On the Number of Generators of Cohen-Macaulay Ideals

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Abstract

Several bounds on the number of generators of Cohen-Macaulay ideals known in the literature follow from a simple inequality which bounds the number of generators of such ideals in terms of mixed multiplicities. Results of Cohen and Akizuki, Abhyankar, Sally, Rees and Boratynski-Eisenbud-Rees are deduced very easily from this inequality.

1 Introduction

The objective of this note is to present a novel approach to several results for the number of generators of Cohen-Macaulay ideals in Cohen-Macaulay local rings. Let

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(R, m) be a Cohen-Macaulay local ring of dimension d. An ideal I of R is called a Cohen-Macaulay ideal if R/I is a Cohen-Macaulay ring. To state our main result we need to recall the basic notation for mixed multiplicities of ideals. Let (R, m) be a local ring. Let I be an ideal of positive height. Consider the function $C(r, s) = \ell(m^r I^s/m^{r+1} I^s)$. This function is given by a polynomial $Q(r, s)$ in two variables $r$ and $s$ for all large values of $r$ and $s$ [B]. This polynomial can be written as

$$Q(r, s) = \sum_{i+j\leq d-1} e_{ij} \binom{r+i}{i} \binom{s+j}{j}$$

where $e_{ij}$ are integers for all $i, j = 0, 1, 2, \ldots, d-1$. When $i + j = d - 1$ we write $e_{i+j} = e_j(m|I)$. These integers which appear with the monomials of degree $d - 1$ in $Q(r, s)$ are nonnegative and they are called the mixed multiplicities of $m$ and $I$. Let $\mu(I)$ denote the minimum number of generators for $I$. The principal result in this paper is the following:

**Theorem 1.1** Let (R, m) be a Cohen-Macaulay local ring of dimension d. Let I be a Cohen-Macaulay ideal of R of positive height $h$. Then for $q = 0, 1, 2, \ldots, h$,

$$\mu(I) \leq h - q + (q - 1)e(R/I) + e_{h-q}(m|I).$$

We shall recover and generalize several results known in the literature which give upper bounds for the minimum number of generators of Cohen-Macaulay ideals. This will be done quite easily by invoking the above inequality and then applying standard results about mixed multiplicities.

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## 2 Bound for $\mu(I)$ for $m$-primary ideals

The proof of theorem 1.1 is by induction on $s = \dim R/I$. When $s = 0$, the ideal $I$ is $m$-primary. Therefore we treat this case first in this section. We begin by recalling a
few facts about mixed multiplicities of ideals.

(1) Let $I$ and $J$ be $m$-primary ideals in a $d$-dimensional local ring $(R, m)$. The function $B(r, s) = \ell(R/I^rJ^s)$ is called the Bhattacharya function of $I$ and $J$. Bhattacharya [B] proved that for large values of $r$ and $s$ the Bhattacharya function is given by a polynomial $P(r, s)$ of total degree $d$ in $r$ and $s$ with rational coefficients. Moreover it can be written as

$$P(r, s) = \sum_{i+j \leq d} e_{ij} \binom{r+i}{i} \binom{s+j}{j}.$$ 

The coefficients $e_{ij}$ are integers and the ones for which $i+j = d$ are positive and they are called mixed multiplicities of $I$ and $J$. We will use the notation $e_j(I|J) = e_{ij}$ for the mixed multiplicities of $I$ and $J$.

(2) Rees [R1] showed that $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$. Here $e(.)$ denotes the Hilbert-Samuel multiplicity.

(3) Risler and Teissier [T1] provided an interpretation of the other mixed multiplicities. They showed that the $j$th mixed multiplicity $e_j(I|J)$ is the multiplicity of an ideal generated by $d-j$ elements of $I$ and $j$ elements of $J$ chosen sufficiently generally.

(4) Rees [R2] introduced the important concept of joint reductions of ideals which helps in calculation of mixed multiplicities. An ideal $K \subset J$ is called a reduction of $J$ if there exists an $n \in \mathbb{N}$ such that $KJ^n = J^{n+1}$ [NR]. Let $I_1, I_2, \ldots, I_d$ be $m$-primary ideals of $R$. A set of elements $(x_1, x_2, \ldots, x_d)$ where $x_i \in I_i$, $i = 1, 2, \ldots, d$ is called a joint reduction of the set of ideals $(I_1, I_2, \ldots, I_d)$ if the ideal $\sum_{i=1}^{i=d} x_i I_1 I_2 \ldots I_{i-1} I_{i+1} \ldots I_d$ is a reduction of $I_1 I_2 \ldots I_d$. Rees showed [R3] that if $R/m$ is infinite then joint reductions exist. The following is a crucial result in the theory of mixed multiplicities:

**Theorem 2.1 (Rees’s Mixed Multiplicity Theorem [R3])** Let $(x_1, x_2, \ldots, x_d)$ be a joint reduction of the set of ideals $(I, I, \ldots, I, J, J, \ldots, J)$ where $I$ is repeated $d - q$ times and $J$ is repeated $q$ times. then $e_q(I|J) = e((x_1, x_2, \ldots, x_d))$.

Now we prove our main theorem for $m$–primary ideals.

3
Theorem 2.2 Let \((R, m)\) be a \(d\)-dimensional Cohen-Macaulay local ring and \(I\) be an \(m\)-primary ideal of \(R\). Then for \(q = 0, 1, 2, \ldots, d\),
\[
\mu(I) \leq d - q + (q - 1) \ell(R/I) + e_{d-q}(m|I).
\]

Proof. We may assume that \(R/m\) is infinite. Let \((x_1, x_2, \ldots, x_q, a_1, a_2, \ldots, a_{d-q})\) be a joint reduction of \((m, m, \ldots, m, I, I, \ldots, I)\) where \(m\) is repeated \(q\) times and \(I\) is repeated \(d-q\) times. Let \(\underline{x}\) and \(\underline{a}\) denote the ideals \((x_1, x_2, \ldots, x_q)\) and \((a_1, a_2, \ldots, a_{d-q})\) respectively. Consider the \(R\)-module homomorphism
\[
\phi : (R/m)^{d-q} \oplus (R/I)^q \rightarrow \frac{(\underline{x}, \underline{a})}{\underline{x}I + \underline{a}m},
\]
given by
\[
\phi(y'_1, \ldots, y'_{d-q}, b'_1, \ldots, b'_q) = (y_1a_1 + \ldots + y_{d-q}a_{d-q} + b_1x_1 + \ldots + b_qx_q)',
\]
where primes denote the residue classes. Hence
\[
\ell \left( \frac{(\underline{x}, \underline{a})}{\underline{x}I + \underline{a}m} \right) \leq d - q + q \ell(R/I).
\]
But
\[
\ell \left( \frac{(\underline{x}, \underline{a})}{\underline{x}I + \underline{a}m} \right) = \ell(R/\underline{x}I + \underline{a}m) - \ell(R/(\underline{x} + \underline{a})).
\]
\[
= \ell(R/I) + \ell(I/Im) + \ell(Im/\underline{x}I + \underline{a}m) - e_{d-q}(m|I).
\]
Hence \(\mu(I) \leq d - q + (q - 1) \ell(R/I) + e_{d-q}(m|I)\).

Corollary 2.3 (Akizuki [Ak], Cohen [C]) Let \((R, m)\) be a one-dimensional Cohen-Macaulay local ring. Then for any \(m\)-primary ideal \(I\) of \(R\), \(\mu(I) \leq e(m)\).

Proof. Put \(d = q = 1\) to get \(\mu(I) \leq e_0(m|I) = e(m)\).

The next result was proved by Abhyankar [A] for the maximal ideal.
Corollary 2.4 Let \((R, m)\) be a \(d\)-dimensional Cohen-Macaulay local ring. Let \(I\) be an \(m\)-primary ideal. Then

\[ \mu(I) \leq e(I) - \ell(R/I) + d. \]

Proof. Put \(q = 0\) to get

\[ \mu(I) \leq d - \ell(R/I) + e_d(m|I) = d - \ell(R/I) + e(I). \]

Recall that the nilpotency degree of a nilpotent ideal \(I\) is the smallest integer \(t\) for which \(I^t = 0\). The next result was proved by Sally [S] for \(q = 1\).

Corollary 2.5 Let \((R, m)\) be a \(d\)-dimensional Cohen-Macaulay local ring. Let \(I\) be an \(m\)-primary ideal. Let the nilpotency degree of \(m/I\) be \(t\). Then

\[ \mu(I) \leq d - q + (q - 1) \ell(R/I) + t^{d-q} e(m). \]

Proof. It is easy to prove the following: (1) \(e_i(m^p|I^q) = q^i p^{d-i} e_i(m|I)\) (2) \(e_i(I|I) = e(I)\) and (3) for an ideal \(K \subset I, e_i(m|I) \leq e_i(m|K)\). These imply that

\[ \begin{align*}
\mu(I) & \leq d - q + (q - 1) \ell(R/I) + e_{d-q}(m|m^t) \\
& = d - q + (q - 1) \ell(R/I) + t^{d-q} e_{d-q}(m|m) \\
& = d - q + (q - 1) \ell(R/I) + t^{d-q} e(m).
\end{align*} \]

The next result generalizes a bound due to Boratynski, Eisenbud and Rees. This follows from theorem 2.2 and by the following Minkowski type inequality for mixed multiplicities due to Rees and Sharp [RS] and Teissier [T2].

Theorem 2.6 Let \(I\) and \(J\) be \(m\)-primary ideals of a local ring \((R, m)\) of dimension \(d\). Then for \(i = 0, 1, \ldots, d,\)

\[ e_i(I|J) \leq \sqrt{e(I)^{d-i}e(J)^i}. \]
Corollary 2.7 Let $I$ be an $m$-primary ideal in a $d$-dimensional Cohen-Macaulay local ring. Then for $q = 0, 1, \ldots, d$,

$$
\mu(I) \leq d - q + (q - 1) \ell(R/I) + d^{\sqrt{e(m)q}}e(I)^{d-q}.
$$

3 Bound for $\mu(I)$ for Cohen-Macaulay ideals

The proof of theorem 1.1 is by induction on the dimension of $R/I$. The following lemma of Rees provides us with a tool to pass to one lower dimension.

Lemma 3.1 (Rees’s Lemma [R3]) Let $(R, m)$ be a local ring with infinite residue field $R/m$. Let $(I_1, I_2, \ldots, I_g)$ be a set of ideals of $R$. Let $\mathcal{P}$ be a finite set of prime ideals so that none of the primes in $\mathcal{P}$ contain any of the ideals $I_1, I_2, \ldots, I_g$. Then there exist integers $s_i \geq 0$ and elements $x_i \in I_i \setminus \bigcup\{p : p \in \mathcal{P}\}$ where $i = 1, 2, \ldots, g$ so that for all $r_i \geq s_i$ and for all $r_j \geq 0$, $j \neq i$,

$$
x_i R \cap I_1^{r_1} I_2^{r_2} \cdots I_g^{r_g} = x_i I_1^{s_1} I_2^{s_2} \cdots I_i^{r_i - 1} \cdots I_g^{s_g}.
$$

Definition 3.2 The element $x_i \in I_i$ in Rees’s lemma is called superficial for the set of ideals $I_1, I_2, \ldots, I_g$.

Lemma 3.3 [KV] Let $(R, m)$ be a local ring. Let $I$ be an ideal of positive height $h$. If $x \in m$ is superficial for $I$ and $m$ then for $i = 0, \ldots, h$,

$$
e_i(m|I) = e_i\left(\frac{m}{xR} \mid I + xR \right).
$$

Proof of the theorem 1.1 Apply induction on $s = \dim R/I$. If $s = 0$ then $I$ is $m$-primary. Thus $\ell(R/I) = e(R/I)$. Therefore the theorem follows theorem 2.2. Suppose that $s \geq 1$. Then we can choose a nonzerodivisor $x \in m$ which is superficial for $m$
and $I$ and its image is a nonzerodivisor in and superficial for $m/I$. Put $\overline{R} = R/xR$ and $\overline{I} = I/xR$. Then
\[
\mu(\overline{I}) = \dim \overline{I}/\overline{m}\overline{I} = \dim \frac{I + xR}{mI + xR} = \dim \frac{I}{mI + (xR \cap I)} = \mu(I).
\]

It is easy to see that $ht(\overline{I}) = ht(I)$. Since $x$ is superficial for $R/I$, $e(R/I) = e(\overline{R}/\overline{I})$.

By lemma 3.3, $e_{h-q}(\overline{m}|\overline{I}) = e_{h-q}(m|I)$. The theorem follows by induction.

**Corollary 3.4** Let $I$ be a Cohen-Macaulay ideal of positive height in a $d$-dimensional Cohen-Macaulay local ring $(R, m)$. Then for all $q = 0, 1, \ldots, h$,
\[
\mu(I) \leq h - q + (q - 1) e(R/I) + e(R/I)^{h-q}e(R).
\]

**Proof.** Put $s = \dim(R/I)$. Suppose $s = 0$. Then $I$ is $m$-primary. Let the nilpotency degree of $m/I$ be $t$.

Hence
\[
\begin{align*}
\ell_{d-q}(m|I) &\leq \ell_{d-q}(m|m^t) = \ell^{d-q}e(R) \\
&\leq \ell(R/I)^{d-q}e(R)
\end{align*}
\]

\[
= e(R/I)^{d-q}e(R).
\]

Now let $s \geq 1$. Pick $x \in m \setminus I$ so that it is superficial for $m$ and $I$ and $x$ is a nonzerodivisor in $R/I$ and it is superficial for $m/I$. Then
\[
\mu(I) = \mu(\overline{I}) \leq h - q + (q - 1) e(\overline{R}/\overline{I}) + e(\overline{R}/\overline{I})^{h-q}e(R) = h - q + (q - 1) e(R/I) + e(R/I)^{h-q}e(R).
\]

**Corollary 3.5** (Sally[S]) $\mu(I) \leq h - 1 + e(R/I)^{h-1}e(R)$.

**Corollary 3.6** (Rees[R4]) Suppose that $ht(I) = 2$. Then $\mu(I) \leq e(R) + e(R/I)$.

**Proof.** Put $h = q = 2$.

**Corollary 3.7** (Rees[R4]) Suppose that $ht(I) = 1$. Then $\mu(I) \leq e(R)$.

**Proof.** Put $h = q = 1$. 
4 Comparison with other bounds

In this section we present some examples to show that our bounds can sometimes give better results than the previously known bounds.

First we consider a bound found by Valla in [V].

**Theorem 4.1** Let \((R,m)\) be a CM local ring of dimension \(d\) and multiplicity \(e\). Let \(I\) be a CM ideal of height \(h\). Suppose that \(e(R/I) = \epsilon\). Put \(r = \min(e, \epsilon)\). Then
a) If \(h = 0\) then \(\mu(I) \leq e - \epsilon\).
b) If \(h > 0\) then \(\mu(I) \leq e + \epsilon(h-1)^2/h + r(h-1)/h\).
c) If \(h \geq 2\), and \(I \subseteq m^2\), then \(\mu(I) \leq e + \epsilon(h-1)^2/h + \min(r + h, \epsilon)(h-1)/h - \binom{h}{2}\).

Let \((R,m)\) be the three dimensional regular local ring \(k[[x,y,z]]\) where \(k\) is any field. Consider the ideal \(I = (x^2, xy, y^n)\) of the power series ring \(k[[x,y]]\) over a field \(k\). Then \(e_1(m|I) = 3\). To calculate Valla’s bound notice that \(e = 1, \epsilon = e(R/p^{(2)}) = 9\), by the associativity formula. Thus Valla’s bound gives \(\mu(I) \leq 6\). Our bound in Theorem 1.1 gives \(\mu(I) \leq 4\). In fact \(I\) is four generated. Next we consider a very appealing bound found in [DGV].

**Theorem 4.2** Let \((R,m)\) be a CM local ring of dimension \(d \geq 1\). Let \(I\) be an \(m\)-primary ideal such that \(m^s \subseteq I\). Then

\[
\mu(I) \leq e(R) \binom{s + d - 2}{d - 1} + \binom{s + d - 2}{d - 2}.
\]

Consider the ideal \(I = (x^2, xy, y^n)\), \(n \geq 2\) of the power series ring \(k[[x,y]]\) over a field \(k\). Then \(e_1(m|I) = 2\), hence the bound in Theorem 1.1 tells us that \(I\) is generated by atmost 3 elements. The bound in [DGV] tells us that \(I\) is generated by atmost \(n+1\) elements. On the other hand the bound in [DGV] is often better for large powers of ideals. If \((R,m)\) is a regular local ring of dimension \(d\) then Theorem 1.1 implies that \(m^n\) is generated by atmost \(d - 1 + nd - 1\) elements. The bound in [DGV]
gives the exact number of generators. Hence our bound is inferior to the bound in [DGV] in this case.

References


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