1. Introduction

Our objective in these notes is to present Stanley’s solution of the Anand-Dumir-Gupta (ADG) conjecture concerning enumeration of doubly stochastic matrices or magic squares. Let \( \mathbb{N} \) denote the set of nonnegative integers and let \( \mathbb{P} \) denote the set of positive integers. An \( n \times n \) matrix \( M \) is called a magic square if its entries are in \( \mathbb{N} \) and the sum of entries in any row or column is a given integer \( r \). The number \( r \) is called the line sum of \( M \). It is clear that 
\[
H_1(r) = 1 \quad \text{and} \quad H_2(r) = r + 1.
\]
MacMahon \cite{2} and independently Anand-Dumir-Gupta \cite{1} showed that the number of \( 3 \times 3 \) magic squares with line sum \( r \) is given by
\[
H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.
\]
Inspired by these formulas they proposed the following conjectures in 1966. \cite{1}:

**Conjecture 1.1 (Anand-Dumir-Gupta).** Fix \( n \geq 1 \). Then

1. \( H_n(r) \in \mathbb{C}[r] \).
2. \( \deg H_n(r) = (n-1)^2 \).
3. \( H_n(i) = 0 \) for \( i = -1, -2, \ldots, -(n-1) \).
4. \( H_n(-n-r) = (-1)^{n-1}H_n(r) \) for all \( r \).

We will see that the above four assertions about \( H_n(r) \) are equivalent to the following:
\[
\sum_{r=0}^{\infty} H_n(r)\lambda^r = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^{(n-1)^2+1}},
\]
where \( h_0, h_1, \ldots, h_d \) are integers, \( d = (n-1)^2 + 1 - n \), \( h_0 + h_1 + \cdots + h_d \neq 0 \) and \( h_{d-i} = h_i \) for \( i = 0, 1, \ldots, d \). Stanley made the additional conjectures that

1. \( h_i \geq 0 \) for all \( i \) and
2. \( h_0 \leq h_1 \leq \cdots \leq h_{[d/2]} \).

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Stanley settled (1)-(5) in 1973 [4]. A geometric proof based on Ehrhart polynomials of integral polytopes appears in Stanley’s Red Book [5]. The conjecture (6) is still open.

2. Linear homogeneous Diophantine equations

Let $x_{ij}$; $i, j = 1, 2, \ldots, n$ be indeterminates. The entries of an $n \times n$ magic square are solutions to the following system of linear homogeneous Diophantine equations:

\[
\begin{align*}
    x_{11} + x_{12} + \cdots + x_{1n} &= \sum_{j=1}^{n} x_{ij} \text{ for } i = 2, 3, \ldots, n. \\
    x_{11} + x_{12} + \cdots + x_{1n} &= \sum_{i=1}^{n} x_{ij} \text{ for } j = 2, 3, \ldots, n.
\end{align*}
\]

Thus the problem of counting magic squares is a special case of counting nonnegative integer solutions of a system of linear Diophantine equations. Let $\Phi$ be an $r \times n \mathbb{Z}$-matrix. Let $x_1, x_2, \ldots, x_n$ be indeterminates. Let $X$ denote the column vector $(x_1, x_2, \ldots, x_n)^t$. We are interested in the $\mathbb{N}$-solutions to the system $\Phi X = 0$. We gather all the solutions in the semigroup

\[ E_{\Phi} = \{ \beta \in \mathbb{N}^n : \Phi \beta = 0 \}. \]

Let $k$ be any field. For $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^t$, put $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$. With $E_{\Phi}$ we can associate the semigroup ring

\[ R_{\Phi} = k[x^\beta : \beta \in E_{\Phi}]. \]

Stanley studied the semigroup ring $R_{\mu}$ where $\mu$ is the $(2n - 2) \times n^2$ coefficient matrix of the system (1). In particular he showed that the ring $R_{\mu}$ is Gorenstein and calculated its canonical module and thus its $a$-invariant. We shall see that these observations are enough to settle the conjectures (1)-(5). Let us begin by observing the

**Theorem 2.1.** The semigroup ring $R_{\Phi}$ is a finitely generated $k$-algebra.

**Proof.** Let $I$ denote the ideal in $R = k[x_1, x_2, \ldots, x_n]$ generated by the set

\[ P = \{ x^\beta : 0 \neq \beta \in E_{\Phi} \}. \]

Since $R$ is Noetherian, $I$ is generated by a finite subset $G = \{ x^{\delta_1}, x^{\delta_2}, \ldots, x^{\delta_t} \}$ of $P$. We claim that

\[ R_{\Phi} = k[x^\delta : x^\delta \in G]. \]

Indeed, Any $x^\beta \in R_{\Phi}$ can be written as $x^\beta = x^{\delta_i} x^\gamma$ for some $i$ and $x^\gamma \in R$. Thus $\gamma = \beta - \delta_i \in E_{\Phi}$. The argument can be repeated for $x^\gamma$, eventually yielding an expression for $x^\beta$ in terms of $x^{\delta_i}$ for $i = 1, 2, \ldots, t$. \qed

As far as the structure of $R_{\mu}$ is concerned, we have more precise information due to
Theorem 2.2 (Birkhoff-von Neumann Theorem). Every $n \times n$ magic square is an $\mathbb{N}$-linear combination of the $n \times n$ permutation matrices.

Thus $R$ is generated by $n!$ degree $n$ monomials. Let $[R_\mu]_r$ denote the $k$-subspace of $R_\mu$ generated by the monomials of degree $nr$. These monomials are in one-to-one correspondence with magic squares of line sum $r$. Moreover, $R_\mu = \bigoplus_{r=0}^\infty [R_\mu]_r$. Thus

$$H(R_\mu, r) = \dim_k [R_\mu]_r = H_n(r).$$

This observation will eventually lead to the conclusion that $H_n(r)$ is a polynomial in $r$ for all $r$. But for the time being we can see that it is so for all large values of $r$ in view of the Hilbert-Serre theorem.

Lemma 2.3. If $\Phi X = 0$ has a positive solution, then $\dim R_\Phi = n - \text{rank} \Phi$.

Proof. We show that the vectors $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are $\mathbb{Q}$-linearly independent if and only if $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ are algebraically independent over $k$. Suppose $\beta_1, \beta_2, \ldots, \beta_d \in E_\Phi$ are linearly independent over $\mathbb{Q}$. Let

$$\sum_\alpha a_\alpha (x^{\beta_1})^{\alpha_1} (x^{\beta_2})^{\alpha_2} \ldots (x^{\beta_d})^{\alpha_d} = 0,$$

for certain $a_\alpha \in k$ and distinct vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. Since $\beta_1, \beta_2, \ldots, \beta_d$ are linearly independent over $\mathbb{Q}$, the vectors $\alpha_1 \beta_1 + \cdots + \alpha_d \beta_d$ are distinct. Hence $a_\alpha = 0$ for all $\alpha$.

Conversely let $x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_d}$ be algebraically independent over $k$. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}$ such that $\alpha_1 \beta_1 + \cdots + \alpha_d \beta_d = 0$. Without loss of generality we may assume that $\alpha_1, \alpha_2, \ldots, \alpha_p > 0$ and $\alpha_{p+1}, \ldots, \alpha_d < 0$. Then

$$\alpha_1 \beta_1 + \cdots + \alpha_p \beta_p = \alpha_{p+1} \beta_{p+1} + \cdots + \alpha_d \beta_d.$$

This yields the algebraic dependency relation $x^{\alpha_1 \beta_1} \cdots x^{\alpha_p \beta_p} = x^{\alpha_{p+1} \beta_{p+1}} \cdots x^{\alpha_d \beta_d}$.

Let $\alpha \in \mathbb{P}^n \cap E_\Phi$. Let $d = n - \text{rank} \Phi$. Pick linearly independent solutions $\beta_1, \beta_2, \ldots, \beta_d \in \mathbb{Z}^n$ of $\Phi X = 0$. Let $t \in \mathbb{Q}_+$. If $\alpha - t \beta_1, \alpha - t \beta_2, \ldots, \alpha - t \beta_d$ are linearly dependent over $\mathbb{Q}$, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Z}$, not all zero such that

$$a_1 (\alpha - t \beta_1) + a_2 (\alpha - t \beta_2) + \cdots + a_d (\alpha - t \beta_d) = 0.$$

We have unique rational numbers $b_1, b_2, \ldots, b_d$ such that $\alpha = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_d \beta_d$. Put $\alpha = \sum_{i=1}^d a_i$. Then $\sum_{i=1}^d (ab_i - ta_i) \beta_i = 0$. Let $a_p \neq 0$. Then $t = ab_p/a_p$. Hence by selecting $t \in \mathbb{Q}_+$ sufficiently small, we get a contradiction. This proves that $\delta_1 = \alpha - t \beta_1, \delta_2 = \alpha - t \beta_2, \ldots, \delta_d = \alpha - t \beta_d$ are linearly independent solutions in $\mathbb{P}^n$. Hence $x^{\delta_1}, \ldots, x^{\delta_d}$ are algebraically independent elements of $R_\Phi$.

Corollary 2.4. The function $H_n(r)$ is a polynomial in $r$ for large $r$ of degree $(n - 1)^2$.

Proof. The ring $R_\mu$ is a standard graded $k$-algebra. The $r^{th}$ graded component of it is generated by monomials of degree $nr$ corresponding to magic squares of line sum $r$. Hence $H_n(r)$ is a polynomial for large $r$. We show that

$$\dim R_\mu = (n - 1)^2 + 1.$$
By the above lemma, \( \dim R_\mu = \text{nullity } \mu \). Note that to construct a magic square, we may assign any nonnegative values to the variables \( x_{ij} \), for \( i, j = 1, 2, \ldots, (n - 1) \) and a value for \( x_{1n} \) will determine the rest of the entries. Thus \( \text{nullity } \mu = (n - 1)^2 + 1 \).

\[ \square \]

**Proposition 2.5** (MacMahon [2], Anand-Dumir-Gupta [1]). The number of \( 3 \times 3 \) magic squares with line sum \( r \) is given by

\[
H_3(r) = \binom{r + 4}{4} + \binom{r + 3}{4} + \binom{r + 2}{4}.
\]

**Proof.** By the above corollary, the dimension of the semigroup ring \( R \) generated over a field \( k \) by the monomials corresponding to the six \( 3 \times 3 \) permutation matrices is \( (n - 1)^2 + 1 = 5 \). Let \( S = k[y_1, y_2, \ldots, y_6] \). Put

\[
M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},
\]

\[
M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Note that \( M_1 + M_2 + M_3 = M_4 + M_5 + M_6 \). Let \( f = y_1 y_2 y_3 - y_4 y_5 y_6 \). Hence \( S/(f) \simeq R \). Therefore

\[
H(S/(f), \lambda) = (1 - \lambda^3)/(1 - \lambda)^6 = (1 + \lambda + \lambda^2)/(1 - \lambda)^5.
\]

This yields the desired formula.

\[ \square \]

### 3. Cohen-Macaulay Property of \( R_\Phi \)

In this section we show that \( R_\Phi \) is a Cohen-Macaulay ring. This is done by showing that it is a ring of invariants of an algebraic torus acting linearly on a polynomial ring. A well-known theorem of Hochster then implies that it is Cohen-Macaulay.

Write the \( r \times n \) matrix \( \Phi = [\gamma_1, \gamma_2, \ldots, \gamma_n] \) where \( \gamma_i \) is the \( i \)th column vector of \( \Phi \). Let \( k^\times \) denote the multiplicative group of \( k \). Consider the algebraic torus

\[
T = \{ \text{diag}(u^{\gamma_1}, u^{\gamma_2}, \ldots, u^{\gamma_n}) : u = (u_1, u_2, \ldots, u_r) \in (k^\times)^r \}.
\]

\( T \) acts on \( R = k[x_1, x_2, \ldots, x_n] \) via the automorphisms \( \tau_u : x_i \mapsto u^{\gamma_i} x_i, \quad i = 1, 2, \ldots, n \). Let \( \beta \in \mathbb{N}^n \). Then

\[
\tau_u(x^{\beta}) = (u^{\gamma_1} x_1)^{\beta_1} (u^{\gamma_2} x_2)^{\beta_2} \cdots (u^{\gamma_n} x_n)^{\beta_n} = u^{\beta_1 \gamma_1 + \beta_2 \gamma_2 + \cdots + \beta_n \gamma_n} x^{\beta}.\]
Hence $\tau_u(x^\beta) = x^\beta$ if and only if $\beta \in E \Phi$. Hence $R \Phi$ is the ring of invariants of the torus $T$ acting linearly on $R$. By Hochster’s theorem $[3]$ $R \Phi$ is Cohen-Macaulay.

We can now dispose the conjecture (5) of Stanley. Since $R_{\mu}$ is Cohen-Macaulay homogeneous ring of dimension $d = (n - 1)^2 + 1$, there exists an h.sop $a$ for $R_{\mu}$ of elements of degree one. Hence
\[
F(R_{\mu}, \lambda) = \frac{F(R_{\mu}/(a), \lambda)}{(1 - \lambda)^d}.
\]
Hence the numerator of the above Hilbert series is a polynomial with positive coefficients.

4. Macaulay’s theorem for Gorenstein graded rings

The purpose of this section is to recall the basic definitions and facts about Gorenstein graded rings and provide a proof of Macaulay’s theorem concerning their Hilbert series.

Let $R$ be an $\mathbb{N}$-graded ring. Let $\mathcal{M}$ be the category of $\mathbb{Z}$-graded $R$-modules. Let $M = \bigoplus M_n$ and $N = \bigoplus N_n \in \mathcal{M}$. An $R$-linear map $f : M \to N$ is a morphism in $\mathcal{M}$ if $f(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. By $M(n)$ we mean the module $M$ with grading defined by $[M(n)]_m = M_{m+n}$ for all $m \in \mathbb{Z}$. Put
\[
*\text{Hom}(M, N)_n = \{ f : M \to N(n) \} \quad \text{and} \quad *\text{Hom}(M, N) = \bigoplus_{n \in \mathbb{Z}} *\text{Hom}(M, N)_n.
\]
It is easy to check that if $M$ is finitely generated then $*\text{Hom}(M, N) = \text{Hom}(M, N)$.

**Proposition 4.1.** Let $A = k[x_1, x_2, \ldots, x_s]$ be polynomial ring over a field $k$. Let $I$ be a homogeneous ideal of $A$. Let $A/I$ be Cohen-Macaulay. Then
\[
\text{Ext}^i(A/I, A) \neq 0 \iff i = h = \text{ht}(I).
\]

**Proof.** By Auslander-Buchsbaum formula $\text{pd}(A/I) = \text{depth } A - \dim A/I = s - (s - h) = h$. Write a graded minimal resolution of $A/I$ as an $A$-module:
\[
0 \to A^{\delta_h} \to A^{\delta_{h-1}} \to \cdots \to A^{\delta_1} \to A \to A/I \to 0.
\]
Thus $\text{Ext}^i(A/I, A) = 0$ for $i > h$. Since $A$ is Cohen-Macaulay, $\text{Ext}^i(A/I, A) = 0$ for $i < h$. \hfill $\square$

**Definition 4.2.** The $A$-module $K_A/I = \text{Ext}^h(A/I, A)$ is called the canonical module of $A/I$. The ring $A/I$ is called Gorenstein if $K_A/I \simeq A/I(a)$, for some $a \in \mathbb{Z}$. The integer $a$ is called the $a$-invariant of $A/I$.

**Theorem 4.3.** Put $R = A/I$ and $d = \dim(R)$. Let the degree of $x_i = e_i \in \mathbb{P}$ for $i = 1, 2, \ldots, s$. Then as rational functions of $\lambda$
\[
F(K_R, \lambda) = (-1)^d F(R, 1/\lambda) \lambda^{-\sum_{i=1}^s e_i}.
\]

**Proof.** Write a minimal free resolution of $R$ as an $A$-module:
\[
0 \to M_h \xrightarrow{\phi_h} M_{h-1} \xrightarrow{\phi_{h-1}} \cdots \to M_1 \xrightarrow{\phi_1} M_0 \xrightarrow{\phi_0} R \to 0.
\]
Apply Hom(−, A) to the above resolution to get the complex:

\[ 0 \longrightarrow \text{Hom}(M_0, A) \xrightarrow{\phi_0^*} \text{Hom}(M_1, A) \xrightarrow{\phi_1^*} \cdots \xrightarrow{\phi_h^*} \text{Hom}(M_h, A) \longrightarrow 0. \]

Thus \( K_R \cong \text{Hom}(M_h, A)/\text{Im}(\phi_h^*). \) Hence we have the following minimal free resolution for \( K_R \) as an \( A \)-module:

\[ 0 \longrightarrow \text{Hom}(M_0, A) \xrightarrow{\phi_0^*} \cdots \xrightarrow{\phi_h^*} \text{Hom}(M_h, A) \longrightarrow K_R \longrightarrow 0. \]

It is easy to see that for integers \( m \) and \( n, \)

\[ F(M(n), \lambda) = \lambda^{-n} F(M, \lambda) \quad \text{and} \quad \text{Hom}(A(m), A) \cong A(-m). \]

Let \( \text{rank}(M_i) = \beta_i \) and \( M_i = \bigoplus_{j=1}^{\beta_i} A(-g_{ij}) \) for \( i = 0, 1, \ldots, h. \) Put \( D(\lambda) = \prod_{p=1}^{s}(1 - \lambda^{e_p}) \) and \( N_i(\lambda) = \sum_{j=1}^{\beta_i} \lambda^{g_{ij}}. \) Now we calculate the Hilbert series of \( R \) and \( K_R \) from their minimal free resolutions written above. Put \( e = \sum_{i=1}^{s} e_i. \)

\[ F(M_i, \lambda) = \sum_{j=1}^{\beta_i} F(A(-g_{ij}), \lambda) = \frac{\sum_{j=1}^{\beta_i} \lambda^{g_{ij}}}{\prod_{p=1}^{s}(1 - \lambda^{e_p})} = \frac{N_i(\lambda)}{D(\lambda)}. \]

Hence \( F(R, \lambda) = \sum_{i=0}^{h} N_i(\lambda)/D(\lambda)(-1)^i. \) To find \( F(K_R, \lambda), \) note that

\[ F(K_R, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} F(M_i^*, \lambda) = \sum_{i=0}^{h} (-1)^{i+h} F(A(g_{ij}), \lambda) = \sum_{i=0}^{h} (-1)^{i+h} N_i(\lambda^{-1})/D(\lambda). \]

Since \( D(\lambda^{-1}) = (-1)^s D(\lambda) \lambda^{-e}, \) we get

\[ F(R, 1/\lambda) = \sum_{i=0}^{h} (-1)^i \frac{N_i(\lambda^{-1})}{D(\lambda^{-1})} = (-1)^{s-h} \lambda^e F(K_R, \lambda) = (-1)^d \lambda^e F(K_R, \lambda). \]

\[ \square \]

**Corollary 4.4 (Macaulay’s Theorem).** If the ring \( R = A/I \) is Gorenstein of dimension \( d \) then for some \( \sigma \in \mathbb{Z}, \)

\[ F(R, 1/\lambda) = (-1)^d \lambda^\sigma F(R, \lambda). \]

If \( R \) is standard Gorenstein with \( F(R, \lambda) = (h_0 + h_1 \lambda + \cdots + h_g \lambda^g)/(1 - \lambda)^d, \) and \( h_g \neq 0, \) then

1. \( h_i = h_{g-i}, \) for all \( i = 0, 1, \ldots, g. \)
2. \( \sigma = d - g. \)
3. If \( \sigma \geq 1, \) then \( H(n) = \dim R_n \) is a polynomial \( P(n) \) for all \( n, \)
   - (a) \( P(-i) = 0 \) for all \( i = 1, 2, \ldots, (\sigma - 1), \) and
   - (b) \( P(n) = (-1)^{d-1} P(-\sigma - n) \) for all \( n \in \mathbb{Z}. \)

**Proof.** (1) and (2): Put \( e = \sum_{i=1}^{s} e_i. \) Suppose \( R \) is Gorenstein. Then \( K_R \cong R(a), \) for some \( a \in \mathbb{Z}. \) Hence

\[ F(K_R, \lambda) = \lambda^{-a} F(R, \lambda) = (-1)^d \lambda^{-\sigma} F(R, 1/\lambda). \]

Hence \( F(R, \lambda) = \lambda^{\sigma - e}(-1)^d F(R, 1/\lambda). \) Now let \( R \) be standard Gorenstein. Write

\[ F(R, \lambda) = (h_0 + h_1 \lambda + h_2 \lambda^2 + \cdots + h_g \lambda^g)/(1 - \lambda)^d \]
where \( h_g \neq 0 \). Then
\[
F(R, 1/\lambda) = (-1)^d \lambda^{d-g}(h_0 \lambda^g + h_1 \lambda^{g-1} + \cdots + h_g)/(1 - \lambda)^d = \lambda^{-a}(-1)^{d}F(R, \lambda).
\]
Hence \( d - g = e - a = \sigma \) and \( h_i = h_{g-i} \) for all \( i = 0, 1, \ldots, g \).

(3) We know that if \( \sigma \geq 1 \), then \( H(n) \) is a polynomial for all \( n \in \mathbb{Z} \) and as \( \dim R_n = 0 \) for all \( n < 0 \), \( P(n) = 0 \) for all \( n = -1, -2, \ldots, -(\sigma - 1) \), and \( P(-\sigma) \neq 0 \). We have for all \( n \geq -(\sigma - 1) \),
\[
P(n) = h_0 \binom{n + d - 1}{d - 1} + h_1 \binom{n + d - 2}{d - 1} + \cdots + h_g \binom{d - 1 + n - g}{d - 1}.
\]
Now use the fact that \( h_i = h_{g-i} \) for all \( i = 1, 2, \ldots, g \), and \( \binom{n}{p} = (-1)^{p(p-n)} \) as polynomials,
\[
P(n) = \sum_{i=0}^{g} h_i \binom{d - 1 + n - i}{d - 1} = \sum_{i=0}^{g} h_{g-i} \binom{d - 1 + (d - 1 + n - i)}{d - 1}(-1)^{d-1} = \sum_{i=0}^{g} h_i \binom{g - i - n - 1}{d - 1}(-1)^{d-1} = \sum_{i=0}^{g} h_i \binom{d - \sigma - i - n - 1}{d - 1}(-1)^{d-1} = (-1)^{d-1}P(-\sigma - n).
\]

\[\square\]

**Definition 4.5.** The vector \((h_0, h_1, \ldots, h_g)\) is called the **h-vector** of the standard graded algebra \(R\). If the condition \( h_i = h_{g-i} \) is satisfied for all \( i = 0, i, \ldots, g \) then we say that the h-vector of \(R\) is **symmetric**.

**Example 4.6.** The symmetry of the h-vector of a standard graded Cohen-Macaulay algebra \(R\) does not imply that \(R\) is Gorenstein. We construct an example. Consider the ideal \(I = (xyz, xw, zw)\) of the polynomial ring \(A = k[x, y, z, w]\). The ideal \(I\) is generated by the maximal minors of the matrix
\[
M = \begin{bmatrix}
0 & z & x \\
-w & -yz & 0
\end{bmatrix}.
\]
A resolution of \(R = A/I\) as an \(A\)-module is:
\[
0 \longrightarrow A(-3) \oplus A(-4) \overset{f}{\longrightarrow} A(-3) \oplus A(-2)^{2} \overset{g}{\longrightarrow} A \longrightarrow R \longrightarrow 0,
\]
where the maps \(f\) and \(g\) are defined as
\[
f ([r, s]) = [r, s]M \quad \text{and} \quad g ([r, s, t]) = rxyz - sxw + tzw.
\]
It can be shown easily that the above sequence is a minimal resolution of \(R\). Hence by Auslander-Buchsbaum formula, \( \text{depth } R = \text{depth } A - \text{pd } R = 4 - 2 = 2 = \dim R \). Hence \(R\) is Cohen-Macaulay. However it is not Gorenstein as the above resolution shows that \( \text{rank } K_R = 2 \). The Hilbert series
of $R$ can be found from the resolution and it turns out to be $(1 + 2\lambda + \lambda^2)/(1 - \lambda)^2$. Hence the $h$-vector of $R$ is symmetric, although it is not Gorenstein.

**Remark:** The principal result of [6] shows that the symmetry of the $h$-vector implies Gorenstein property provided $R$ is a Cohen-Macaulay domain.

5. **A sketch of Stanley’s solution**

By Corollary 4.4, we need to show that the degree of the Hilbert series of $R_{\mu}$ is $-n$ and it is Gorenstein. By the Grothehdieck-Serre difference formula, the degree of the Hilbert series of $R_{\Phi}$ is the integer $a(R_{\Phi}) = \max\{n : H^d(R_{\Phi})_n \neq 0\}$.

**Theorem 5.1 (Stanley, [4]).**

1. $H^d(R_{\Phi}) = k[x^\beta : \beta \in E_{\Phi}, \text{ and } \beta < 0]$.  
2. $K_{R_{\Phi}} = k[x^\beta : \beta \in E_{\Phi}, \text{ and } \beta > 0]$.  
3. If $\gamma = (1,1,\ldots,1) \in E_{\Phi}$, then $K_{R_{\Phi}} = x^\gamma R_{\Phi}$. Hence in this case, $R_{\Phi}$ is Gorenstein.

For the case of magic squares, the $n \times n$ magic square $J_n$ whose each entry is 1 is the smallest positive solution and by the description of $H^d(R_{\Phi})$, the $a$-invariant of $R_{\mu}$ is $-n$. Hence The degree of its Hilbert series is $-n$. It proves that $H_n(r)$ is a polynomial for all $r > -n$. Moreover $H_n(-n) \neq 0$ and  

$$H_n(-1) = H_n(-2) = \cdots = H_n(-(n-1)) = 0.$$  

By Corollary 4.4, we conclude that  

$$H_n(r) = (-1)^{(n-1)}H_n(-r-n) = (-1)^{n-1}H_n(-r-n).$$

for all $n$.

**References**


**E-mail address:** jkv@math.iitb.ac.in