HILBERT COEFFICIENTS AND DEPTH OF FIBER CONES

A. V. JAYANTHAN AND J. K. VERMA

Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract. Criteria are given in terms of certain Hilbert coefficients for the fiber cone $F(I)$ of an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(R, \mathfrak{m})$ so that it is Cohen-Macaulay or has depth at least $dim(R) - 1$. A version of Huneke's fundamental lemma is proved for fiber cones. S. Goto's results concerning Cohen-Macaulay fiber cones of ideals with minimal multiplicity are obtained as consequences.

1. Introduction

Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring having infinite residue field $R/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal. The objective of this paper is to explore some connections between the coefficients of the polynomial $P_{\mathfrak{m}}(I, n)$ corresponding to the function $H_{\mathfrak{m}}(I, n) = \lambda(R/\mathfrak{m}I^n)$ and depth of the fiber cone $F(I) = \oplus_{n \geq 0} I^n/\mathfrak{m}I^n$ of I. The relation between Hilbert coefficients and depth has been a subject of several papers in the context of associated graded rings and Rees algebras of ideals. That conditions on Hilbert coefficients could force high depth for associated graded rings was first observed by J. D. Sally in [S1]. Since then numerous conditions have been provided for the Hilbert coefficients so that the associated graded ring of I, $G(I) = \oplus_{n \geq 0} I^n/I^{n+1}$, is either Cohen-Macaulay or has almost maximal depth, i.e. the grade of the maximal homogeneous ideal of $G(I)$ is at least $d-1$.

Let J be a minimal reduction of I. In their elegant paper [HM] S. Huckaba and T. Marley provided necessary and sufficient conditions on the coefficients of the Hilbert polynomial $P(I, n)$ corresponding to the Hilbert function $H(I, n) = \lambda(R/I^n)$ so that $G(I)$ is Cohen-Macaulay and has almost maximal depth. We shall write the Hilbert polynomial $P(I, n)$ in the following way:

$$P(I, n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d.$$

Huckaba and Marley showed:
(i) $G(I)$ is Cohen-Macaulay if and only if $e_1(I) = \sum_{n \geq 1} \lambda(I^n + J/J)$
(i) $G(I)$ has almost maximal depth if and only if $e_1(I) = \sum_{n \geq 1} \lambda(I^n/JI^{n-1})$.

Their results unify several results known in the literature and provide an effective approach to dealing with such questions. In the paper [JSV], we have provided elementary proofs of these theorems of Huckaba and Marley.

The relation between Hilbert coefficients and depth of $F(I)$ is not well understood. The papers [CZ], [G], [DV] and [DRV] provide sufficient conditions in terms of certain Hilbert coefficients for the

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Cohen-Macaulay property of $F(I)$. The form ring $G(I)$ and the fiber cone $F(I)$ coincide when $I = \mathfrak{m}$. This indicates that there may be analogues of results of Huckaba and Marley for the fiber cone. The first guess for the appropriate Hilbert function to be used is naturally the Hilbert function of $F(I)$. However we have observed that this does not seem to be of much help in predicting depth. We will show that the coefficients of the polynomial $P_{\mathfrak{m}}(I,n)$ corresponding to the function $\lambda(R/\mathfrak{m}I^n)$ control the Cohen-Macaulay and almost maximal depth properties of $F(I)$.

We now describe the main results of this paper. Write the polynomial $P_{\mathfrak{m}}(I,n)$ as

$$P_{\mathfrak{m}}(I,n) = g_0(I)\binom{n+d-1}{d} - g_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d(I).$$

Let grade $G(I)_+ \geq d-1$. In sections 4 and 5 we shall prove that
(i) $F(I)$ is Cohen-Macaulay if and only if $g_1(I) = \sum_{n\geq 1} \lambda(\mathfrak{m}I^n + JI^{n-1}/JI^{n-1}) - 1$ and
(ii) $F(I)$ has almost maximal depth if and only if $g_1(I) = \sum_{n\geq 1} \lambda(\mathfrak{m}I^n/\mathfrak{m}JI^{n-1}) - 1$.

It can be seen that the minimal number of generators of I, $\mu(I) \leq e_0(I) + d - \lambda(R/I)$. S. Goto in [G] defined an ideal I in a Cohen-Macaulay local ring to have *minimal multiplicity* if equality holds in the above inequality. He showed that if I has minimal multiplicity then $F(I)$ is Cohen-Macaulay if and only if $G(I)$ has almost maximal depth. We shall recover this result in section 6 as a consequence of our criterion for Cohen-Macaulayness in terms of $g_1(I)$. In fact we shall prove that I has minimal multiplicity if and only if $g_1(I) = -1$.

Since the criteria for Cohen-Macaulay and almost maximal depth properties of $F(I)$ require one to know the coefficient $g_1(I)$, it is desirable to have an effective method of its computation. In Section 5 we shall show that in a one dimensional Cohen-Macaulay local ring $g_1(I) = \sum_{n\geq 1} \lambda(\mathfrak{m}I^n/J\mathfrak{m}I^{n-1}) - 1$. We will also present a simple proof of a criterion due to T. Cortadellas and S. Zarzuela [CZ] for the sequence of initial forms in $F(I)$ of elements in a regular sequence in R to be a regular sequence in $F(I)$.

In Section 3 we shall generalize the *fundamental lemma of Huneke* in [H] for finding a formula for $g_1(I)$. However, we need a modified version of this lemma so that it works for the function $\lambda(R/\mathfrak{m}I^n)$.

In the second section we will discuss the technical topic of superficial and regular elements in fiber cones.

As no extra effort is required, we will prove our results for filtrations of ideals. In a subsequent paper on fiber cones we will see that it is useful to develop the criteria for filtrations as sometime we need to deal with filtrations to prove results about the I-adic filtration.

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2. Superficial and regular elements in fiber cones of filtrations

In this section we will gather some results on superficial and regular elements in fiber cones. Throughout this paper $(R,\mathfrak{m})$ will denote a Noetherian local ring of positive (Krull) dimension d, with maximal ideal $\mathfrak{m}$ and infinite residue field $R/\mathfrak{m}$. A sequence of ideals $\mathcal{F} = \{I_n\}_{n\geq 0}$ is called a filtration if $I_0 = R$, $I_1 \neq R$, $I_{n+1} \subseteq I_n$, and $I_n I_m \subseteq I_{n+m}$ for all $n,m \geq 0$. The Rees algebra $\mathcal{R}(\mathcal{F})$ and and the associated graded ring $G(\mathcal{F})$ are defined to be the graded rings:

$$\mathcal{R}(\mathcal{F}) = R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots, G(\mathcal{F}) = R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots$$

The filtration $\mathcal{F}$ is called Noetherian if $\mathcal{R}(\mathcal{F})$ is a Noetherian ring. Throughout the paper we will assume that $\mathcal{F}$ is Noetherian and $I_n \neq 0$ for all $n \geq 0$. The ideal generated by elements of positive degree in $G(\mathcal{F})$ will be denoted by $G(\mathcal{F})_+$. The filtration $\mathcal{F}$ is called I_1-good if $\mathcal{R}(\mathcal{F})$ is a finite module over the Rees algebra $\mathcal{R}(I_1)$. An I_1-good filtration is called a Hilbert filtration if I_1 is $\mathfrak{m}$-primary.

The fiber cone of $\mathcal{F}$ with respect to an ideal K containing I_1 is the graded ring

$$F_K(\mathcal{F}) = R/K \oplus I_1/KI_1 \oplus I_2/KI_2 \oplus \cdots.$$

For $x \in I_1$, let x^* and x^o denote the initial form in degree one component of $G(\mathcal{F})$ and $F_K(\mathcal{F})$ respectively. We will always assume that $I_{n+1} \subseteq KI_n$ for all $ngeq0$. This is required in Lemma 2.3 which is required in all the arguments that apply induction on the dimension of R in the subsequent sections.

Definition 2.1. *An element $x \in I_1$ is said to be superficial in $F_K(\mathcal{F})$ if there exists $c > 0$ such that $(0 : x^o) \cap F_K(\mathcal{F})_n = 0$ for all $n > c$.*

It can easily be seen that x^o is superficial in $F_K(\mathcal{F})$ if and only if there exists $c > 0$ such that $(KI_{n+1} : x) \cap I_n = KI_n$ for all $n > c$. We first show the existence of superficial elements in $F_K(\mathcal{F})$ and proceed to prove some of their properties. The existence of superficial elements in a graded ring is well-known. Since in our case we need existence of elements which are superficial in $F_K(\mathcal{F})$ as well as $G(\mathcal{F})$ simultaneously, we give a proof of the following result for the sake of completeness.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let $\mathcal{F}$ be an I_1-good filtration and K be an $\mathfrak{m}$-primary ideal containing I_1. Then there exists $x \in I_1 \backslash KI_1$ such that x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$.*

Proof : Let the set of associated primes of $G(\mathcal{F})$ and $F_K(\mathcal{F})$ be

$$\operatorname{Ass}(G(\mathcal{F})) = \{P_1, \ldots, P_r, P_{r+1}, \ldots, P_s\} \text{ and } \operatorname{Ass}(F_K(\mathcal{F})) = \{Q_1, \ldots, Q_l, Q_{l+1}, \ldots, Q_m\}$$

such that for all $n \gg 0$, $I_n/I_{n+1} \subseteq P_i$ for $r+1 \leq i \leq s$ and $I_n/KI_n \subseteq Q_j$ for $l+1 \leq j \leq m$. The associated graded ring $G(\mathcal{F})$ and the fiber cone $F_K(\mathcal{F})$ are both homomorphic images of the extended Rees algebra $\mathcal{R}(\mathcal{F})(t^{-1})$ since $G(\mathcal{F}) = \mathcal{R}(\mathcal{F})(t^{-1})/(t^{-1})$ and $F_K(\mathcal{F}) = \mathcal{R}(\mathcal{F})(t^{-1})/(t^{-1}, K)$. Let $\mathcal{P} = \{P'_1, \ldots, P'_s, Q'_1, \ldots, Q'_m\}$ be the collection of prime ideals in the extended Rees algebra $\mathcal{R}(\mathcal{F})(t^{-1})$ which are the pre-images of the ideals $P_1, \ldots, P_s$ in $\operatorname{Ass}(G(\mathcal{F}))$ and $Q_1, \ldots, Q_m$ in $\operatorname{Ass}(F_K(\mathcal{F}))$ respectively. Since $R/\mathfrak{m}$ is infinite, $\mathcal{R}_1 \neq \mathfrak{m}I_1 t \cup_{i=1}^r P'_i \cup_{i=1}^l Q'_i$. Choose $xt \in \mathcal{R}_1 \backslash \mathfrak{m}I_1 t \cup_{i=1}^r P'_i \cup_{i=1}^l Q'_i$. We show that $0 \neq x^o \in F_K(\mathcal{F})_1$ is superficial in $F_K(\mathcal{F})$ and $x^* \in G(\mathcal{F})_1$ is superficial in $G(\mathcal{F})$. We need to show that $(0 : x^o) \cap F_K(\mathcal{F})_n = 0$ for $n \gg 0$. Let $y^o \in (0 : x^o)$. Let $(0) = N_1 \cap \cdots \cap N_m$ be a primary decomposition of (0) in $F_K(\mathcal{F})$ such that N_i is Q_i-primary for $i = 1, \ldots, m$. Then $y^o x^o \in N_i$ for all $1 \leq i \leq l$ and $x^o \notin Q_i$ for $i = 1, \ldots, l$. Therefore $y^o \in N_i$ for $i = 1, \ldots, l$. Thus $(0 : x^o) \subseteq N_1 \cap \cdots \cap N_l$. For $l+1 \leq j \leq m$, $F_K(\mathcal{F})_n \subseteq Q_j$ for $n \gg 0$. Therefore there exists a $c > 0$ such that $\oplus_{n \geq c} F_K(\mathcal{F})_n \subseteq N_{l+1} \cap \cdots \cap N_m$. Therefore for all $n \geq c$

$$(0 : x^o) \cap F_K(\mathcal{F})_n \subseteq N_1 \cap \cdots \cap N_l \cap N_{l+1} \cap \cdots \cap N_m = (0).$$

Therefore x^o is superficial in $F_K(\mathcal{F})$. A similar argument shows that x^* is superficial in $G(\mathcal{F})$. $\square$

In the next lemma, we characterize the property of an element being superficial in fiber cone in terms of its properties in the local ring.

Lemma 2.3. *(i) If there exists a $c > 0$ such that $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$, then x^o is superficial in $F_K(\mathcal{F})$.*

(ii) If x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$, then there exists $c > 0$ such that $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$. Moreover if x is regular in R, then $KI_n : x = KI_{n-1}$ for all $n \gg 0$.

Proof : (i) Suppose $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$. Then $(KI_n : x) \cap I_{n-1} \subseteq (KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$. Therefore x^o is superficial in $F_K(\mathcal{F})$.

(ii) Suppose $x \in I_1$ is such that x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$. Then there exist c_1, c_2 such that for all $n > c_1$, $(0 : x^*) \cap G(\mathcal{F})_n = 0$ and $(0 : x^o) \cap F_K(\mathcal{F})_n = 0$ for all $n > c_2$. Choose $c = \max\{c_1, c_2\} + 1$.

Claim : $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$.

Let $y \in (KI_n : x) \cap I_c$. Without loss of generality one can assume that $y \in I_c \backslash I_{c+1}$. We consider two cases here.

Case I : $y \notin KI_c$. If $xy \in KI_{c+1}$, then $y^o \in (0 : x^o) \cap F_K(\mathcal{F})_c = 0$ which is a contradiction. Therefore, $xy \notin KI_{c+1}$. Since $xy \in KI_n$, $n < c + 1$ so that $n \leq c$. Therefore $y \in I_c \subseteq I_n \subseteq KI_{n-1}$.

Case II : $y \in KI_c$. Since $yx \in KI_n \subseteq I_n$, $y \in (I_n : x) \cap I_c = I_{n-1}$. Since $y \notin I_{c+1}$, $n - 1 < c + 1$, $n \leq c + 1$. Therefore $y \in KI_c \subseteq KI_{n-1}$.

Suppose that x is regular in R. Then by the Artin-Rees lemma, there exists a c such that $KI_n \cap (x) = I_{n-c}(KI_c \cap (x)) \subseteq xI_{n-c} \subseteq xI_c$ for all $n \geq 2c$. Therefore $KI_n : x \subseteq I_c$ for $n \gg 0$. Hence for all $n \gg 0$, $KI_n : x = (KI_n : x) \cap I_c = KI_{n-1}$. $\square$

For an element $x \in I$ such that x^* is superficial in $G(I)$, it is known that $x \in I \backslash I^2$. In the following result we show that a similar property is true for superficial elements in $F_K(\mathcal{F})$.

Lemma 2.4. *If $x^o \in I_1/KI_1$ is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$, then $x \in I_1 \setminus KI_1$.*

Proof : Since $x^o \in F_K(\mathcal{F})$ and $x^* \in G(\mathcal{F})$ are superficial, by Lemma 2.3, there exists $c > 0$ such that $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$. Put $n = c + 1$. Then $(KI_{c+1} : x) = KI_c$. Suppose $x \in KI_1$. Let $y \in I_c$. Then $xy \in KI_{c+1}$ so that $y \in (KI_{c+1} : x) = KI_c$. Therefore $I_c = KI_c$. By Nakayama Lemma $I_c = 0$ which is a contradiction to the fact that $\mathcal{F}$ is a Hilbert filtration. Therefore $x \in I_1 \setminus KI_1$. $\square$

For the fiber cone $F_K(\mathcal{F})$, let $H(F_K(\mathcal{F}), n) = \lambda(F_K(\mathcal{F})_n) = \lambda(I_n/KI_n)$ denote its Hilbert function and let $P(F_K(\mathcal{F}), n)$ denote the corresponding Hilbert polynomial.

Proposition 2.5. *Let $x^o \in F_K(\mathcal{F})$ be superficial. Then* $\dim F_K(\mathcal{F})/x^oF_K(\mathcal{F}) = \dim F_K(\mathcal{F}) - 1$.

Proof : Consider the exact sequence

$$0 \longrightarrow (KI_n : x) \cap I_{n-1}/KI_{n-1} \longrightarrow I_{n-1}/KI_{n-1} \xrightarrow{x} I_n/KI_n \longrightarrow I_n/(KI_n + xI_{n-1}) \longrightarrow 0.$$

Then $H(F_K(\mathcal{F}), n) - H(F_K(\mathcal{F}), n-1) = H(F_K(\mathcal{F})/x^oF_K(\mathcal{F}), n) - \lambda((KI_n : x) \cap I_{n-1}/KI_{n-1})$ for all $n \geq 1$. Since x^o is superficial in $F_K(\mathcal{F})$, $(KI_n : x) \cap I_{n-1} = KI_{n-1}$ for $n \gg 0$, so that $P(F_K(\mathcal{F})/(x^o), n) = P(F_K(\mathcal{F}), n) - P(F_K(\mathcal{F}), n-1)$. Hence $\dim F_K(\mathcal{F})/x^oF_K(\mathcal{F}) = \dim F_K(\mathcal{F}) - 1$. $\square$

In the following lemma we provide a characterization for regular elements in $F_K(\mathcal{F})$. It can be seen that this property is quite similar to the behaviour of regular elements in $G(I)$.

Lemma 2.6. *For $x \in I_1 \backslash KI_1$, $x^o \in F_K(\mathcal{F})$ is regular if and only if $(KI_n : x) \cap I_{n-1} = KI_{n-1}$ for all $n \geq 1$. If x^* is regular in $G(\mathcal{F})$ and x^o is regular in $F_K(\mathcal{F})$ then $KI_n : x = KI_{n-1}$ for all $n \geq 1$.*

Proof : Suppose $(KI_n : x) \cap I_{n-1} = KI_{n-1}$ for all $n \geq 1$. In other words, $(0 : x^o) \cap F_K(\mathcal{F})_{n-1} = (0)$ for all $n \geq 1$. Since $(0 : x^o)$ is a homogeneous ideal, $(0 : x^o) = \oplus_{n\geq 0}(0 : x^o) \cap F_K(\mathcal{F})_n = (0)$. Conversely, assume that x^o is a regular element in $F_K(\mathcal{F})$. Then $\alpha_n : F_K(\mathcal{F})_{n-1} \longrightarrow F_K(\mathcal{F})_n$ is an injective map for all $n \geq 1$, where α_n is the multiplication by x^o for all $n \geq 1$. Since $\ker \alpha_n = (KI_n : x) \cap I_{n-1}/KI_{n-1} = 0$, $(KI_n : x) \cap I_{n-1} = KI_{n-1}$ for all $n \geq 1$.

If $KI_n : x = KI_{n-1}$ for all $n \geq 1$, then clearly x^o is regular in $F_K(\mathcal{F})$. Suppose that x^* is regular in $G(\mathcal{F})$ and x^o is regular in $F_K(\mathcal{F})$. Let $y \in KI_n : x$. If there exists a t such that $0 \neq y^o \in I_t/KI_t$, then $0 \neq y^o x^o \in I_{t+1}/KI_{t+1}$ so that $yx \notin KI_{t+1}$. Since $yx \in KI_n$, $n < t+1$. Therefore $y \in I_t \subseteq I_n \subseteq KI_{n-1}$. Suppose we can not find t such that $0 \neq y^o \in I_t/KI_t$. Then, if $y \in I_n$, $y \in KI_n$. Since $y \in KI_n : x$, $y \in I_n : x$. By hypothesis, x^* is regular in G. Hence $I_n : x = I_{n-1}$ for $n \geq 1$. Therefore $y \in I_{n-1}$ and hence $y \in KI_{n-1}$. $\square$

The following lemma is an analogue of Lemma 2.2 of [HM]. This will play a crucial role in induction arguments. This is the so-called Sally-machine for fiber cones.

Lemma 2.7. *Let $x \in I_1$ be such that x^* is superficial in $G(\mathcal{F})$ and $x^o \in F_K(\mathcal{F})$ is superficial in $F_K(\mathcal{F})$. Let $\bar{\mathcal{F}} = \{I_n + xR/xR\}_{n\geq 0}$ and $\bar{K} = K/xR$. If $\operatorname{depth} F_{\bar{K}}(\bar{\mathcal{F}}) > 0$, then x^o is regular in $F_K(\mathcal{F})$.*

Proof : Let $y^o \in I_t/KI_t$ be such that its natural image $\bar{y}^o$ is a regular element in $F_{\bar{K}}(\bar{\mathcal{F}})$. Then $(KI_{n+tj} : y^j) \cap I_n \subseteq (KI_n, x)$ for all $n, j \geq 1$. Since x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$, there exists $c > 0$ such that $(KI_{n+j} : x^j) \cap I_c = KI_n$ for all $n > c$ and $j \geq 1$, by Lemma 2.3 (ii). Let n and j be arbitrary and $p > c/t$. Then $y^p(KI_{n+j} : x^j) \subseteq (KI_{n+pt+j} : x^j) \cap I_c \subseteq KI_{n+pt}$. Thus

$$(KI_{n+j} : x^j) \cap I_n \subseteq (KI_{n+pt} : y^p) \cap I_n \subseteq (KI_n, x).$$

Therefore $(KI_{n+j} : x^j) \cap I_n \subseteq KI_n + x(KI_{n+j} : x^{j+1})$. Iterating this formula $n+1$ times we get,

$$\begin{aligned}(KI_{n+j} : x^j) \cap I_n &\subseteq KI_n + xKI_{n-1} + \cdots + x^{n+1}(KI_{n+j} : x^{n+j+1}) \\ &= KI_n.\end{aligned}$$

Therefore $(KI_{n+j} : x^j) \cap I_n = KI_n$ for all $n \geq 1$ and hence x^o is regular in $F_K(\mathcal{F})$. $\square$

3. Hilbert coefficients for the function $\lambda(R/KI_n)$

Throghout this section $\mathcal{F} = \{I_n\}_{n\geq 0}$ will be a Hilbert filtration of R. Let K be an ideal of R such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$. Let $H(F,n) = \lambda(F_K(\mathcal{F})_n) = \lambda(I_n/KI_n)$ be the Hilbert function of the fiber cone $F = F_K(\mathcal{F})$. Then, $H(F,n) = \lambda(R/KI_n) - \lambda(R/I_n)$. Since both $H(F,n)$ and $\lambda(R/I_n)$ are polynomials for $n \gg 0$, $\lambda(R/KI_n)$ is also a polynomial for $n \gg 0$. Since the coefficients of this polynomial are related with the Hilbert coefficients of the fiber cone and the Hilbert-Samuel coefficients of $\mathcal{F}$, it is expected that their properties will be related with the properties of the fiber cone. Huneke's fundamental lemma [H] provides formulas for the Hilbert coefficients of the Hilbert polynomial of an $\mathfrak{m}$-primary ideal in a two-dimensional Cohen-Macaulay local ring. We will prove an analogue of this lemma for the fiber cones. It will yield formulas for the Hilbert polynomial of the fiber cone once we have access to a minimal reduction of I_1.

Let $H_K(\mathcal{F},n) = \lambda(R/KI_n)$ (resp. $H(\mathcal{F},n) = \lambda(R/I_n)$) and let $P_K(\mathcal{F},n)$ (resp. $P(\mathcal{F},n)$) be the corresponding polynomial. Since $P_K(\mathcal{F},n) = P(\mathcal{F},n) + P(F,n)$, it is a polynomial of degree d with leading coefficient $e_0(I)$. We write the above polynomials in the following way:

$$\begin{aligned}
P(F,n) &= f_0\binom{n+d-2}{d-1} - f_1\binom{n+d-3}{d-2} + \cdots + (-1)^{d-1}f_{d-1},\\
P(\mathcal{F},n) &= e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d}e_{d},\\
P_K(\mathcal{F},n) &= g_0\binom{n+d-1}{d} - g_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d}g_{d}.
\end{aligned}$$

Then $g_0 = e_0$ and $g_i = e_i - f_{i-1}$ for all $1 \leq i \leq d$. For a numerical function $h : \mathbb{Z} \longrightarrow \mathbb{Z}$, let $\Delta h(n) := h(n) - h(n-1)$.

Lemma 3.1. *Let $(R,\mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let $\mathcal{F}$ be a Hilbert filtration, K be an ideal with $I_1 \subseteq K$ and $J = (x,y)$ be a minimal reduction of I_1. Then for all $n \geq 2$,*

$$\Delta^2\left[P_K(\mathcal{F},n) - H_K(\mathcal{F},n)\right] = \lambda\left(\frac{KI_n}{KJI_{n-1}}\right) - \lambda\left(\frac{KI_{n-1}:J}{KI_{n-2}}\right).$$

Proof : Consider the exact sequence

$$0 \longrightarrow \frac{R}{KI_{n-1}:J} \xrightarrow{\ \beta\ } \left(\frac{R}{KI_{n-1}}\right)^2 \xrightarrow{\ \alpha\ } \frac{J}{KJI_{n-1}} \longrightarrow 0,$$

where α is the map $\alpha(\bar{r},\bar{s}) = \overline{xr+ys}$ and $\beta(\bar{r}) = (\bar{y}\bar{r}, -\bar{x}\bar{r})$. It follows that for all $n \geq 2$

$$\begin{aligned}
2\lambda(R/KI_{n-1}) &= \lambda(R/(KI_{n-1}:J)) + \lambda(J/KJI_{n-1})\\
&= \lambda(R/(KI_{n-1}:J)) + \lambda(R/KJI_{n-1}) - \lambda(R/J).
\end{aligned}$$

Therefore $e_0(\mathcal{F}) + 2\lambda(R/KI_{n-1}) = \lambda(R/KJI_{n-1}) + \lambda(R/(KI_{n-1}:J))$. Hence

$$\begin{aligned}
e_0(\mathcal{F}) &- \lambda(R/KI_n) + 2\lambda(R/KI_{n-1}) - \lambda(R/KI_{n-2})\\
&= \lambda(R/KJI_{n-1}) - \lambda(R/KI_n) + \lambda(R/(KI_{n-1}:J)) - \lambda(R/KI_{n-2})\\
&= \lambda(KI_n/KJI_{n-1}) - \lambda(KI_{n-1}:J/KI_{n-2})
\end{aligned}$$

Since $\Delta^2 P_K(\mathcal{F},n) = e_0(\mathcal{F})$,

$$\Delta^2\left[P_K(\mathcal{F},n) - H_K(\mathcal{F},n)\right] = \lambda\left(\frac{KI_n}{KJI_{n-1}}\right) - \lambda\left(\frac{KI_{n-1}:J}{KI_{n-2}}\right).$$

□

Corollary 3.2 ([H], Fundamental Lemma 2.4)**.** *Let $(R,\mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal. Let $J=(x,y)$ be a minimal reduction of I. Let $H(I,n)=\lambda(R/I^n)$ be the Hilbert function of I and let $P(I,n)$ be the corresponding Hilbert polynomial. Then for all $n\geq 2$,*

$$\Delta^2[P(I,n)-H(I,n)] = \lambda\left(\frac{I^n}{JI^{n-1}}\right) - \lambda\left(\frac{I^{n-1}:J}{I^{n-2}}\right).$$

Proof : Set $K=R$, $\mathcal{F}=\{I^n\}$ in Lemma 3.1. Then $H_K(\mathcal{F},n)=H(I,n)$ for all $n\geq 0$ so that $P_K(\mathcal{F},n)=P(I,n)$ and hence the assertion follows. □

As a consequence of the generalization of the Fundamental Lemma, we obtain expressions for the Hilbert coefficients g_1 and g_2.

Corollary 3.3. *Set*

$$v_n = \begin{cases} e_0(\mathcal{F}) \text{ if } n=0 \\ e_0(\mathcal{F}) - \lambda(R/KI_1) + \lambda(R/K) \text{ if } n=1 \\ \lambda(KI_n/KJI_{n-1}) - \lambda(KI_{n-1}:J/KI_{n-2}) \text{ if } n\geq 2. \end{cases}$$

Then $g_1 = \sum_{n\geq 1} v_n$ and $g_2 = \sum_{n\geq 1}(n-1)v_n + \lambda(R/K)$.

Proof : From Lemma 3.1 we have,

$$\sum_{n\geq 0}\Delta^2[P_K(\mathcal{F},n) - H_K(\mathcal{F},n)]t^n = \sum_{n\geq 0} v_n t^n.$$

Write $P_K(\mathcal{F},n) = e_0(\mathcal{F})\binom{n+2}{2} - g_1'(n+1) + g_2'$. Then comparing with the earlier notation, we get $g_1 = g_1' - e_0(\mathcal{F})$ and $g_2 = e_0(\mathcal{F}) - g_1' + g_2'$. Since $P_K(\mathcal{F},n)$ is a polynomial of degree 2, $\Delta^2 P_K(\mathcal{F},n) = e_0(\mathcal{F})$ for all $n\geq 0$ so that $\sum_{n\geq 0}\Delta^2 P_K(\mathcal{F},n)t^n = e_0(\mathcal{F})/(1-t)$. Let $\sum_{n\geq 0} H_K(\mathcal{F},n)t^n = f(t)/(1-t)^3$. Then by Proposition 4.1.9 of [BH], $e_0(\mathcal{F}) = f(1)$, $g_1' = f'(1)$ and $g_2' = f''(1)/2!$. Also we have,

$$\begin{aligned}
\sum_{n\geq 0}\Delta^2 H_K(\mathcal{F},n)t^n &= \sum_{n\geq 0} H_K(\mathcal{F},n)t^n - 2\sum_{n\geq 0} H_K(\mathcal{F},n-1)t^n + \sum_{n\geq 0} H_K(\mathcal{F},n-2)t^n \\
&= \frac{f(t)}{(1-t)^3} - 2H_K(\mathcal{F},-1) - 2t\frac{f(t)}{(1-t)^3} + H_K(\mathcal{F},-2) + tH_K(\mathcal{F},-1) + t^2\frac{f(t)}{(1-t)^3} \\
&= \frac{f(t)}{(1-t)} - 2\lambda(R/K) + \lambda(R/K) + t\lambda(R/K) \\
&= \frac{f(t) - (1-t)^2\lambda(R/K)}{(1-t)}.
\end{aligned}$$

Therefore

$$\sum_{n\geq 0}\Delta^2[P_K(\mathcal{F},n) - H_K(\mathcal{F},n)]t^n = \frac{e_0(\mathcal{F}) - f(t) + (1-t)^2\lambda(R/K)}{(1-t)} = \sum_{n\geq 0} v_n t^n.$$

Therefore

$$e_0(\mathcal{F}) - f(t) + (1-t)^2\lambda(R/K) = (1-t)\sum_{n\geq 0} v_n t^n.$$

Hence

$$f(t) = e_0(\mathcal{F}) + (1-t)^2\lambda(R/K) - (1-t)\sum_{n\geq 0} v_n t^n. \tag{1}$$

Thus, $f(1) = e_0(\mathcal{F})$. Differentiating (1) with respect to t, we get

$$f'(t) = -2(1-t)\lambda(R/K) - (1-t)\sum_{n\geq 0} n v_n t^{n-1} + \sum_{n\geq 0} v_n t^n.$$

Therefore $g_1' = f'(1) = \sum_{n\geq 0} v_n$. Differentiating (1) twice with respect to t, we get

$$f''(t) = 2\lambda(R/K) - (1-t)\sum_{n\geq 0} n(n-1)v_n t^{n-2} + 2\sum_{n\geq 0} n v_n t^{n-1}$$

so that

$$g_2' = f''(1)/2 = \sum_{n\geq 0} n v_n + \lambda(R/K).$$

Therefore

$$g_1 = g_1' - e_0(\mathcal{F}) = \sum_{n\geq 0} v_n - e_0(\mathcal{F}) = \sum_{n\geq 1} v_n$$

and

$$g_2 = g_2' - g_1 = \sum_{n\geq 0} n v_n + \lambda(R/K) - \sum_{n\geq 1} v_n = \sum_{n\geq 1}(n-1)v_n + \lambda(R/K).$$

$\square$

Remark: The above formulas for g_1 and g_2 generalize the formulas for e_1 and e_2 obtained by Huneke as consequences of his fundamental lemma. To obtain Huneke's formulas for e_1 and e_2, one simply puts $K = R$ in the above formulas for g_1 and g_2.

Example 3.4. Let k be any field and let $R = k[\![x,y]\!]$. Let $I = (x^3, x^2y, y^3)$. Then $J = (x^3, y^3)$ is a minimal reduction of I. Then $r_J(I) = 3$, $\mathfrak{m}I^n = \mathfrak{m}JI^{n-1}$ for all $n \geq 2$ and $\mathfrak{m}I^n : J = \mathfrak{m}I^{n-1}$ for all $n \geq 1$. Therefore $v_n = 0$ for all $n \geq 2$. One can also see that $e_0 = 9$ and $\lambda(R/\mathfrak{m}I) = 10$. Hence we have $v_0 = 9, v_1 = e_0 - \lambda(R/mI) + \lambda(R/m) = 9 - 10 + 1 = 0$ Thus $g_1' = v_0 + v_1 = 9$ and $g_2' = v_1 + \lambda(R/m) = 1$ which gives $g_1 = g_1' - e_0 = 0$ and $g_2 = g_2' - g_1' + e_0 = 1$.

The following lemma shows that the behaviour of the superficial elements in $F_K(\mathcal{F})$ is quite similar to that of superficial elements in $G(\mathcal{F})$.

Lemma 3.5. *Let x be a regular element in I_1 such that x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$. Let $\bar{g}_i$ denote the coefficients of the polynomial corresponding to the function $\lambda(\bar{R}/\bar{K}\bar{I}_n)$, where "$-$" denotes* modulo$(x)$. *Then $\bar{g}_i = g_i$ for all $i = 0, \ldots, d-1$.*

Proof : Consider the exact sequence

$$0 \longrightarrow \frac{KI_n : x}{KI_n} \longrightarrow R/KI_n \xrightarrow{x} R/KI_n \longrightarrow R/(KI_n + xR) \longrightarrow 0.$$

Then $\lambda(\bar{R}/\bar{K}\bar{I}_n) = \lambda(R/(KI_n + xR)) = \lambda(KI_n : x/KI_n)$. Since x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$, $KI_n : x = KI_{n-1}$ for $n \gg 0$, by Lemma 2.3. Hence, $\lambda(R/(KI_n + xR)) = \lambda(R/KI_n) - \lambda(R/KI_{n-1})$. for $n \gg 0$. Therefore

$$\begin{aligned} P_{\bar{K}}(\bar{\mathcal{F}}, n) &= P_K(\mathcal{F}, n) - P_K(\mathcal{F}, n-1) \\ &= \sum_{i=0}^{d} (-1)^i g_i \binom{n+d-i-1}{d-i} - \sum_{i=0}^{d} (-1)^i g_i \binom{n+d-i-2}{d-i} \\ &= \sum_{i=0}^{d-1} (-1)^i g_i \binom{n+d-i-2}{d-1-i}. \end{aligned}$$

□

4. Cohen-Macaulay fiber cones

In this section we obtain a lower bound for the Hilbert coefficient g_1. We will characterize the Cohen-Macaulayness of $F_K(\mathcal{F})$ in terms of g_1. In this characterization, We need to assume that $G(\mathcal{F})$ has almost maximal depth. We will show by an example that we need this assumption for such a characterization.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with infinite residue field and let $J \subseteq I_1$ be a minimal reduction of I_1 and let K be an ideal such that $I_{n+1} \subseteq KI_n$. Then $g_1 \geq \sum_{n \geq 1} \lambda(KI_n + J/J) - \lambda(R/K)$.*

Proof : We apply induction on d. Let $d = 1$ and let $(x) = J$. For $i \geq 0$, from the exact sequence

$$0 \longrightarrow \frac{(KI_{i+1} : x) \cap I_i}{KI_i} \longrightarrow I_i/KI_i \xrightarrow{x} I_{i+1}/KI_{i+1} \longrightarrow I_{i+1}/(KI_{i+1} + xI_i) \longrightarrow 0.$$

it follows that

$$\lambda\left(\frac{I_{i+1}}{KI_{i+1}}\right) - \lambda\left(\frac{I_i}{KI_i}\right) = \lambda\left(\frac{I_{i+1}}{(KI_{i+1} + xI_i)}\right) - \lambda\left(\frac{(KI_{i+1} : x) \cap I_i}{KI_i}\right) \qquad \cdots\cdots \ (E_i).$$

Summing up $E_0, E_1, \cdots, E_{n-1}$, we get

$$\lambda(I_n/KI_n) - \lambda(R/K) = \sum_{i=1}^{n} \lambda(I_i/(KI_i + xI_{i-1})) - \sum_{i=1}^{n} \lambda((KI_i : x) \cap I_{i-1}/KI_{i-1}).$$

Thus for all $n \gg 0$,

$$f_0 = \lambda(R/K) + \sum_{i \geq 1} \lambda(I_i/(KI_i + xI_{i-1})) - \sum_{i \geq 1} \lambda((KI_i : x) \cap I_{i-1}/KI_{i-1}).$$

Since $\gamma(\mathcal{F}) \geq d-1$, by Theorem 4.7 of [HM], $e_1(\mathcal{F}) = \sum_{i \geq 1} \lambda(I_i/xI_{i-1})$. Thus

$$\begin{aligned} g_1 &= e_1(\mathcal{F}) - f_0 \\ &= \sum_{i\geq 1} \lambda(I_i/xI_{i-1}) - \lambda(R/K) - \sum_{i \geq 1} \lambda(I_i/(KI_i + xI_{i-1})) + \sum_{i\geq 1} \lambda((KI_i : x) \cap I_{i-1}/KI_{i-1}) \\ &= \sum_{i \geq 1} \lambda((KI_i + xI_{i-1})/xI_{i-1}) + \sum_{i \geq 1} \lambda((KI_i : x) \cap I_{i-1}/KI_{i-1}) - \lambda(R/K). \end{aligned}$$

This implies that $g_1 \geq \sum_{i \geq 1} \lambda((KI_i + xI_{i-1})/xI_{i-1}) - \lambda(R/K) \geq \sum_{i \geq 1} \lambda(KI_i + J/J) - \lambda(R/K)$. Hence the result is true for $d = 1$. Let us assume that $d > 1$ and the assertion is true for $d-1$. Choose the generators $x_1, \ldots, x_d$ of J such that x_1^o (resp. $x_1^* \in G(\mathcal{F})$) is superficial in $F_K(\mathcal{F})$ (resp. $G(\mathcal{F})$). Let " $-$ " denote images modulo(x_1). Then by Lemma 3.5, $\bar{g_1} = g_1$. By induction

$$\begin{aligned} \bar{g_1} &\geq \sum_{n \geq 1} \lambda(\bar{K}\bar{I_n} + \bar{J}/\bar{J}) - \lambda(\bar{R}/\bar{K}) \\ &= \lambda((KI_n + xR) + (J + xR)/(J + xR)) - \lambda(R/K) \\ &= \sum_{n \geq 1} \lambda(KI_n + J/J) - \lambda(R/K). \end{aligned}$$

□

Now we prove a characterization for Cohen-Macaulayness of the fiber cone in terms of g_1.

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ with $R/\mathfrak{m}$ infinite. Let $\mathcal{F} = \{I_n\}$ be a Hilbert filtration of R and let K be an ideal of R containing I_1. Let J be a minimal reduction of I_1. Assume that $\gamma(\mathcal{F}) \geq d-1$. Then $F_K(\mathcal{F})$ is Cohen-Macaulay if and only if $g_1 = \sum_{n \geq 1} \lambda(KI_n + JI_{n-1}/JI_{n-1}) - \lambda(R/K)$.*

Proof. Suppose that $F_K(\mathcal{F})$ is Cohen-Macaulay. As $J + KI_1/KI_1$ is generated by a homogeneous system of parameters of degree 1, $f_0 = \lambda(F_K(\mathcal{F})/JF_K(\mathcal{F})) = \sum_{i \geq 0} \lambda(I_i/(KI_i + JI_{i-1}))$. Since $\gamma(\mathcal{F}) \geq d-1$, by Theorem 4.7 of [HM], $e_1 = \sum_{i \geq 1} \lambda(I_i/JI_{i-1})$. Therefore

$$\begin{aligned} g_1 &= e_1 - f_0 = \sum_{n \geq 1} \lambda(I_n/JI_{n-1}) - \sum_{n \geq 0} \lambda(I_n/(KI_n + JI_{n-1})) \\ &= \sum_{n \geq 1} \lambda(I_n/JI_{n-1}) - \sum_{n \geq 1} \lambda(I_n/(KI_n + JI_{n-1})) - \lambda(R/K) \\ &= \sum_{n \geq 1} \lambda(KI_n + JI_{n-1}/JI_{n-1}) - \lambda(R/K). \end{aligned}$$

Conversely, suppose $g_1 = \sum_{n \geq 1} \lambda(KI_n + JI_{n-1}/JI_{n-1}) - \lambda(R/K)$. Then by reversing the above steps, one can see that $f_0 = \sum_{n \geq 0} \lambda(I_n/(KI_n + JI_{n-1})) = \lambda(F_K(\mathcal{F})/JF_K(\mathcal{F}))$. Therefore $F_K(\mathcal{F})$ is Cohen-Macaulay. □

The following example shows that the assumption in Theorem 4.2 that depth $G(\mathcal{F}) \geq d-1$ cannot be dropped.

Example 4.3. Let $R = k[\![x,y]\!], \mathfrak{m} = (x,y)$ and $I = (x^4, x^3y, xy^3, y^4)$. Then $J = (x^4, y^4)$ is a minimal reduction of I and $I^3 = JI^2$. Note that $I^n = \mathfrak{m}^{4n}$ for all $n \geq 2$. We compute the Hilbert coefficients of I. Since $I^n = \mathfrak{m}^{4n}$ for all $n \geq 2$, $\lambda(R/I^n) = \lambda(R/\mathfrak{m}^{4n}) = \binom{4n+1}{2} = e_0(I)\binom{n+1}{2} - e_1(I)\ n + e_2(I)$. Solving the equation by putting various values for n, we get $e_0(I) = 16,\ e_1(I) = 6,\ e_2(I) = 0$. From direct computations one can see $\lambda(I/J) = 5$ and $\lambda(I^2/JI) = 2$. Hence $e_1(I) < \sum_{n\geq 1} \lambda(I^n/JI^{n-1})$. Therefore $\operatorname{depth} G(I) = 0$. Since $I^n = \mathfrak{m}^{4n}$ for all $n \geq 2$, $\mu(I^n) = \lambda(\mathfrak{m}^{4n}/\mathfrak{m}^{4n+1}) = \binom{4n+1}{1}$ for all $n \geq 1$. Therefore $f_0 = 4$ so that $g_1 = e_1 - f_0 = 2$. Also, one can see that $\lambda(\mathfrak{m}I + J/J) \neq 0$ and $\mathfrak{m}I^n \subseteq JI^{n-1}$ for all $n \geq 2$. Then $\lambda(\mathfrak{m}I + J/J) = \lambda(\mathfrak{m}I/\mathfrak{m}I \cap J) = \lambda(\mathfrak{m}I/\mathfrak{m}J) = 3$. Therefore $\sum_{n\geq 1} \lambda(\mathfrak{m}I^n + JI^{n-1}/JI^{n-1}) - 1 = 2 = g_1$. The Hilbert Series of the fiber cone is given by

$$H(F(I), t) = \frac{1 + 2t + 2t^2 - t^3}{(1-t)^2}.$$

Since the numerator contains a negative coefficient, $F(I)$ is not Cohen-Macaulay.

5. Fiber cones with almost maximal depth

In this section we present a characterization for the fiber cone $F_K(\mathcal{F})$ to have almost maximal depth in terms of g_1. This is an analogue of the Huckaba-Marley characterization for the associated graded ring to have almost maximal depth referred above. We will need to invoke a result due to T. Cortadellas and S. Zarzuela from [CZ] which gives a criterion for a sequence of degree one elements in $F_K(\mathcal{F})$ to be a regular sequence. We take this opportunity to present a simple proof of their result since it is a very basic result for detecting regular sequences in fiber cones.

Theorem 5.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathcal{F}$ a filtration of ideals, K an ideal containing I_1 and $x_1, \ldots, x_k \in I_1$. Assume that*

(i) *$x_1, \ldots, x_k$ is a regular sequence in R.*
(ii) *$x_1^* \ldots, x_k^* \in G(\mathcal{F})$ is a regular sequence.*
(iii) *$x_1^o, \ldots, x_k^o \in F_K(\mathcal{F})$ is a superficial sequence.*

Then $\operatorname{depth}_{(x_1^o,\ldots,x_k^o)} F_K(\mathcal{F}) = k$ if and only if $(x_1, \ldots, x_k) \cap KI_n = (x_1, \ldots, x_k)KI_{n-1}$ for all $n \geq 1$.

Proof : We induct on k. Let $k = 1$. Let $(x) \cap KI_n = xKI_{n-1}$ for all $n \geq 1$. Then $KI_n : x = KI_{n-1}$ for all $n \geq 1$ and hence x^o is regular in $F_K(\mathcal{F})$. Suppose x^o is regular in $F_K(\mathcal{F})$. Let $n \geq 1$ and $y \in (x) \cap KI_n$. Let $y = rx$ for some $r \in R$. Then $r \in KI_n : x \subseteq I_n : x$. Since x^* is a nonzerodivisor in $G(\mathcal{F})$, $I_n : x = I_{n-1}$ for all $n \geq 1$. Therefore $r \in I_{n-1}$. Hence $r \in (KI_n : x) \cap I_{n-1} = KI_{n-1}$ so that $y \in xKI_{n-1}$. Now assume that $k > 1$ and the result is true for all $l \leq k-1$. Put $J = (x_1, \ldots, x_k)$, $J^o = (x_1^o, \ldots, x_k^o) \subseteq F_K(\mathcal{F})$ and $J^* = (x_1^*, \ldots, x_k^*) \subseteq G(\mathcal{F})$. Let " $-$ " denote images modulo (x_1). Then $F_K(\mathcal{F})/x_1^o F_K(\mathcal{F}) \cong F_{\bar{K}}(\bar{\mathcal{F}})$. Assume $J \cap KI_n = JKI_{n-1}$ for all $n \geq 1$. Then

$$\bar{J} \cap \bar{K}\bar{I}_n = (J + x_1R) \cap (KI_n + x_1R) = J \cap (KI_n + x_1R) = (J \cap KI_n) + x_1R = JKI_{n-1} + x_1R = \bar{J}\bar{K}\bar{I}_{n-1}.$$

By induction $\operatorname{depth}_{\bar{J}^o} F_{\bar{K}}(\bar{\mathcal{F}}) = k-1$. Thus x_1^o is regular in $F_K(\mathcal{F})$ and hence, $\operatorname{depth}_{J^o}(F_K(\mathcal{F})) = k$, by Lemma 2.7. Conversely assume that $\operatorname{depth}_{J^o}(F_K(\mathcal{F})) = k$. Since x_1^o is superficial in $F_K(\mathcal{F})$ and

$\operatorname{depth} F_K(\mathcal{F}) > 0$, x_1^o is regular in $F_K(\mathcal{F})$. Then $\operatorname{depth}_{\bar{J}^o}(F(\bar{I})) = k-1$. Hence, by induction, $\bar{J} \cap \bar{K}\bar{I}_n = \bar{J}\bar{K}\bar{I}_{n-1}$. Therefore, $J \cap KI_n + x_1R = JKI_{n-1} + x_1R$. Hence

$$\begin{aligned} J \cap KI_n &= JKI_{n-1} + (x_1R \cap (J \cap KI_n)) \\ &= JKI_{n-1} + (x_1R \cap KI_n) \\ &= JKI_{n-1} + (x_1KI_{n-1}) = JKI_{n-1}. \end{aligned}$$

Therefore $J \cap KI_n = JKI_{n-1}$. □

We need the following lemma in the proof of the characterization for the fiber cone to have depth at least $d-1$.

Lemma 5.2. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{I_n\}$ be a Hilbert filtration of R such that $\gamma(\mathcal{F}) \geq d-1$. Let K be an ideal of R containing I_1. Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of I_1 such that $x_1^*, \ldots, x_{d-1}^*$ is a regular sequence in $G(\mathcal{F})$. If $KI_n \cap (x_1, \ldots, x_{d-1}) \subseteq JKI_{n-1}$ for all $n \geq 1$, then $x_1^o, \ldots, x_{d-1}^o$ is a regular sequence in $F_K(\mathcal{F})$.*

Proof : Since $x_1^*, \ldots, x_{d-1}^*$ is a regular sequence in $G(\mathcal{F})$, by Theorem 5.1, it is enough to show that $KI_n \cap (x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{d-1})KI_{n-1}$ for all $n \geq 1$. Induct on n. Let $z \in KI_1 \cap (x_1, \ldots, x_{d-1})$. Write $z = \sum_{i=1}^{d-1} r_i x_i$ for $r_i \in R$. Since $KI_1 \cap (x_1, \ldots, x_{d-1}) \subseteq (x_1, \ldots, x_d)K$, $z = \sum_{i=1}^{d-1} s_i x_i + px_d$ for some $p, s_i \in K$. Then $px_d \in (x_1, \ldots, x_{d-1})$ and hence $p \in (x_1, \ldots, x_{d-1})$. Since $x_d \in K$, $z \in (x_1, \ldots, x_{d-1})K$. Now assume that $n \geq 2$ and for all $l < n$,

$$KI_l \cap (x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{d-1})KI_{l-1}.$$

Let $z \in KI_n \cap (x_1, \ldots, x_{d-1}) \subseteq JKI_{n-1}$. Write $z = \sum_{i=1}^{d-1} r_i x_i = \sum_{i=1}^{d-1} s_i x_i + px_d$, where $r_i \in R$, $p, s_i \in KI_{n-1}$. Then $px_d \in (x_1, \ldots, x_{d-1})$ and hence $p \in (x_1, \ldots, x_{d-1})$. Therefore

$$KI_n \cap (x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{d-1})KI_{n-1} + x_d(KI_{n-1} \cap (x_1, \ldots, x_{d-1})).$$

By induction $KI_{n-1} \cap (x_1, \ldots, x_{d-1}) \subseteq (x_1, \ldots, x_{d-1})KI_{n-2}$. Hence $p \in (x_1, \ldots, x_{d-1})KI_{n-2}$ so that $px_d \in (x_1, \ldots, x_{d-1})KI_{n-1}$. Therefore $z \in (x_1, \ldots, x_{d-1})KI_{n-1}$. □

We prove a necessary and sufficient condition for the fiber cone to have depth at least $d-1$ in terms of g_1 in the following theorem.

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let $\mathcal{F}$ be a Hilbert filtration, K an ideal such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$ and J a minimal reduction of I_1. Assume that $\gamma(\mathcal{F}) \geq d-1$. Then $g_1 = \sum_{n \geq 1} \lambda(KI_n/JKI_{n-1}) - \lambda(R/K)$ if and only if* $\operatorname{depth} F_K(\mathcal{F}) \geq d-1$.

Proof : We induct on d. Let $d = 1$. Let (x) be a reduction of I_1. From the proof of Proposition 4.1 we get,

$$g_1 = \sum_{n \geq 1} \lambda((KI_n + xI_{n-1})/xI_{n-1}) + \sum_{n \geq 1} \lambda((KI_n : x) \cap I_{n-1}/KI_{n-1}) - \lambda(R/K).$$

Claim : For all $n \geq 1$, $(KI_n : x) \cap I_{n-1}/KI_{n-1} \cong xI_{n-1} \cap KI_n/xKI_{n-1}$.
Consider the multiplication map $\mu_x : (KI_n : x) \cap I_{n-1}/KI_{n-1} \longrightarrow xI_{n-1} \cap KI_n/xKI_{n-1}$. Let $y = xs \in xI_{n-1} \cap KI_n$ for some $s \in I_{n-1}$. Then $s \in (KI_n : x) \cap I_{n-1}$. Therefore, μ_x is surjective. Let $y \in (KI_n : x) \cap I_{n-1}$ and $xy \in xKI_{n-1}$. Since x is regular in R, $y \in KI_{n-1}$. Hence μ_x is injective so that μ_x is an isomorphism. Therefore we have,

$$\begin{aligned} g_1 &= \sum_{i\geq 1}[\lambda(KI_i/(KI_i \cap xI_{i-1})) + \lambda((xI_{i-1} \cap KI_i)/xKI_{i-1})] - \lambda(R/K) \\ &= \sum_{i\geq 1}\lambda(KI_i/xKI_{i-1}) - \lambda(R/K). \end{aligned}$$

Assume now that $d > 1$. Choose $x_1, \ldots, x_d$ such that $J = (x_1, \ldots, x_d)$, $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(\mathcal{F})$ and $(x_1^o, \ldots, x_d^o)$ is a superficial sequence in $F_K(\mathcal{F})$. Since $\gamma(\mathcal{F}) \geq d-1$, $x_1^*, \ldots, x_{d-1}^*$ is a regular sequence in $G(\mathcal{F})$ (existence of such a generating set can be derived from Proposition A.2.4 of [Ma]). Suppose $g_1 = \sum_{n\geq 1}\lambda(KI_n/JKI_{n-1}) - \lambda(R/K)$. Let " $-$ " denote images modulo $(x_1, \ldots, x_{d-1})$. Then by Lemma 3.5, $g_1 = \bar{g_1}$ and

$$\begin{aligned} \bar{g_1} &= \sum_{n\geq 1}\lambda(\bar{K}\bar{I}_n/\bar{J}\bar{K}\bar{I}_{n-1}) - \lambda(\bar{R}/\bar{K}) \qquad \text{(Since } \dim \bar{R} = 1) \\ &= \sum_{n\geq 1}\lambda((KI_n + (x_1, \ldots, x_{d-1}))/(JKI_{n-1} + (x_1, \ldots, x_{d-1}))) - \lambda(R/K) \\ &= \sum_{n\geq 1}\lambda(KI_n/(JKI_{n-1} + KI_n \cap (x_1, \ldots, x_{d-1}))) - \lambda(R/K). \end{aligned}$$

By assumption $g_1 = \sum_{n\geq 1}\lambda(KI_n/JKI_{n-1}) - \lambda(R/K)$. Therefore, $(x_1, \ldots, x_{d-1}) \cap KI_n \subseteq JKI_{n-1}$. Hence by Lemma 5.2, $x_1^o, \ldots, x_{d-1}^o$ is a regular sequence in $F_K(\mathcal{F})$.

Conversely, let depth $F_K(\mathcal{F}) \geq d-1$. Choose $x_1 \in I_1$ such that x_1^* is regular in $G(\mathcal{F})$ and x_1^o is regular in $F_K(\mathcal{F})$. Hence $F_K(\mathcal{F})/x_1^oF_K(\mathcal{F}) \cong F_{\bar{K}}(\bar{\mathcal{F}})$ and $g_1 = \bar{g_1}$, where " $-$ " denote images modulo (x_1). Then depth $F_K(\mathcal{F})/x_1^oF_K(\mathcal{F}) \geq d-2$. By induction

$$\begin{aligned} \bar{g_1} &= \sum_{n\geq 1}\lambda(KI_n + x_1R/JKI_{n-1} + x_1R) - \lambda(R/K) \\ &= \sum_{n\geq 1}\lambda(KI_n/(JKI_{n-1} + (x_1R \cap KI_n))) - \lambda(R/K). \end{aligned}$$

Since x_1^o is regular in $F_K(\mathcal{F})$ and x_1^* is regular in $G(\mathcal{F})$, $x_1R \cap KI_n = x_1KI_{n-1}$. Therefore

$$\bar{g_1} = \sum_{n\geq 1}\lambda(KI_n/(JKI_{n-1} + x_1KI_{n-1})) - \lambda(R/K) = \sum_{n\geq 1}\lambda(KI_n/JKI_{n-1}) - \lambda(R/K).$$

Since $g_1 = \bar{g_1}$, the assertion follows. $\square$

6. Cohen-Macaulay fiber cones of ideals with minimal multiplicity

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R and J be a minimal reduction of I. Let K be an ideal containing I. Let $F_K(I)$ be the fiber cone of I with respect to K and let $G(I)$ be the associated graded ring of I. Let $\mu(I)$ denote the minimum number of generators of I. It is known that $e_0(I)+d-\lambda(R/I)\geq \mu(I)$, [G]. Shiro Goto defined an ideal I to have minimal multiplicity if $e_0(I)+d-\lambda(R/I)=\mu(I)$. It can be seen that I has minimal multiplicity if and only if for any minimal reduction J of I, $I\mathfrak{m}=J\mathfrak{m}$. We generalize this notion. An ideal I is said to have minimal multiplicity with respect to K if $KI=KJ$ for any minimal reduction J of I. Let $H_K(I,n)=\lambda(R/KI^n)$ and $P_K(I,n)=\sum_{i=0}^{d}(-1)^i g_i\binom{n+d-i-1}{d-i}$ be the corresponding polynomial.

Proposition 6.1. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d.*
(i) If I has minimal multiplicity with respect to K, then $g_1=-\lambda(R/K)$.
(ii) If $KI\cap J=KJ$ and $g_1=-\lambda(R/K)$ then I has minimal multiplicity.
(iii) The ideal I has minimal multiplicty if and only if $g_1=-1$.

Proof. (i) Suppose I has minimal multiplicity with respect to K. Let J be a minimal reduction of I. Then $KI^n=KJ^n$ for all $n\geq 1$. Therefore for all $n\geq 1$,

$$\begin{aligned}\lambda(R/KI^n) &= \lambda(R/KJ^n)=\lambda(R/J^n)+\lambda(J^n/KJ^n)\\ &= e_0(I)\binom{n+d-1}{d}+\lambda(R/K)\binom{n+d-1}{d-1}.\end{aligned}$$

Hence $g_1=-\lambda(R/K)$.
(ii) Suppose $g_1=-\lambda(R/K)$ and $KI\cap J=KJ$. By Proposition 4.1, $g_1\geq\sum_{n\geq 1}\lambda(KI^n+J/J)-\lambda(R/K)$. Hence $KI\subseteq J$. Thus $KI=KI\cap J=KJ$. Hence I has minimal multiplicity with respect to K.
(iii) Follows from (i) and (ii) since $\mathfrak{m}I\cap J=\mathfrak{m}J$.

□

In the next result we generalize Proposition 2.5 of [G].

Proposition 6.2. *Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and let I be an $\mathfrak{m}$-primary ideal with minimal multiplicity with respect to $K\supseteq I$. Suppose $KI\cap J=KJ$ Then $F_K(I)$ is Cohen-Macaulay if and only if $\gamma(I)\geq d-1$.*

Proof. Suppose that I has minimal multiplicity with respect to K and $F_K(I)$ is Cohen-Macaulay. Since $e_1(I)=f_0+g_1$. By Theorem 4.7 of [HM], it is enough to show that $e_1(I)=\sum_{n\geq 1}\lambda(I^n/JI^{n-1})$. Since $F_K(I)$ is Cohen-Macaulay, by [DRV], $f_0=\sum_{n\geq 0}\lambda(I^n/JI^{n-1}+KI^n)=\lambda(R/K)+\sum_{n\geq 1}\lambda(I^n/JI^{n-1})$, the last equality holds since $KI^n\subseteq JI^{n-1}$ for all $n\geq 1$. Since I has minimal multiplicity with respect to K, $g_1=-\lambda(R/K)$. Therefore $e_1(I)=\lambda(R/K)+\sum_{n\geq 1}\lambda(I^n/JI^{n-1})-\lambda(R/K)=\sum_{n\geq 1}\lambda(I^n/JI^{n-1})$. Hence $\gamma(I)\geq d-1$.
Conversely, assume that $\gamma(I)\geq d-1$. Then $e_1(I)=\sum_{n\geq 1}\lambda(I^n/JI^{n-1})$. Since I has minimal multiplicity with respect to K, $KI^n\subseteq JI^{n-1}$ for all $n\geq 1$ so that $\sum_{n\geq 1}\lambda(KI^n+JI^{n-1}/JI^{n-1})=0$. Therefore $\sum_{n\geq 1}\lambda(KI^n+JI^{n-1}/JI^{n-1})-\lambda(R/K)=-\lambda(R/K)=g_1$. Therefore, by Theorem 4.2, $F_K(I)$ is Cohen-Macaulay. □

Proposition 6.3. *Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal of R such that $\gamma(I)\geq d-1$. Let J be a minimal reduction of I such that $KI\cap J=KJ$ and $KI^2=KJI$. Then $F_K(I)$ is Cohen-Macaulay.*

Proof. Apply induction on d. Let $d=1$. Then $KI^n = Kx^{n-1}I$ for all $n \geq 2$. For $n \geq 2$,

$$\begin{aligned}\lambda(R/KI^n) &= \lambda(R/Kx^{n-1}I) = \lambda(R/x^{n-1}R) + \lambda(x^{n-1}R/Kx^{n-1}I) \\ &= (n-1)e_0(I) + \lambda(R/KI) = e_0(I)\ n - [\lambda(R/(x)) - \lambda(R/KI)]\,.\end{aligned}$$

Therefore

$$g_1 = \lambda(R/(x)) - \lambda(R/KI) = \lambda(R/xK) - \lambda(xR/xK) - \lambda(R/KI) = \lambda(KI/KJ) - \lambda(R/K).$$

Since $KI^n = KJI^{n-1} \subseteq JI^{n-1}$, $\lambda(KI^n + JI^{n-1}/JI^{n-1}) = 0$ for $n \geq 2$. Therefore by Theorem 4.2, $F_K(I)$ is Cohen-Macaulay.

Now Let $d > 1$ and let $J = (x_1, \ldots, x_d)$ is chosen such that $x_1^* \ldots, x_{d-1}^* \in G(I)$ is a regular sequence. Let "$-$" denote modulo $(x_1, \ldots, x_{d-1})$. Then $\bar{I}^2\bar{K} = \bar{J}\bar{I}\bar{K}$ and $\bar{K}\bar{I} \cap \bar{J} = \bar{K}\bar{J}$. Therefore, by induction $F_{\bar{K}}(\bar{I})$ is Cohen-Macaulay. Therefore depth $F_{\bar{K}}(\bar{I}) = 1$. Hence, by Lemma 2.7, depth $F_K(I) = d$.

□

Theorem 6.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal of R with minimal multiplicity. Then the following statements are equivalent:*

1. *$G(I)$ is Cohen-Macaulay.*
2. *$F(I)$ is Cohen-Macaulay and $r(I) \leq 1$.*
3. *$r(I) \leq 1$.*

Proof. $(1) \Rightarrow (2)$: Since I has minimal multiplicity, $I\mathfrak{m} = J\mathfrak{m}$ for any minimal reduction J of I. Assume that $G(I)$ is Cohen-Macaulay. Then by Proposition 6.2, $F(I)$ is Cohen-Macaulay. Therefore

$$\begin{aligned} f_0 &= \sum_{n\geq 0} \lambda(I^n/JI^{n-1} + \mathfrak{m}I^n) = 1 + \sum_{n \geq 1} \lambda(I^n/JI^{n-1}) \text{ (since } \mathfrak{m}I^n \subseteq JI^{n-1} \text{ for all } n \geq 1) \\ &= 1 + \lambda(I/J) + \sum_{n\geq 2} \lambda(I^n/JI^{n-1}) \\ &= 1 + e_0(I) - \lambda(R/I) + \sum_{n \geq 2} \lambda(I^n/JI^{n-1}) \\ &= 1 + \mu(I) - d + \sum_{n\geq 2} \lambda(I^n/JI^{n-1}) \end{aligned}$$

Since $I^2 \subseteq I\mathfrak{m} = J\mathfrak{m} \subseteq J$, $I^2 = I^2 \cap J = IJ$, as $G(I)$ is Cohen-Macaulay.
$(2) \Rightarrow (3)$: Clear.
$(3) \Rightarrow (1)$: This is known but we recall the argument. Assume that $r(I) \leq 1$. Then $I^{n+1} = JI^n$ for all $n \geq 1$ so that $JI^n = I^{n+1} \cap J$ for all $n \geq 1$. Therefore by [VV], $x_1^*, \ldots, x_d^*$ is a regular sequence in $G(I)$. Hence $G(I)$ is Cohen-Macaulay. □

Example 6.5. Let $R = k[\![t^4, t^5, t^6, t^7]\!]$ and let $I = (t^4, t^5, t^6)$. Then $J = (t^4)$ is a minimal reduction of I. It can easily be checked that $I\mathfrak{m} = J\mathfrak{m}$. Hence $F(I)$ is Cohen-Macaulay. Since I has minimal multiplicity, $g_1 = -1$. Since $\mathfrak{m}I^n \subseteq J$ for all $n \geq 1$, $g_1 = -1 = \sum_{n\geq 1} \lambda(\mathfrak{m}I^n + JI^{n-1}/JI^{n-1}) - \lambda(R/\mathfrak{m})$.

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, India - 400076

E-mail address: jayan@math.iitb.ac.in

E-mail address: jkv@math.iitb.ac.in