Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (May 10, 2019)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice:

Math	Stat

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks. Some of the questions have two parts of 1 mark each. There will be no partial credit.*
- *Simplify all your answers. In particular, the answer should not be in the form of a sum or product.*
- *The answer to each question is a number (or a tuple of numbers), a set, or a function. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*
- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*
- *Only the question paper will be graded. Write only the final answers on the question paper at the space provided below the questions. You are being given a separate work-sheet for solving the problems. This work sheet will not be graded.*
- *If your score at least 16 marks in the written test, then you will surely be called for the interview. Final selection will be based on written test marks and interview marks.*

Probability - 6 Questions

1. Let X and Y be independent and identically distributed random variables with common probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = \min(X, Y)$. Then $E(U)$ is ____________________.

2. Let U_1, U_2, and U_3 be independent and identically distributed copies of the random variable Uniform(0,θ) for $\theta > 0$. Then $P[\max(U_1, U_2, U_3) > \theta/2]$ is equal to ____________________.

3. Let U be a Uniform(0,1) random variable. Given $U = u$, the random variable X is Poisson(u). Then, $E(X) =$____________________.

4. Assume that X_1, X_2, X_3 are discrete random variables defined on a common probability space Ω and taking values in $\{-1, 1\}$. Further, assume that $E(X_1) = E(X_2) = E(X_3) = E(X_1X_2) = E(X_2X_3) = E(X_3X_1) = 0$. Given this, what is the maximum possible value of $E(X_1X_2X_3)$?

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5. For each $n \geq 1$, assume we have discrete random variables $X_1, \ldots, X_n$ that are independent and identically distributed; further, each X_i is uniform over the set $\{-1, 1\}$. Let $p_n = P[|\sum_{i=1}^{n} X_i| \geq (n/3)]$. Then, $\lim_{n\to\infty} p_n =$____________________.

6. Let X be a random variables with probability density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

and let $Y = aX + b$ where $a, b > 0$. Let $M_Y(t)$ denote the moment generating function of Y. The domain of convergence of M_Y is ____________________.

Linear Algebra - 8 Questions

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 6 & 7 \end{pmatrix}$, the set of all $x \in \mathbb{R}^3$ satisfying $Ax = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$ is given by the equation(s):

   ____

   A vector $b \in \mathbb{R}^3$ such that $Ax = b$ is <u>inconsistent</u> (i.e. has no solution) is:

   ____.

8. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$, and V be a 4-dimensional vector space over $\mathbb{F}$.

   The number of elements in V is ________________________, and the number of 4×4 invertible matrices with entries in $\mathbb{F}$ is ________________________.

9. Let Id_5 is the 5×5 identity matrix, and A be a 5×5 real matrix with eigenvalues 0, 3, and 4, and minimal polynomial $x^2(x-3)(x-4)$.

   The eigenvalues and minimal polynomial of $B = A - 3\mathrm{Id}_5$ are

   ________________________ and

   ____ respectively.

10. A 3×3 orthogonal matrix A whose first two columns are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ is:

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

11. Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ k & 1 & 0 \\ 5 & k-2 & 1 \end{pmatrix}.$$

If A is diagonalizable, then $k =$__________________________.

12. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & a \end{pmatrix}.$$

For which values of a is the matrix A diagonalizable *over* $\mathbb{R}$?

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13. Find a 2×2 matrix A whose eigenvalues are 1 and 4, and whose eigenvectors are

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

respectively.

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14. Let $\mathbb{F}$ be an algebraically closed field of characteristic 5. Take the 5×5 matrix over $\mathbb{F}$ all whose entries are 1. Its Jordan Canonical form is

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Real Analysis - 6 Questions

15. Given that for certain constants A and B,

$$A\Gamma(x) = B^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\frac{1}{2}\right), \quad \text{for all } x > 0,$$

where the function $\Gamma(x)$ denotes the Gamma function.

Then $A =$ ___________ and $B =$ ____________.

16. Given $f(x) = \tan^{-1}\exp(-x - x^{-1})$ on $x > 0$, the image of the function f is

______________________________.

17. Find all real numbers c such that the equation $x^5 - 5x = c$ has three distinct real roots.

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18. Let $\{a_n\}$ be defined as follows:

$$a_1 > 0, \quad a_{n+1} = \ln\frac{e^{a_n}-1}{a_n} \quad \text{for} \quad n \geq 1.$$

Then the sum

$$\sum_{n=1}^{\infty} a_1 a_2 \cdots a_n$$

is ________________.

(Hint: $\{a_n\}$ is a strictly decreasing sequence of positive terms converging to 0).

19. The domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n4^n}{3^n} x^n (1-x)^n$$

is __________________________.

20. If f is the function defined by

$$f(t) = \begin{cases} \frac{1}{t \ln 2} - \frac{1}{2^t - 1}, & t \neq 0 \\ \frac{1}{2}, & t = 0, \end{cases}$$

then the derivative $f'(0)$ of f at 0 is __________________.