^1} \circ p_*$ is the zero map whereby $H_1(p) \circ \Pi_G$ is the zero map and Π_G being surjective, we infer $H_1(p)$ vanishes. \square

3 Connections with complex analysis:

The definition of basis of a free group mirrored the notion of basis of a vector space but the analogy quickly breaks down. If W is a vector subspace of V , the dimension of W does not exceed the dimension of V but the corresponding result for free groups fails rather spectacularly as Artin's theorem shows. Massey [6] (p. 202) has illustrated via covering spaces that the free group on seven generators embeds in the free group on two generators. We shall provide here an example of a covering projection which shows that the free group on four generators embeds in the free group on two generators. This example incidentally is an irregular covering and comes from the theory of Riemann surfaces [8] and [3] (p. 39).

Theorem 7: The map $p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$ given by

$$p(w) = w^3 - 3w$$

is a three sheeted irregular covering projection.

Proof: The derivative $p'(w) = 3(w^2 - 1)$ vanishes precisely at $w = \pm 1$ and so the inverse function theorem says that p is a local homeomorphism on $\mathbb{C} - \{\pm 1\}$. Now $p(1) = -2$ and $p(-1) = 2$ and so the equation $w^3 - 3w = z$ has a double root precisely when $z = \pm 2$ and three distinct roots for all other values of z . The removal of the points $z = \pm 2$ from the target space and the corresponding removal of the four points $p^{-1}\{\pm 2\}$ from the domain now ensures that the cardinality of the fibers $p^{-1}(z)$ is three throughout $z \in \mathbb{C} - \{\pm 2\}$. The map $p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$ is also surjective and a proper map and in fact it is a covering projection [4] (pp. 127-128). Let us determine its group of deck transformations.

Each deck-transformation being a lift of the holomorphic function p , is itself a holomorphic function $\phi : \mathbb{C} - \{\pm 1, \pm 2\} \rightarrow \mathbb{C} - \{\pm 1, \pm 2\}$ satisfying

$$(\phi(z))^3 - 3\phi(z) = z^3 - 3z. \tag{4}$$

We see that $\phi(z)$ must be bounded in a neighborhood of the punctured points $\pm 1, \pm 2$. Riemann's removable singularities theorem now implies that $\phi(z)$ extends as an entire function. One can now factorize¹ equation (4) to conclude that $\phi(z) = z$ is the only holomorphic possibility. A more interesting approach would be to observe that $|\phi(z)| = O(|z|)$ for large $|z|$ whereby $\phi(z) = a + bz$. The values of a and b are easily determined as being 0 and 1 respectively. In any case the group of deck-transformations is trivial and by virtue of theorem 4(ii), the covering is irregular. \square

Theorem 8 (Nielsen-Schreier): If G is a free group then all its subgroups are also free. If G has rank k and H is a subgroup of G of rank l then,

$$l = (k - 1)[G : H] + 1 \tag{5}$$

We mention here that Jakob Nielsen proved the first part of the theorem [1] (p. 84) when the subgroup is finitely generated² and was generalized by Otto Schreier who also gave the formula stated in the theorem [1] (pp. 96-97). The proof via covering spaces is available on p. 204 of [6]. If we have a covering projection $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ for which $\pi_1(X, x_0)$ is a free group, then we may take $G = \pi_1(X, x_0)$, $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and the index $[G : H]$ is then the cardinality of the fiber $p^{-1}(x_0)$. Recalling that the fundamental group of the plane punctured at k points is precisely F_k , we have $k = 2$ and $l = 4$ for the covering projection of theorem 7. Nielsen-Schreier formula gives the value of the index $[G : H]$ as three which is precisely the cardinality of the fibers. One can of course take up more exotic examples such as $p(w) = w^5 - 5w$ in lieu of the one in theorem 7 for which the numbers are $k = 4$, $l = 16$ and $[G : H] = 5$.

For the example in Massey's book (p. 202) the Nielsen-Schreier formula predicts that the covering must be six sheeted which is indeed the case. For the infinite grid we have $[G : H] = \infty$ and the Nielsen-Schreier formula predicts that the subgroup H is not finitely generated.

We conclude this note by drawing attention to the close analogy between the Nielsen-Schreier formula and the Riemann-Hurwitz formula for unbranched finite coverings of a compact Riemann surface [3] (p. 140). These connections are developed in [5] (p. 133 ff.).

¹This is possible only because of the fortuitous circumstance of obtaining a quadratic for $\phi(z)$

²The authors of [1] mention on p. 84 that prior to Nielsen, Max Dehn gave the topological proof of the fact that a subgroup of a free group is free.

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