EXERCISES IN
MA 510 : COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY
SPRING 2006

1. **Examples of algebraic varieties**

(1) Find all algebraic varieties in $\mathbb{A}^1_k$.
(2) Show that for a finite field F, all subsets of $\mathbb{A}^n_F$ are algebraic varieties.
(3) Show that the set $\{(t, t^2, t^3) \mid t \in k\}$ is an algebraic variety in $\mathbb{A}^3_k$.
(4) Show that the following sets are not algebraic varieties: (a) $\{(x, y) \in \mathbb{A}^2_{\mathbb{R}} \mid y = \sin x\}$. (b) $\{(\cos t, \sin t, t) \in \mathbb{A}^3_{\mathbb{R}} \mid t \in \mathbb{R}\}$. (c) $\{(x, e^x) \in \mathbb{A}^2_{\mathbb{R}} \mid x \in \mathbb{R}\}$.
(5) Let $p = (a_1, a_2, \ldots, a_n)$ and $q = (b_1, b_2, \ldots, b_n)$ be distinct points of $\mathbb{A}^n_k$. Show that the line $L_{pq} = \{tp + (1-t)q \mid t \in k\}$ is an algebraic variety of $\mathbb{A}^n_k$ defined by the set of the linear polynomials

$$\{(a_i - b_i)(x_j - b_j) - (a_j - b_j)(x_i - b_i) \mid 1 \le i, j \le n\}.$$

(6) Show that if $V \subseteq \mathbb{A}^n_k$ is an algebraic variety then $L_{pq} \subseteq V$ or $L_{pq} \cap V$ is finite.
(7) Let $M_n(k)$ denote the set of all $n \times n$ matrices with entries from a field k. Show that the sets of symmetric, skew-symmetric, orthogonal, trace 0 and diagonal matrices are algebraic varieties.
(8) Show that the set of all matrices in $M_n(k)$ of rank atmost r, where $r \le n$ is an algebraic variety.
(9) If a field k is not algebraically closed, then any algebraic variety in $\mathbb{A}^n_k$ can be written as $V(g)$ where g is a polynomial. **Hint:** Show that there a polynomial $\phi(x_1, x_2, \ldots, x_m) \in k[x_1, x_2, \ldots, x_m]$, such that $V(\phi) = \{0\}$. Let $V = V(f_1, f_2, \ldots, f_m)$. Consider the solutions to $\phi(f_1, f_2, \ldots, f_m) = 0$.
(10) Let $V \subseteq \mathbb{A}^n_{\mathbb{C}}$ be an algebraic variety such that $\mathbb{Z}^n \subseteq V$. Show that $V = \mathbb{A}^n_{\mathbb{C}}$.

2. **Noetherian rings**

Let R be a nonzero commutative ring and k be a field.
(11) Show that the ring of continuous real valued functions on $[0, 1]$ is not Noetherian.

(12) Let X be an infinite set. Show that $R = \{f : X \to \mathbb{F}_2 \mid f \text{ is a function}\}$ is not Noetherian.

(13) Show that if R is Noetherian then so is the formal power series ring $R[[x]]$.

(14) This exercise outlines a proof of Cohen's theorem: If all prime ideals of R are finitely generated then R is Noetherian.

(a) Prove that if the collection $\mathcal{C}$ of ideals of R that are not finitely generated is nonempty, then it contains a maximal element I and that R/I is Noetherian.

(b) Show that the ideal I in (a) is a prime ideal. **Hint:** Let $x, y \in R$ be such that $xy \in I$ and neither x nor y is in I. Note that (I, x) and $(I : x)$ are finitely generated. Let J be a finitely generated ideal such that $(J, x) = (I, x)$. Now show that $I = J + x(I : x)$. This contradicts the fact that I is not finitely generated.

(15) Let R be a Noetherian ring and $f : R \to R$ be a surjective ring homomorphism. Show that f is an isomorphism. **Hint:** Consider the ascending chain $\mathrm{Ker}(f^n)$.

3. **Decomposition of algebraic varieties**

(16) Decompose the complex varieties $V(y - x^2)$ and $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3)$ as subsets $\mathbb{A}^2_{\mathbb{C}}$.

(17) Show that $f(x, y) = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial but $V(f)$ is reducible.

(18) Let $V \subset W \subset \mathbb{A}^n_k$ be algebraic varieties. Show that each irreducible component of V is contained in an irreducible component of W.

(19) Show that $\mathbb{A}^n_k$ is irreducible if and only if k is infinite.

(20) Find the irreducible components of $V(y^2 - xy - x^2y - y)$ in $\mathbb{A}^2_{\mathbb{R}}$ and in $\mathbb{A}^2_{\mathbb{C}}$. Do the same for $V(y^2 - x(x^2 - 1))$.

4. **Integral extensions, Noether normalization and Nullstellensatz**

(21) Show that a UFD is integrally closed in its quotient field.

(22) Find the integral closures of $k[x, y]/(y^2 - x^3)$ and $k[x, y]/(x - y^2)$ in their quotient fields.

(23) Let R be a UFD and let $P = (t)$ be a principal proper prime ideal of R. Show that there is no nonzero prime ideal Q of R such that Q is properly contained in P.

(24) Let $V = V(f)$ be an irreducible hypersurface in $\mathbb{A}^n$. Show that there is no irreducible algebraic variety W such that W is properly between V and $\mathbb{A}^n$.

(25) Let $P = (x^2 - y^3, y^2 - z^3)$ be an ideal in $R = k[x, y, z]$ where k is a field. Define the ring homomorphism $f : R \to k[t]$ by $f(x) = t^9, f(y) = t^6, f(z) = t^4$. Show that the kernel of f is P and hence P is a prime ideal. Show that $I(V(P)) = P$. Is R integrally closed in its quotient field ? Find $f \in R/P$ which is transcendental over k such that R/P is a finite $k[f]$-algebra.

(26) Let $R = k[x, y]/(y^2 - x^3 + x)$. Find an algebraically independent $f \in R$ such that R is integral over $k[f]$.

(27) Let S/R be an integral ring extension where S is a finite R-algebra generated by n elements. Let m be a maximal ideal of R. Show that there are atmost n maximal ideals in S containing mS.

(28) Let k be an algebraically closed field and $R = k[x_1, x_2, \ldots, x_n]$. Let I be an ideal of R. Suppose that $V(I) = \{P_1, P_2, \ldots, P_r\}$. Consider the map $\phi : R \to k^r$ defined by $\phi(f) = (f(P_1), f(P_2), \ldots, f(P_r))$. Show that f is a surjective linear transformation and find its kernel.

(29) Let the notation be as in the above exercise. Show that $V(I)$ is finite if and only if R/I is a finite dimensional k-vector space.

(30) Let k be an algebraically closed field and $R = k[x_1, x_2, \ldots, x_n]$. Let I be a proper ideal of R. Show that $\sqrt{I} = \cap\{m_a \mid a \in V(I)\}$. Here $m_a = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ for $a = (a_1, a_2, \ldots, a_n)$. Using this show that for any non-maximal prime ideal P of R, $V(P)$ is infinite.

5. **Polynomial and rational functions and maps of affine varieties**

(31) Let $C = V(y^2 - x^3)$. Show that the map $\phi : \mathbb{A}^1_{\mathbb{C}} \to C, \quad \phi(t) = (t^2, t^3)$, is a homeomorphism in Zariski topology but it is not an isomorphism of affine varieties.

(32) Show that the hyperbola $V(xy - 1) \subset \mathbb{A}^2_{\mathbb{C}}$ is not isomorphic to the complex affine line.

(33) Give an example to show that the image of a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^m$ need not be an affine variety.

(34) Let $X = V(x, y)$ and $Y = V(z, w)$ be subvarieties of $\mathbb{A}^4_{\mathbb{C}}$. Show that the ideal of $X \cup Y$ cannot be generated by two polynomials.

(35) Let $k = \overline{\mathbb{F}_p}$. Consider the *Frobenius map:*

$$F : \mathbb{A}^n_k \to \mathbb{A}^n_k, \quad F(a_1, a_2, \ldots, a_n) = (a_1^p, a_2^p, \ldots, a_n^p).$$

Show that F is a bijective morphism. Is it an isomorphism of varieties ?

(36) Let $\phi : V \to W$ be a polynomial map of affine varieties. Let $\phi^* : k[W] \to k[V]$ be the corresponding k-algebra homomorphism of their coordinate rings. Let $\phi(p) = q$ where $p \in V$. Show that ϕ^* extends uniquely to a ring homomorphism $\delta : \mathcal{O}_{W,q} \to \mathcal{O}_{V,p}$ and δ maps the unique maximal ideal of $\mathcal{O}_{W,q}$ into that of $\mathcal{O}_{V,p}$.

(37) Let V be an irreducible affine variety and $f \in k(V)$. The pole set of f is defined to be the set of points of V where f is not defined. Show that the pole set of f is an algebraic subvariety of V.

(38) let $V = V(Y^2 - X^2(X+1)) \subset \mathbb{A}^2_k$. Let x and y be residues of X and Y respectively in $k[V]$. Let $z = y/x$. Find the set of poles of z and z^2.

(39) Let V be an affine variety and $p \in V$. Show that there is a one-to-one correspondence between prime ideals in $\mathcal{O}_{V,p}$ and and subvarieties of V which pass through p. **Hint:** Show that if P is a prime ideal of $\mathcal{O}_{V,p}$, then $P \cap k[V]$ is a prime ideal of $k[V]$ and P is generated by $P \cap k[V]$.

(40) Let $C = V(Y^2 - X^2 - X^3)$. Show that $\phi : \mathbb{A}^1_k \to C$ given by $\phi(t) = (t^2 - 1, t^3 - t)$ is not an isomorphism. Is $\phi : \mathbb{A}^1_k \setminus \{1\} \to C$ an isomorphism ?

6. **Projective varieties**

(41) Let $T : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be an invertible linear transformation. Then T maps lines through origin to lines through origin. Hence T determins a map of $\mathbb{P}^n$ called a *projective change of coordinates.* Let $T = (T_1, T_2, \ldots, T_n)$ where $T_1, \ldots, T_n$ are linear forms. Let $V = V(F_1, F_2, \ldots, F_r)$ where $F_1, F_2, \ldots, F_r$ are forms in $S = k[x_0, x_1, \ldots, x_n]$. Show $T^{-1}(V) = V(G_1, G_2, \ldots, G_r)$ where $G_i = F_i(T_1, T_2, \ldots, T_n)$ for $i = 1, 2, \ldots, r$. Let $S(V)$ denote the homogeneous coordinate ring of a projective variety V. Show
(i) $S(V)$ is isomrphic to $S(T^{-1}V)$,
(ii) $k(V)$ is isomorphic to $k(T^{-1}(V))$ and
(iii) $\mathcal{O}_{V,p}$ is isomorphic to $\mathcal{O}_{T^{-1}(V),T^{-1}(p)}$.

(42) Consider the real affine quadrics: $C = V(x^2 + y^2 - 1)$, $H = V(x^2 - y^2 - 1)$ and $P = V(x^2 - y)$.
(i) Determine the intersections of their projective closures C^*, H^* and P^* with the line at infinity.
(ii) Show that C^* and H^* are projectively equivalent to P^*.

(43) Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ be given by $\phi((x_0 : x_1)) = (x_0^2 : x_0x_1 : x_1^2)$. Show that $C = \phi(\mathbb{P}^1)$ and $\mathbb{P}^1$ are isomorphic as projective varieties but their homogeneous coordinate rings are not.

(44) The variety defined by a linear form is called a hyperplane. Show that the intersection of m hyperplanes in $\mathbb{P}^n$ is nonempty for $m \le n$.

(45) Show that two distinct lines in $\mathbb{P}^2$ intersect in one point.

(46) Let $z \in k(V)$ where V is a projective variety. A point $p \in V$ is called a pole of z if z is not regular at p. Show that the set of poles of z is a projective subvariety of V.

(47) Let $R = k[x_0, x_1, \ldots, x_{n-1}]$ and $S = k[x_0, x_1, \ldots, x_n]$. For $f \in R$, let f^* denote its homogenization with respect to x_n. For $F \in S$, Let $F_* = F(x_0, x_1, \ldots, x_{n-1}, 1)$. For

a homogeneous ideal I of S, let I_* denote the ideal generated by $\{F_* \mid F \in I\}$ and $V_* = V(I_*)$. For an ideal J of R, let J^* denote the ideal generated by $\{f^* \mid f \in R\}$ and $V^* = V(I^*)$.

(i) Let $H_\infty = \mathbb{P}^n \setminus U_{n+1}$. Let V be a proper affine subvariety of $\mathbb{A}^n$. Show that no irreducible component of V^* lies in or contains H_∞.

(ii) Let V be a projective variety in $\mathbb{P}^n$ so that no irreducible component of V lies in or contains H_∞. Show that V_* is a proper subvariety of $\mathbb{A}^n$ and $(V_*)^* = V$.

(48) Show that if $V \subset W \subset \mathbb{P}^n$ are projective varieties and V is a hypersurface, then $W = V$ or $W = \mathbb{P}^n$.

(49) Suppose that $V \subset \mathbb{P}^n$ is a projective variety and $H_\infty \subset V$. Show that either $V = \mathbb{P}^n$ or $V = H_\infty$. If $V = \mathbb{P}^n$ then $V_* = \mathbb{A}^n$ and if $V = H_\infty$, then $V_* = \emptyset$.

(50) Let $V = V(y - x^2, z - x^3) \subset \mathbb{A}^3$. Prove:

(i) $I(V) = (y - x^2, z - x^3)$.

(ii) $xy - zw \in I(V)^* \setminus (wy - x^2, w^2 z - x^3)$.

7. **Noetherian Modules**

Let R be a commutative ring.

(51) Let M be a Noetherian R-module. Let $u : M \to M$ be a module homomorphism. Show that if u is surjective then, u is an isomorphism. Hint: Consider the submodules $\ker(u^n)$.

(52) Let M be an R-module and N_1, N_2 be submodules of M. Show that if M/N_1 and M/N_2 are Noetherian, then so is $M/(N_1 \cap N_2)$.

(53) The annihilator of an R-module M is defined by $\operatorname{ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. Show that if M is a Noetherian R-module, then $R/\operatorname{ann} M$ is a Noetherian ring. Hint: Let $M = Rm_1 + Rm_2 + \cdots + Rm_n$ and $M_i = M$ for all $i = 1, 2, \ldots, n$. Consider the map $f : R \to M_1 \oplus M_2 \oplus \cdots \oplus M_n$ defined by $f(r) = (rm_1, rm_2, \ldots, rm_n)$.

(54) Show that a vector space V over a field k is a Noetherian k-module if and only if it is finite dimensional.

(55) Let p be a fixed prime number. Let G be the subgroup of $\mathbb{Q}/\mathbb{Z}$ whose order is p^n for some n. Show that G has exactly one subgroup G_n of order p^n for each n. Show that G is not a Noetherian $\mathbb{Z}$-module.

8. **Morphisms of projective varieties**

(56) Define $f : \mathbb{P}^1 \to \mathbb{P}^m$ by $[u : v] \mapsto [u^m : u^{m-1}v : u^{m-2}v^2 : \cdots : v^m]$. Prove:

(i) f is a morphism of projective varieties.

(ii) The image C of f is the set of points $[x_0 : x_1 : \cdots : x_m] \in \mathbb{P}^m$ such that

$[x_0 : x_1] = [x_1 : x_2] = \cdots [x_{m-1} : x_m]$.
(iii) The variety C is defined by the polynomials which are 2×2 minors of the matrix with indeterminate entries:

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{m-1} \\ x_1 & x_2 & x_3 & \cdots & x_m \end{bmatrix}.$$

(iv) The variety C and $\mathbb{P}^1$ are isomorphic.

(57) Take $m = 3$ in the above exercise. The curve C is called the twisted cubic It is defined by three quadrics:

$$Q_1 = V(xz - y^2), \quad Q_2 = V(xt - yz), \quad Q_3 = V(yt - z^2).$$

Show that the intersection of any of the two quadrics above is the union of C and a line. Therefore C is not the intersection of any of the three quadrics.

(58) Let $F = V(xt^2 - 2yzt + z^3)$. Show that $C = Q_1 \cap F$.

(59) Find the group of automorphims of $\mathbb{P}^1$.

(60) Two subvarieties V and W of $\mathbb{P}^n$ are called projectively equivalent if there is a projective change of coordinates of $\mathbb{P}^n$ which defines an isomorphism of V and W.
(i) Show that homogeneous coordiante rings of projectively equivalent subvarieties of $\mathbb{P}^n$ are isomorphic. (ii) Give an example of two projective plane curves that are isomorphic but not projectively equivalent.

9. **Resultants and Bezout's theorem**

(61) If in Pascal's theorem, if we let some vertices coincide (the side being a tangent), we get many new theorems.
(a) State and sketch what happens if $P_1 = P_2,\ P_3 = P_4\ P_5 = P_6$.
(b) Let $P_1 = P_2$ and the other four points distinct. Deduce a rule for constructing a tangent to a given conic at a given point, using only a ruler.

(62) Let C be an irreducible cubic. Let L be a line which intersects C at three distinct points P_1, P_2 and P_3. Let L_i be the tangent to C at P_i, and $L_i \cap C = \{P_i, Q_i\}$ for $i = 1, 2, 3$. Show that Q_1, Q_2, Q_3 are collinear. Hint: L^2 is a conic.

(63) Let F be a field and $f(x), g(x) \in F[x]$. Let K be a splitting field of fg so that in $K[x]$, $f(x) = a(x - a_1)(x - a_2)\ldots(x - a_n)$, $\quad g(x) = b(x - b_1)(x - b_2)\ldots(x - b_m)$. Show that $R(f, g) = a^m b^n \prod_{i=1}^n \prod_{j=1}^m (a_i - b_j)$.

(64) Show that (a) $R(g, f) = (-1)^{mn} R(f, g)$.
(b) $R(f, g) = a^{degg} \prod_{i=1}^n g(a_i)$.
(c) If $g = fq + r$, then $R(f, g) = a^{degg - degr} R(f, r)$.

(65) The discriminant $D(f)$ of f is defined by $D(f) = (-1)^{\binom{n}{2}} R(f, f')$.
(a) Let $f(x) = x^2 + ax + b$. Show that $D(f) = a^2 - 4b$.
(b) Let $f(x) = x^3 + px + q$. Show that $D(f) = -4p^3 - 27q^2$.
(c) Show that $D(fg) = D(f)D(g)(R(f, g))^2$.

10. **Tangent space at a point of an affine variety**

(66) Let $V \subset \mathbb{A}^n$ be an affine variety and $p \in V$. For each $r \in \mathbb{N}$, put

$$S_r(V) = \{q \in V \mid \dim T_q(V) \geq r\}.$$

Show that $S_r(V)$ is a closed set in V.
(67) Let V be an irreducible affine variety. Show that there is an open dense subset $W \subset V$ such that all points of W are smooth points of V.
(68) Consider the morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^4$ defined by $\varphi(t) = (t^4, t^5, t^6, t^7)$. Show that $C = \varphi(\mathbb{A}^1)$ is an algebraic curve. Find the tangent space of C at origin. Show that C is not isomorphic to a curve in affine 3-space.
(69) Show that $V(x_0^d + x_1^d + x_2^d + \cdots + x_n^d) \subset \mathbb{A}_k^n$ is nonsingular if char k does not divide d.
(70) Prove that the intersection of a hypersurface V, which is not a hyperplane, with T_pV is singular at $p \in V$.

11. **Modules of finite length**

(71) Let M be a module over a ring R and $N \subseteq M$ a submodule. Suppose that M/N has finite length. Let $x \in R$ such that $\mu_x : M \to M$ is injective and M/xM has finite length. Show that $\ell(M/xM) = \ell(N/xN)$.
(72) Let k be a field and $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring. Let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$. Find $\ell(R/\mathfrak{m}^n)$.
(73) Let I and J be comaximal ideals of a ring R. Show that if $\ell(R/(I \cap J)) < \infty$, then $\ell(R/(I \cap J) = \ell(R/I) + \ell(R/J)$.
(74) Let $S = k[x, y]$ and $R = S/(x^2, y^2, xy)$. Show that R is an S-module of finite length. Find $\ell(R)$. Show that R is an Artinian ring.
(75) Show that an injective endomorphism of an Artinian module M is an automorphism of M.

12. **Dimension of algebraic varieties**

(76) Show that if k is algebraically closed then $\mathbb{A}_k^n$ and $\mathbb{P}^n$ are n-dimensional.
(77) Show that an irreducible hypersurface in $\mathbb{A}^n$ is $(n-1)-$dimensional.
(78) Let V be a d-dimensional irreducible affine variety in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$ such that $V \cap H \neq \emptyset$ and V is not contained in H. Show that all irreducible components of $V \cap H$ have dimension $d-1$.
(79) Show that an irreducible affine variety is zero-dimensional if and only if it is a point.
(80) Show that a irreducible subvariety of the affine plane is one-dimensional if and only if it a plane curve.

QUIZ I : MA 510: Algebraic Geometry

Duration: 11.35-12.30 **Max. Marks:** 10
Date: Feb 6, 2006 **Weightage:** 10 %

(1) Let $f, g \in k[x, y]$ be coprime polynomials. Show that $V(f) \cap V(g)$ is a finite set. [2]
(2) Find a Noether normalization of $R = k[X, Y]/(XY - 1)$. [2]
(3) Let $R = k[x_1, x_2, \ldots, x_n]$ and I be an ideal of R. Show that if R/I is a finite dimensional k-vectorspace then $V(I)$ is a finite set. [2]
(4) Let $R = k[x, y, z]$ and $J = (xy, yz, xz)$. Find the generators of $I(V(J))$. Show that J cannot be generated by two polynomials in R. Find $V(I)$ where $I = (xy, xz - yz)$. Show that $\sqrt{I} = J$. [4]

QUIZ II : MA 510: Algebraic Geometry

Duration: 5-6 p.m. **Max. Marks:** 10
Date: March 18, 2006 **Weightage:** 10 %

(1) Let $F \in S = k[x_0, x_1, \ldots, x_n]$ be an irreducible homogeneous polynomial. Let $V(F) \subset W \subset \mathbb{P}^n$ where W is a projective variety. Show that $W = V(F)$ or $W = \mathbb{P}^n$. [2]
(2) Consider the map $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$\varphi([s : t]) = [s^3 : s^2t : st^2 : t^3].$$

Show: Image$(\varphi) = V(xz - y^2, xw - yz, yw - z^2)$. [3]
(3) Let I be an ideal of the polynomial ring $R = k[x_0, x_1, \ldots, x_{n-1}]$. For a polynomial $f \in R$ of degree d let $f^* = x_n^d f(x_0/x_n, x_1/x_n, \ldots, x_{n-1}/x_n)$ denote its homogenization. Let I^* denote the ideal in S generated by f^* for all $f \in I$. Let $\varphi : \mathbb{A}^n \to \mathbb{P}^n$ be the map

$$\varphi((a_0, a_1, \ldots, a_{n-1})) = [a_0 : a_1 : \cdots : a_{n-1} : 1].$$

Prove the following statements:
(a) The closure V^* of $\varphi(V)$ in $\mathbb{P}^n$ in Zariski topology is $V(I(V)^*)$. [3]
(b) If V is irreducible, then so is V^*. [2]

QUIZ III : MA 510: Algebraic Geometry

Duration: 5.30-6.30 p.m. **Max. Marks:** 10

Date: 3 April 2006 **Weightage:** 10 %

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(1) Let M be an R-module and N and P be submodules of M. Show that M/N and M/P are Noetherian R-modules if and only if $M/(N \cap P)$ and $M/(N+P)$ are Noetherian R-modules. [3]

(2) Define $f : \mathbb{P}^1 \to \mathbb{P}^3$ by $[u : v] \mapsto [u^3 : u^2v : uv^2 : v^3]$. Let $C = f(\mathbb{P}^1)$. Show that C is defined by three quadrics:

$$Q_1 = V(xz - y^2), \quad Q_2 = V(xt - yz), \quad Q_3 = V(yt - z^2).$$

Show that $Q_1 \cap Q_2$ is the union of C and a line. [4]

(3) Let $F = V(xt^2 - 2yzt + z^3)$. Show that $C = Q_1 \cap F$. [3]

Mid-Semester Examination : MA 510: Algebraic Geometry

Duration: 9.30-11.30 **Max. Marks:** 30

Date: Feb 26, 2006 **Weightage:** 30 %

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(1) Let $f(X,Y,Z) = XY + YZ + ZX$ and $R = k[X,Y,Z]/(f)$. Find a Noether normalization of R using a linear change of co-ordinates.

(2) Let $f : V \to W$ be a polynomial map of affine varieties. Show that f is continuous in Zariski topology.

(3) Let V be an irreducible affine variety and $p \in V$. Consider the ring

$$\mathcal{O}_{V,p} = \{f \in k(V) \mid f \text{ is defined at } p\}.$$

Show that $\mathcal{O}_{V,p}$ is a local Noetherian domain.

(4) Let $V = V(XY - ZW)$ and $k[V] = k[x,y,z,w] = k[X,Y,Z,W]/I(V)$. Find the domain of $f = x/z$.

(5) Prove that a polynomial map $F : V \to W$ of affine varieties V and W is an isomorphism of V onto $F(V)$ if and only if $F^* : k[W] \to k[V]$ is surjective.

(6) Define $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ by $\varphi(t) = (t^3, t^4, t^5)$. Show that the image of φ is the space curve $C = V(Y^2 - XZ, Z^2 - X^2Y, X^3 - YZ)$.

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End-Semester Examination : MA 510 : Algebraic Geometry

Duration: 2.30-5.30 **Max. Marks:** 40
Date: April 24, 2006 **Weightage:** 40 %

Let k be an algebraically closed field. Let $\mathbb{A}^n$ and $\mathbb{P}^n$ denote the n-dimensional affine and projective spaces over k respectively.

(1) Let F and G be homogeneous polynomials of degree m and n respectively in the polynomial ring $R = k[x, y, z]$. Let $V(F)$ and $V(G)$ be the plane projective curves defined by F and G in $\mathbb{P}^2$. Show that $V(F) \cap V(G) \neq \emptyset$. [4]

(2) Find the singular points of the affine variety $V = V(x_1^d + x_2^d + \cdots + x_n^d) \subset \mathbb{A}^n$. [4]

(3) Let $V \subset \mathbb{A}^n$ be an affine variety and $p \in V$. Show that there is a one-to-one correspondence between prime ideals in the local ring $\mathcal{O}_{V,p}$ and subvarieties of V containing p. [4]

(4) Using Pascal's theorem, describe a procedure for constructing a tangent line to a conic by using ruler and compass. [4]

(5) Let $V \subset \mathbb{A}^n$ be an affine variety. Let $\mathcal{O}_{V,p}$ be the local ring of V at p. Let $\mathfrak{m}_p$ denote its unique maximal ideal. Show that the dimension of the tangent space $T_p(V)$ is $n - \text{rank } J(p)$ where J denotes the Jacobian matrix of V at p. [6]

(6) Show that the intersection W of a hypersurface $V \subset \mathbb{A}^n$ and its tangent space $T_p(V)$ at $p \in V$ is singular at p. [6]

(7) Show that an irreducible projective variety $V \subset \mathbb{P}^n$ has dimension $n - 1$ if and only if $V = V(f)$ for an irreducible homogeneous polynomial f. [6]

(8) Let $C \subset \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $p = (a, b) \in \mathbb{A}^2$. Make a linear change of coordinates so that $p = (0, 0)$. Write $f = f_0 + f_1 + \cdots + f_d$ where f_i is homogeneous of degree i in x, y. Define the multiplicity $\mu_p(C)$ of C at p to be the least r such that $f_r \neq 0$. Show that $\mu_p(C) = 1$ if and only if p is a smooth point of C. [6]

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