When does the equality $I^3 = QI^2$ hold true
for the ideals $I = Q : m^2$?

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This is a joint work [GMT] with Naoyuki Matsuoka and Ryo Takahashi.

Let $(A, m)$ be a Cohen-Macaulay local ring with the maximal ideal $m$ and $d = \text{dim} A$. Let $Q$ be a parameter ideal in $A$ and we put $I = Q : m^2$. We can then ask the following questions:

**Question 1.**

(1) Is $Q$ a reduction of $I$?

(2) When this is the case, what is the reduction number $r_Q(I) = \min\{n \geq 0 | I^{n+1} = QI^n\}$ of $I$ with respect to $Q$?

(3) Are the graded rings $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$, $R(I) = \bigoplus_{n \geq 0} I^n$, and $F(I) = \bigoplus_{n \geq 0} I^n/mI^n$ Cohen-Macaulay rings?

We may also ask the same questions for the ideals $Q : m^n$ where $n$ is an arbitrary positive integer. In this talk let me, however, restrict my attention to the case where $n = 2$.

If we take $I = Q : m$ (not $Q : m^2$), then we have a complete answer to the questions; the following theorem is due to Alberto Corso, Claudia Polini, Craig Huneke, Wolmer Vasconcelos, and a little bit to myself.

**Theorem 2** ([CHV], [CP], [CPV], [G]). Let $Q$ be a parameter ideal in a Cohen-Macaulay local ring $(A, m)$ with the maximal ideal $m$ and let $I = Q : m$. Then the following three conditions are equivalent to each other.

(1) $I^2 \neq QI$.

(2) $\overline{Q} = Q$.

(3) $A$ is a RLR and $\mu_A(m/Q) \leq 1$.

**Corollary 3.** Let $(A, m)$ be a Cohen-Macaulay local ring and assume that $A$ is not a RLR. Then $I^2 = QI$ for every parameter ideal $Q$ in $A$, so that $G(I)$ and $F(I)$ are Cohen-Macaulay rings, where $I = Q : m$. The Rees algebra $R(I)$ is also a Cohen-Macaulay ring, if $d = \text{dim} A \geq 2$.  

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The equality \( I^2 = QI \) remains true in certain cases, even though the local ring \( A \) is not a Cohen-Macaulay ring. For example, Sakurai and myself [GSa1, GSa2, GSa3] investigated the case where \( A \) is a Buchsbaum local ring and gave the following result.

**Theorem 4** ([GSa1, GSa2, GSa3]). Let \( (A, \mathfrak{m}) \) be a Buchsbaum local ring and assume that either \( \dim A \geq 2 \) or \( \dim A = 1 \) but \( e_0^0(\mathfrak{m}) \geq 2 \). Then there exists an integer \( n > 0 \) such that for every parameter ideal \( Q \) of \( A \) which is contained in \( \mathfrak{m}^n \), you have the equality \( I^2 = QI \), so that the graded rings \( G(I) \), \( R(I) \), and \( F(I) \) are all Buchsbaum rings, where \( I = Q : \mathfrak{m} \).

In my talk I shall give a natural generalization of these results. Unfortunately, my answer will show you that we can expect such a generalization only in the case where the local ring \( A \) is a Gorenstein ring. Unless \( A \) is Gorenstein, the situation is totally different and at this moment I have no idea to control the non-Gorenstein case. (I will give two examples, which shows the non-Gorenstein case is wild.)

Let me now state my own result.

**Theorem 5.** Let \( (A, \mathfrak{m}) \) be a Gorenstein local ring with \( e_0^0(\mathfrak{m}) \geq 3 \) and \( d = \dim A > 0 \). Let \( Q \) be a parameter ideal in \( A \) and put \( I = Q : \mathfrak{m}^2 \). We then have the following.

1. \( m^2I = m^2Q \) and \( I^3 = QI^2 \).
2. \( G(I) \) and \( F(I) \) are both Cohen-Macaulay rings.
3. \( R(I) \) is a Cohen-Macaulay ring, if \( d \geq 3 \).

As a direct consequence of this result, we get the following.

**Corollary 6.** Let \( (A, \mathfrak{m}) \) be a Gorenstein local ring with \( d = \dim A \geq 3 \). Then \( I^2 = QI \) for every parameter ideal \( Q \) of \( A \) which is contained in \( \mathfrak{m}^2 \), where \( I = Q : \mathfrak{m}^2 \).

**References**


