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Mixed Hilbert Coefficients of Homogeneous d-Sequences and Quadratic Sequences

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0. INTRODUCTION

This paper is the outcome of our attempt to answer the following question. Let $R = k[X]$ be the polynomial ring obtained by adjoining to a field k the entries of an $m \times n$ matrix X of indeterminates (we assume $m \leq n$), let M denote the maximal ideal of R generated by the entries of X, and let I be the ideal of R generated by the $m \times m$ minors of X. It is well known (see the next paragraph for hint of a proof) that the dimension of $M^r I^s / M^{r+1} I^s$ as a vector space over $R/M = k$ is, for large r and s, given by a polynomial in r and s of total degree one less than the dimension of R:

$$\dim_k \frac{M^r I^s}{M^{r+1} I^s} \underset{r,s \gg 0}{=} \sum_{i+j \leq \dim R - 1} a_{ij}(M \mid I) \binom{r+i}{i} \binom{s+j}{j}. \tag{1}$$

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The coefficients $a_{ij}(\mathsf{M}|I)$ of top degree, that is, with $i+j=\dim R-1$, are called the *mixed multiplicities* of the ordered pair $(\mathsf{M}|I)$. The problem of obtaining formulas for the mixed multiplicities of $(\mathsf{M}|I)$ in terms of m and n was what started us out on this paper.

Our approach to mixed multiplicities is through the following slightly more general problem. The associated graded ring $\mathrm{gr}_{(\mathsf{M},It)} R[It]$ of the Rees algebra $R[It]$ with respect to the maximal ideal (M, It) has a natural bigrading on it, and the graded piece of degree (r,s) in this bigrading is $\mathsf{M}^r I^s/\mathsf{M}^{r+1}I^s$. Thus the problem of computing this bigraded Hilbert series in terms of m and n would, if solved, lead to formulas for all the coefficients $a_{ij}(\mathsf{M}|I)$, not just for those of top degree. We solve this problem. The expression we obtain for the bigraded Hilbert series

$$\mathbb{H}\big(\mathrm{gr}_{(\mathsf{M},It)} R[It];\lambda_1,\lambda_2\big)=\sum_{r,s\geq 0}\dim_k \frac{\mathsf{M}^r I^s}{\mathsf{M}^{r+1}I^s}\lambda_1^r\lambda_2^s$$

is (see Corollary 3.2.1 and Example 3.2.3)

$$\mathbb{H}\big(\mathrm{gr}_{(\mathsf{M},It)} R[It];\lambda_1,\lambda_2\big)=H(R;\lambda_1)+\lambda_2\sum_{\omega\in\Omega}H\left(\frac{R}{R\Pi^\omega};\lambda_1\right)H(F_\omega;\lambda_2), \tag{2}$$

where

$H(R;\lambda_1)$ is the Hilbert series $\sum_{i\geq 0}(\dim_k R_i)\lambda_1^i$ of the graded k-algebra R;

Π is the poset of all minors of all sizes of the matrix X;

Ω is the ideal of Π consisting of the $m\times m$ minors;

$\Pi^\omega=\{\pi\in\Pi\mid \pi\not\geq\omega\}$, the ideal of Π "cogenerated by ω";

$R\Pi^\omega$ is the ideal of R generated by the elements of Π^ω;

F_ω is the face ring with coefficients in the field k of the poset $\Pi_\omega=\{\pi\in\Pi\mid\pi\leq\omega\}$.

The face rings F_ω are determined entirely by the combinatorics of the underlying poset Ω; recall that the face ring of a poset P, with coefficients in a field k, is the quotient of the polynomial ring $k[x_p\mid p\in P]$, where the x_p are indeterminates, by the ideal $(x_px_q\mid(p,q)\in\Delta)$, where Δ is the set of all incomparable pairs of elements in P. The rings $R/R\Pi^\omega$ are rather well studied quotients of R—formulas for the Hilbert series of these rings have been obtained by Abhyankar [A] and by several authors after him. Equation (2) therefore gives an effective method for calculating the bigraded Hilbert series in terms of m and n.

Using the fact that all maximal chains in Π have the same length, we deduce from eq. (2) the following expression for the mixed multiplicities of $(\mathfrak{M} \mid I)$ (see Corollary 3.2.2):

$$a_{ij}(\mathfrak{M} \mid I) = \sum_{\substack{\omega \in \Omega \\ \mathrm{rk}(\omega) = j+1}} \mathrm{e}\left(\frac{R}{R\Pi^{\omega}}\right) \mathrm{e}(\omega) \qquad \text{for } i + j = \dim R - 1, \quad (3)$$

where

$\mathrm{e}(R/R\Pi^{\omega})$ is the multiplicity as a graded k-algebra of $R/R\Pi^{\omega}$;

$\mathrm{rk}(\omega)$ is the length of any maximal chain $\omega_1 \lneq \cdots \lneq \omega_p = \omega$ in Ω;

$\mathrm{e}(\omega)$ is the number of distinct maximal chains of the type $\omega_1 \lneq \cdots \lneq \omega_p = \omega$.

The referee has urged that the authors provide for the reader's convenience explicit expressions for the above mixed multiplicities in terms of m and n. Unfortunately, however, we do not know a closed-form expression for the right-hand side of eq. (3). For more information and for more admissions of ignorance about closed-form expressions, the reader is referred to Examples 2.6 and 2.8 of [RS].

Equation (2) holds more generally (see Corollary 3.2.1) for an ideal I generated by elements of a straightening-closed ideal Ω of a graded algebra R with straightening law over a field k on a finite poset Π, provided that degree $\omega \leq$ degree ω' as elements of R whenever $\omega \leq \omega' \in \Omega$. (Here $\mathfrak{M}$ denotes the graded maximal ideal of R.) In fact, the setup we work with is even more general and includes within its ambit ideals generated by d-sequences and Huckaba–Huneke ideals of analytic deviation 1 and 2 (see Sect. 3).

Having discussed above the results of the paper, we now give a more formal introduction to its content and technique. We restrict our attention in this introduction to the Rees algebra alone, but statements analogous to the ones to be made here hold for the associated graded ring and the extended Rees algebra.

We begin with a basic definition. Let R be a Noetherian ring, $\mathfrak{M}$ a maximal ideal of R, and I an ideal of R. There exists a natural bigrading on the associated graded ring $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ of the Rees algebra $R[It]$ with respect to the maximal ideal $(\mathfrak{M}, It)$, and the graded piece of degree (r, s) is $\mathfrak{M}^s I^s / \mathfrak{M}^{r+1} I^s$. The dimension of $\mathfrak{M}^r I^s / \mathfrak{M}^{r+1} I^s$ as a vector space over $R/\mathfrak{M}$ is therefore a polynomial in r and s for large values of r and s. We write

$$\dim_{R/\mathfrak{M}} \frac{\mathfrak{M}^r I^s}{\mathfrak{M}^{r+1} I^s} \underset{r,s \gg 0}{=} \sum a_{ij}(\mathfrak{M} \mid I) \binom{r+i}{i} \binom{s+j}{j}$$

and christen the a_{ij} the *mixed Hilbert coefficients* of the ordered pair $(\mathfrak{M} \mid I)$.

Let R be a quotient by a graded ideal of a polynomial ring in finitely many variables over a field k. Let $\mathfrak{M}$ denote the irrelevant maximal ideal of R. Let $x_1, \ldots, x_n$ be a d-sequence of homogeneous elements in R with $\deg(x_1) \leq \cdots \leq \deg(x_n)$. Let I denote the ideal $(x_1, \ldots, x_n)$. The multiplicity of the Rees algebra $R[It]$ of I with respect to the maximal ideal $(\mathfrak{M}, It)$ is computed in [HTU] by Herzog, Trung, and Ulrich. The answer is in terms of the multiplicities as graded algebras over k of R and the quotients of R by the *related ideals* $I_j = ((x_1, \ldots, x_{j-1}) : x_j)$, $1 \leq j \leq n$, of the d-sequence. In [RS] the technique and results of [HTU] are extended to linearizations of quadratic sequences with nondecreasing degrees. Here we push the work in [HTU] and [RS] to its logical conclusion: we write down the Hilbert series and Hilbert polynomial of $R[It]$ with respect to $(\mathfrak{M}, It)$ in terms of those of R and R/I_j as graded algebras over k. In fact, we write down the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$. From this we get not only the (singly graded) Hilbert series and Hilbert polynomial of $R[It]$ with respect to $(\mathfrak{M}, It)$ but also formulas for the mixed Hilbert coefficients of the pair $(\mathfrak{M} \mid I)$.

Our starting point here is the main technical lemma of [HTU] and this we describe now. The authors of that paper are interested in the multiplicity of $R[It]$ with respect to $(\mathfrak{M}, It)$, that is, the multiplicity of $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ as a graded algebra over the base field k. To calculate this multiplicity—or for that matter the Hilbert series of $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ as a graded k-algebra—we are free to look instead at the associated graded ring $\mathrm{gr}_{\mathcal{F}}(R[It])$ of $R[It]$ with respect to $\mathcal{F}$, where $\mathcal{F}$ is any filtration on $R[It]$ finer than the $(\mathfrak{M}, It)$-adic filtration. Now they look at a particular such filtration. To describe this, write $R[It]$ as a homomorphic image of the polynomal ring $\mathcal{A} = R[T_1, \ldots, T_n]$ under the map $T_i \mapsto x_i t$, and let $\mathcal{N}$ denote the maximal ideal $(\mathfrak{M}, T_1, \ldots, T_n)$ of $\mathcal{A}$. The degree-lexicographic filtration $\mathcal{F}$ on $\mathcal{A}$ is finer than the $\mathcal{N}$-adic filtration. Since $(\mathfrak{M}, It)$ is the image of $\mathcal{N}$ under the above map from $\mathcal{A}$ to $R[It]$, the induced filtration $\mathcal{F}$ on $R[It]$ is finer than the $(\mathfrak{M}, It)$-adic filtration. This is the filtration they consider. The associated graded ring $\mathrm{gr}_{\mathcal{F}}(R[It])$ is presented as a quotient of $\mathrm{gr}_{\mathcal{F}}(\mathcal{A}) \cong \mathcal{A}$, and under the hypothesis that $x_1, \ldots, x_n$ is a d-sequence with nondecreasing degrees, the presentation ideal is shown to have the form of a monomial ideal. This is their main technical result. From this the formula for the multiplicity of $R[It]$ is deduced by first writing the presentation ideal as an intersection and then using the associativity formula for multiplicities. The argument in [RS] runs exactly parallel to the argument in [HTU] just described.

Our work here is based on two observations. First, the bigrading on $\mathrm{gr}_{(\mathcal{M},It)} R[It]$ is induced from the one on $\mathcal{A}$, and since $\mathcal{F}$ respects the bigrading on $\mathcal{A}$, the bigraded Hilbert series of $\mathrm{gr}_{\mathcal{F}}(R[It])$ is the same as that of $\mathrm{gr}_{(\mathcal{M},It)} R[It]$. Second, the associativity formula for multiplicities is a false lead—it gives the formula for the multiplicity all right but it hides other information that can just as readily be extracted from the presentation ideal. In our main result, namely, Theorem 2.4 below, we deduce from the presentation ideal an expression for the bigraded Hilbert series of $\mathrm{gr}_{(\mathcal{M},It)} R[It]$. The argument is not any more difficult than the argument in [HTU] and [RS] based on the associativity formula.

The organization of this paper should be clear from the titles of its sections. We would like to draw the reader's attention to our rather special notation for the Hilbert series and the Hilbert polynomial, to the remarks in Sect. 4, and to the question posed in connection with Example 3.2.3.

To end this section, here are some bibliographical notes. Mixed multiplicities were introduced first by Teissier and Risler [T] in their work on Milnor numbers of hypersurface singularties. They calculated them in terms of sufficiently general elements. Later Rees [Re] introduced joint reductions to replace sufficiently general elements. This made the task of calculating the mixed multiplicities easier. Paul Roberts [Ro] has used mixed multiplicities to define multiplicity of a homomorphism between free modules. The remarks in Sect. 4 below are in the spirit of the formula in [KV] for the multiplicity of the extended Rees algebra in terms of mixed multiplicities. For more information about mixed multiplicities, see [KV] and the references therein. There are similarities between the proof of Theorem 2.4 below and the proof in [JR].

1. NOTATION FOR HILBERT SERIES AND HILBERT POLYNOMIALS

Let k be a field and let $k[U_1, \ldots, U_p]$ be the polynomial ring in p variables over k with the usual grading. Let $R = \oplus_{i \geq 0} R_i$ be a graded quotient of $k[U_1, \ldots, U_p]$. It is well known (e.g., it follows immediately from Hilbert's syzygy theorem) that the *Hilbert series* $H(R; t) := \sum_{m \geq 0} \dim_k(R_m) t^m$ is a rational function of the form $Q(t)/(1-t)^p$, where $Q(t)$ is a polynomial with integer coefficients. Thus $\dim_k(R_m)$ is a polynomial $P(R; m)$ for large m. This is the *Hilbert polynomial* of R and we write

$$P(R; m) = \sum_{i \geq 0} c_i \binom{m+i}{i}.$$

The coefficients c_i are all integers and of course $c_i = 0$ for large i since this is a polynomial. To calculate the Hilbert polynomial from the Hilbert series, write the Hilbert series as a Laurent polynomal in $1 - t = X$. The following proposition shows that the Hilbert polynomial can be read off the principal part of this Laurent polynomial.

PROPOSITION 1.1. *If $H(R; X) = \sum a_i X^i$ is the Hilbert series written as a Laurent polynomial in $X = 1 - t$, then the Hilbert polynomal is given by $P(R;m) = \sum_{i \geq 0} a_{-i-1} \binom{m+i}{i}$.*

Proof. Note that $1/(1-t)^m = \sum_{j \geq 0} \binom{m-1+j}{j} t^j$ for $m \geq 1$. ∎

For a bigraded quotient R of the bigraded polynomial ring $k[U_1, \ldots, U_p, V_1, \ldots, V_q]$, where each U_i has degree $(1,0)$ and each V_j has degree $(0,1)$, the *bigraded Hilbert series* $\mathbb{H}(R; t_1, t_2) := \sum_{r,s \geq 0} \dim_k(R_{r,s}) t_1^r t_2^s$ is similarly a rational function of the form $Q(t_1, t_1)/(1-t_1)^p(1-t_2)^q$, and the *bigraded Hilbert polynomial* $\mathbb{P}(R; m, n)$, whose value for large r and s equals $\dim_k R_{r,s}$ can similarly be read off the principal part of the Laurent polynomial form of writing the Hilbert series: if $\mathbb{H}(R; X, Y) = \sum a_{ij} X^i Y^j$, where $X = 1 - t_1$ and $Y = 1 - t_2$, then

$$\mathbb{P}(R; m, n) = \sum_{i,j \geq 0} a_{-i-1,-j-1} \binom{m+i}{i} \binom{n+j}{j}.$$

All Hilbert series appearing in the sequel are written as Laurent polynomials in the above fashion: $H(R; X) = \sum a_i(R) X^i$ and $\mathbb{H}(R; X, Y) = \sum a_{ij}(R) X^i Y^j$. We define $a_{ij}(M \mid I) := a_{ij}(\mathrm{gr}_{(M, It)} R[It])$ and $b_{ij}(M \mid I) := a_{ij}(\mathrm{gr}_{(M/I, I/I^2)}(\mathrm{gr}_I(R)))$, where M and I are ideals in a ring R and the bigradings on the rings appearing on the right-hand side are to be specified later by context.

Any bigraded ring can be viewed as a singly graded ring by considering elements of degree (i, j) to have degree $i + j$. Under this process, the singly graded Hilbert series is obtained from the bigraded Hilbert series by setting $Y = X$. Thus the coefficient a_{ij} in the bigraded Hilbert polynomial contributes towards the coefficient a_{i+j} in the singly graded Hilbert polynomial. But terms of the bigraded Hilbert series of the form $X^i Y^j$ with $i + j < 0$ but $i \geq 0$ or $j \geq 0$, while they contribute to the singly graded Hilbert polynomial, are irrelevant to the bigraded Hilbert polynomial.

2. THE THEOREM

In this section we prove the main result (Theorem 2.4), which unfortunately is rather technical. The starting point of the proof is the expression proved in [RS] for the ideal of leading forms, with respect to the degree-lexicographic filtration $\boldsymbol{F}$, of the ideal defining the Rees algebra as a quotient of the polynomial ring over the base ring. This ideal of leading forms has the form of a monomial ideal which enables us to express it as an intersection of simpler ideals and deduce from there the formula for the bigraded Hilbert series.

We begin by recalling from [R2] the definition of a quadratic sequence. A subset Λ of a finite poset $(\Omega, \leq)$ is an *ideal* if

$$\lambda \in \Lambda, \qquad \omega \in \Omega, \quad \text{and} \quad \omega \leq \lambda \quad \Rightarrow \quad \omega \in \Lambda.$$

If Λ is an ideal of Ω and $\omega \in \Omega \setminus \Lambda$ is such that $\lambda \in \Lambda$ for every $\lambda \lneq \omega$, then (Λ, ω) is a *pair* of Ω. Given a set $\{x_\omega \mid \omega \in \Omega\}$ of elements of a ring R and $\Lambda \subseteq \Omega$, denote by X_Λ the ideal $(x_\lambda \mid \lambda \in \Lambda)$ of R ($X_\Lambda = 0$ if Λ is empty) and by I the ideal $X_\Omega = (x_\omega \mid \omega \in \Omega)$.

Definition 2.1 [*R*2, *Definition* 3.3]. A set $\{x_\omega \mid \omega \in \Omega\} \subseteq R$ is a *quadratic sequence* if for every pair (Λ, ω) of Ω there exists an ideal Θ of Ω such that

1. $(X_\Lambda : x_\omega) \cap I = X_\Theta$;
2. $x_\omega X_\Theta \subseteq X_\Lambda I$.

Such an ideal Θ is said to be *associated* with the pair (Λ, ω). This association need not be unique—the set $\{x_{\omega \mid \omega \in \Omega}\}$ of generators of I may not be unshortenable—but X_Θ is unique by 1.

Definition 2.2. A *linerization* of a poset Ω of cardinality n is a bijective map $\#\colon \Omega \to [1, n] := \{1, \ldots, n\}$ such that $\omega \leq \omega' \Rightarrow \#(\omega) \leq \#(\omega')$.

Let $\{x_\omega \mid \omega \in \Omega\}$ be a quadratic sequence and, let $\#\colon \Omega \to [1, n]$ be a fixed linearization. Identify Ω with $[1, n]$ via $\#$. Then $([1, j-1], j)$ is a pair of Ω for every $j \in [1, n]$. Let

$$\Theta_j = \text{an ideal of } \Omega \text{ associated with } \big([1, j-1], j\big),$$

$$I_j = \big((x_1, \ldots, x_{j-1}) : x_j\big),$$

$$\Delta = \big\{(j, k) \mid 1 \leq j \leq k \leq n,\ x_j \cdot x_k \in (x_1, \ldots, x_{j-1})\big\}.$$

For $k \in [1, n]$, set

$$\Psi_k = [1, k-1] \bigcup_{\substack{j \le k \\ (j,k) \notin \Delta}} \Theta_j \quad \text{and} \quad \mathfrak{A}_k = X_{\Psi_k}.$$

Note that Ψ_k is an ideal of Ω and that $\mathfrak{A}_k$ is independent of the choices of Θ_j.

Definition 2.3. A linearization $\#\colon \Omega \to [1, n]$ of the indexing poset Ω of a quadratic sequence is *stable* if $I_k = (\mathfrak{A}_k : x_k)$ for every k, $1 \le k \le n$.

We can now state our main theorem.

THEOREM 2.4. *Let R be a standard graded algebra over a field k, that is, $R = \oplus_{i \ge 0} R_i = R_0[R_1]$ with $R_0 = k$. Let $\{x_\omega \mid \omega \in \Omega\} \subseteq R$ be a quadratic sequence consisting of homogeneous elements of R, let $\#\colon \Omega \to [1, n]$ be a stable linearization, and suppose that*

$$\deg(x_1) \le \cdots \le \deg(x_n).$$

Let I denote the ideal $(x_1, \ldots, x_n)$ and $\mathsf{M} = \oplus_{i > 0} R_i$ the irrelevant maximal ideal. Then the bigraded Hilbert series of $\mathrm{gr}_{(\mathsf{M}, It)}(R[It])$ is given by

$$\begin{aligned}
&\mathbb{H}\big(\mathrm{gr}_{(\mathsf{M}, It)} R[It]; X, Y\big) \\
&\quad = H(R; X) + (1 - Y)\Bigg[\sum_{\substack{l=1 \\ (l,l) \in \Delta}}^{N} H\left(\frac{R}{I_l}; X\right) \frac{H(F_l; Y)}{2 - Y} \\
&\qquad\qquad + \sum_{\substack{l=1 \\ (l,l) \notin \Delta}}^{n} H\left(\frac{R}{I_l}; X\right) H(F_l; Y)\Bigg],
\end{aligned}$$

where Δ is defined as above and

$$F_l = \frac{k\big[T_i, T_l \mid i < l, (i,l) \notin \Delta\big]}{\begin{aligned}\big(T_j T_k \mid 1 \le j \le k \le l, (j,k) \in \Delta, (j,l) \notin \Delta, (k,l) \notin \Delta\big) \\ + \big(T_l^2 \mid \textit{only if } (l,l) \in \Delta\big)\end{aligned}}.$$

The formula for the bigraded Hilbert series of $\mathrm{gr}_{(\mathsf{M}/I, I/I^2)}(\mathrm{gr}_I(R))$ is the same as the one above except that on the right-hand side we replace R by R/I and

R/I_l *by* $R/(I+I_l)$:

$$\mathbb{H}\big(\mathrm{gr}_{(\mathcal{M}/I,\, I/I^2)}(\mathrm{gr}_I(R));X,Y\big)$$
$$= H\left(\frac{R}{I};X\right) + (1-Y)\left[\sum_{\substack{l=1\\(l,l)\in\Delta}}^{n} H\left(\frac{R}{I+I_l};X\right)\frac{H(F_l;Y)}{2-Y}\right.$$
$$\left. + \sum_{\substack{l=1\\(l,l)\notin\Delta}}^{n} H\left(\frac{R}{I+I_l};X\right)H(F_l;Y)\right].$$

Proof. We prove only the formula for the Rees algebra. If in this proof we replace R by R/I and R/I_j by $R/(I+I_j)$, we get the proof of the formula for the associated graded ring.

Map $R[T_1,\dots,T_n]\to R[It]$ by sending T_i to x_it. Let $\mathcal{F}$ be the filtration on $R[T_1,\dots,T_n]$ defined in Sect. 1 of [RS]. The associated graded ring $\mathrm{gr}_{\mathcal{F}}(R[It])$ of $R[It]$ with respect to the induced filtration $\mathcal{F}$ has, by Theorem 1.4 of [RS], the following presentation:

$$\mathrm{gr}_{\mathcal{F}}(R[It]) \cong \frac{R[T_1,\dots,T_n]}{\big(I_1T_1,\dots,I_nT_n,T_jT_k \mid (j,k)\in\Delta\big)}. \tag{4}$$

Since $\mathrm{gr}_{(\mathcal{M},\, It)}(R[It])$ and $\mathrm{gr}_{\mathcal{F}}(R[It])$ have the same bigraded Hilbert series, we may work with the latter ring.

The ideal $(I_1T_1,\dots,I_nT_n,T_jT_k \mid (j,k)\in\Delta)$ is a "block monomial ideal," i.e., it behaves like a monomial ideal if we treat $I_1,\dots,I_n$ as blocks. To calculate the Hilbert series, we treat it as such. The first step then is to write it as an intersection of simpler ideals. In order to do this, we first observe that if $j\le k$, $(j,k)\notin\Delta$, and $a\in I_j$, then by the stability assumption $ax_k\in I_j\cap I = X_{\Theta_j}\subseteq I_k$, so $a\in I_k$. Thus, for $j\le k$, either $(j,k)\in\Delta$ or $I_j\subseteq I_k$.

We claim that

$$\big(I_1T_1,\dots,I_nT_n,T_jT_k \mid (j,k)\in\Delta\big) = \bigcap_{l=0}^{n} J_l,$$

where $J_0 := (T_1,\dots,T_n)$ and

$$J_l := \big(I_l,T_{l+1},\dots,T_n,T_i,T_jT_k \mid i<l,\,(i,l)\in\Delta;$$
$$1\le j\le k\le l,\,(k,l)\in\Delta\big).$$

That $\bigcap J_l$ contains the left-hand side follows from the observation made above. For the converse, suppose $aT_{i_1} \cdots T_{i_p}$ does not belong to the left-hand side, where $a \in R$ and $i_1 \leq \cdots \leq i_p$. Then no pair (i_r, i_s) with $r < s$ belongs to Δ. Thus $aT_{i_1} \cdots T_{i_p}$ does not belong to J_{i_p} and the claim is proved.

Now let $\mathcal{I}_0 := J_0$ and for $1 \leq l \leq n$, $\mathcal{I}_l := \mathcal{I}_{l-1} \cap J_l$. Letting $\mathcal{A}$ denote the polynomial ring $R[T_1, \ldots, T_n]$ we have exact sequences

$$0 \to \frac{\mathcal{A}}{\mathcal{I}_l} \to \frac{\mathcal{A}}{\mathcal{I}_{l-1}} \oplus \frac{\mathcal{A}}{J_l} \to \frac{\mathcal{A}}{\mathcal{I}_{l-1} + J_l} \to 0$$

for $1 \leq l \leq n$. These yield

$$\mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_l}\right) = \mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_{l-1}}\right) + \mathbb{H}\left(\frac{\mathcal{A}}{J_l}\right) - \mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_{l-1} + J_l}\right),$$

where $\mathbb{H}$ stands for bigraded Hilbert series.

Adding these n equations, we get

$$\mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_n}\right) = \mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_0}\right) + \sum_{l=1}^{n} \mathbb{H}\left(\frac{\mathcal{A}}{J_l}\right) - \mathbb{H}\left(\frac{\mathcal{A}}{\mathcal{I}_{l-1} + J_l}\right).$$

It is proved easily by induction that

$$\mathcal{I}_{l-1} = \big(I_1T_1, \ldots, I_{l-1}T_{l-1}, T_l, \ldots, T_n, T_jT_k \mid 1 \leq j \leq k \leq l-1, (j,k) \in \Delta\big).$$

The expression for $\mathcal{I}_{l-1} + J_l$ is now immediate:

$$\mathcal{I}_{l-1} + J_l = \big(I_l, T_l, \ldots, T_n, T_i, T_jT_k \mid i < l, (i,l) \in \Delta;\ 1 \leq j \leq k < l, (j,k) \in \Delta\big).$$

Observe that $\mathbb{H}(\mathcal{A}/\mathcal{I}_n)$ is the desired Hilbert series, that $\mathcal{A}/\mathcal{I}_l \cong R/I_l \otimes_k F_l$, where F_l is as in the statement of the theorem, and that $\mathcal{A}/(\mathcal{I}_{l-1} + J_l) \cong R/I_l \otimes_k \overline{F}_l$, where $\overline{F}_l := F_l/(T_l)$. Either $(l,l) \in \Delta$, in which case $F_l = \overline{F}_l \otimes_k k[T_l]/(T_l^2)$, or $(l,l) \notin \Delta$, in which case $F_l = \overline{F}_l[T_l]$. In the former case, the Hilbert series of F_l and $\overline{F}_l$ are related by $H(F_l; t_2) = H(\overline{F}_l; t_2)(1 + t_2)$ (which, if we set $1 - t_2 = Y$, means $H(F_l; Y) = H(\overline{F}_l; Y)(2 - Y)$) and in the latter case, by $H(F_l; t_2) = H(\overline{F}_l; Y)/(1 - t_2)$ (which, if we set $1 - t_2 = Y$, means $H(F_l; Y) = H(\overline{F}_l; Y)/Y$). We thus get the desired formula. ∎

3. EXAMPLES

3.1. *d-Sequences*

An ordered sequence $x_1, \ldots, x_n$ of elements of a ring is called a *d-sequence* if

$$\big((x_1, \ldots, x_{j-1}) : x_j x_k\big) = \big((x_1, \ldots, x_{j-1}) : x_k\big) \qquad \forall\, 1 \le j \le k \le n. \quad (5)$$

It is easy to see that (see, for example, [R1]) that the above condition is equivalent to

$$\big((x_1, \ldots, x_{j-1}) : x_j\big) \cap I = (x_1, \ldots, x_{j-1}) \qquad \forall\, 1 \le j \le n, \qquad (6)$$

where I is the ideal $(x_1, \ldots, x_n)$. It is clear from this last equation that d-sequences are quadratic sequences: indeed $\Lambda = [1, j-1]$ is itself associated with the pair $(\Lambda, \omega = j)$. It is also clear, since $\Psi_k = [1, k-1]$ for every $k \in [1, n]$, that the (only possible) linearization is stable. Thus Theorem 2.4 is applicable to homogeneous d-sequences whose elements have nondecreasing degrees. For $(j, k) \in \Delta$, eq. (5) shows that I_k is the unit ideal, so the term $T_j T_k$ is redundant in the denominator of the right-hand side of eq. (4). We may therefore assume, in the conclusion of Theorem 2.4, that Δ is empty, hence that $F_l = k[T_1, \ldots, T_l]$, and hence that $H(F_l; Y) = Y^{-l}$. Thus Theorem 2.4 reduces in this case to

COROLLARY 3.1.1. *Let R be a standard graded algebra over a field, let $x_1, \ldots, x_n$ be a d-sequence consisting of homogeneous elements of R, and suppose that*

$$\deg(x_1) \le \cdots \le \deg(x_n).$$

Let I denote the ideal $(x_1, \ldots, x_n)$ and $\mathfrak{M}$ the irrelevant maximal ideal.

Then the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}, It)}(R[It])$ is given by

$$\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}, It)} R[It]; X, Y\big) = H(R; X) + (1 - Y) \sum_{l=1}^{n} H\left(\frac{R}{I_l}; X\right) Y^{-l}.$$

The formula for the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R))$ is the same as the one above except that on the right-hand side we replace R by R/I and R/I_l by $R/(I + I_l)$:

$$\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R); X, Y)\big)$$
$$= H\left(\frac{R}{I}; X\right) + (1 - Y) \sum_{l=1}^{n} H\left(\frac{R}{I + I_l}; X\right) Y^{-l}.$$

In particular,

$$a_p\big(\mathrm{gr}_{(\mathfrak{M},It)}(R[It])\big) = a_p(R) + \sum_{l=1}^{n} a_{p-l}\left(\frac{R}{I_l}\right) - a_{p-l+1}\left(\frac{R}{I_l}\right),$$

$$a_{pq}(\mathfrak{M}\,|\,I) = a_p\left(\frac{R}{I_{q+1}}\right) - a_p\left(\frac{R}{I_{q+2}}\right),$$

$$a_p\big(\mathrm{gr}_{(\mathfrak{M}/T,I/I^2)}\,\mathrm{gr}_I(R)\big) = a_p\left(\frac{R}{I}\right) + \sum_{l=1}^{n} a_{p-l}\left(\frac{R}{I+I_l}\right) - a_{p-l+1}\left(\frac{R}{I+I_l}\right),$$

$$b_{pq}(\mathfrak{M}\,|\,I) = a_p\left(\frac{R}{I+I_{q+1}}\right) - a_p\left(\frac{R}{I+I_{q+2}}\right).$$

For a detailed discussion of examples to which the above corollary can be applied, we refer the reader to [HTU, Sect. 3]. Here we stay content with the simplest application:

EXAMPLE 3.1.2 (regular sequences). Let R be a standard graded algebra over a field and let $x_1, \ldots, x_n$ be a regular sequence of homogeneous elements of respective degrees $d_1 \le \cdots \le d_n$. Let I denote the ideal $(x_1, \ldots, x_n)$ and $\mathfrak{M}$ the irrelevant maximal ideal. Corollary 3.1.1 is applicable in this situation since a regular sequence is, evidently, a d-sequence. Since $I_l = (x_1, \ldots, x_{l-1})$ for $1 \le l \le n$, we have $H(R/I_l; X) = H(R; X)f_1 \cdots f_l$, where $f_i = (1-(1-X)^{d_i})$, so that

$$\begin{aligned}
&\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M},It)}\,R[It]; X, Y\big)\\
&\quad = H(R;X) + (1-Y)\sum_{l=1}^{n} H(R;X)f_1 \cdots f_{l-1}Y^{-l}\\
&\quad = \frac{H(R;X)}{Y^n}\left[f_1 \cdots f_{n-1} + \sum_{l=1}^{n-1} Y^{n-l}(1-f_l)f_1 \cdots f_{l-1}\right].
\end{aligned}$$

Predictably, the formula for the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}/I,I/I^2)}\,\mathrm{gr}_i(R)$ reduces to

$$\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}/I,I/I^2)}\,\mathrm{gr}_I(R); X, Y\big) = \frac{H(R;X)}{Y^n} f_1 \cdots f_n.$$

3.2. *Straightening-Closed Ideals in Graded ASLs*

We borrow the notation, terminology, definitions, and references of the subsection of [RS, Sect. 2] that has the same title as the present one. Throughout our discussion of straightening-closed ideals, R denotes a graded ASL on a finite poset Π over a field $R_0 = k$. For $\Lambda \subseteq \Pi$, the ideal $(\lambda \mid \lambda \in \Lambda)$ of R is denoted $R\Lambda$. Let Ω be a straightening-closed ideal of Π. By Proposition 2.4 of [RS], $\{\omega \mid \omega \in \Omega\}$ is a quadratic sequence in R. By Proposition 2.5 of [RS], any linearization $\#\colon \Omega \to [1, n]$ of Ω is stable. Assuming that $\deg(x_1) \leq \cdots \leq \deg(x_n)$ for a given linearization, we may therefore apply Theorem 2.4. Proposition 2.5 of [RS] gives

$$\Delta = \{(\xi, \nu) \mid \xi, \nu \text{ incomparable elements of } \Omega, \text{ and } \#(\xi) \leq \#(\nu)\},$$

$$I_l = R\Pi^{\omega}, \qquad \text{where } \omega = \#^{-1}(l) \text{ and } \Pi^{\omega} = \{\pi \in \Pi \mid \pi \not\geq \omega\}.$$

Thus F_l is the face ring of the simplicial complex with vertex set $\{\sigma \in \Omega \mid \sigma \leq \omega := \#^{-1}(l)\}$ and faces the square-free standard monomials on this poset. We obtain the following

COROLLARY 3.2.1. *Let R be a graded ASL on a finite poset Π over a field k and let $\mathfrak{M}$ denote the graded maximal ideal of R. If Ω is a straightening-closed ideal of Π which admits a linearization $\#\colon \Omega \to [1, n]$ satisfying*

$$\deg(x_1) \leq \cdots \leq \deg(x_n)$$

then the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M},(R\Omega)t)}(R[(R\Omega)t])$ is given by

$$\begin{aligned}\mathbb{H}\big(&\mathrm{gr}_{(\mathfrak{M},(R\Omega)t)} R[(R\Omega)t]; X, Y\big)\\ &= H(R; Y) + (1 - X) \sum_{\omega \in \Omega} H\left(\frac{R}{R\Pi^{\omega}}; X\right) H(F_{\omega}; Y),\end{aligned}$$

where F_{ω} is the face ring of the poset $\{\sigma \in \Omega \mid \sigma \leq \omega\}$. The formula for the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}/R\Omega, R\Omega/(R\Omega)^2)}(\mathrm{gr}_{R\Omega}(R))$ is the same as the one above except that on the right-hand side we replace R by $R/R\Omega$ and $R/R\Pi^{\omega}$ by $R/R(\Omega \cup \Pi^{\omega})$:

$$\begin{aligned}\mathbb{H}\big(&\mathrm{gr}_{(\mathfrak{M}/R\Omega, R\Omega/(R\Omega)^2)}(\mathrm{gr}_{R\Omega}(R))\big)\\ &= H\left(\frac{R}{R\Omega}; X\right) + (1 - Y) \sum_{\omega \in \Omega} H\left(\frac{R}{R(\Omega \cup \Pi^{\omega})}; X\right) H(F_{\omega}; Y).\end{aligned}$$

In particular,

$$a_p\big(\mathrm{gr}_{(\mathsf{M},(R\Omega)t)}\, R[(R\Omega)t]\big)$$
$$= a_p(R) + \sum_{\omega\in\Omega}\sum_j a_j\left(\frac{R}{R\Pi^\omega}\right)\big[a_{p-j-1}(F_\omega) - a_{p-j}(F_\omega)\big],$$
$$a_{pq}(\mathsf{M}\,|\,R\Omega) = \sum_{\omega\in\Omega} a_p\left(\frac{R}{R\Pi^\omega}\right)\big[a_q(F_\omega) - a_{q+1}(F_\omega)\big],$$
$$a_p\big(\mathrm{gr}_{(\mathsf{M}/R\Omega,\,R\Omega/(R\Omega)^2)}\,\mathrm{gr}_{R\Omega}(R)\big)$$
$$= a_p\left(\frac{R}{R\Omega}\right) + \sum_{\omega\in\Omega}\sum_j a_j\left(\frac{R}{R(\Omega\cup\Pi^\omega)}\right)\big[a_{p-j-1}(F_\omega) - a_{p-j}(F_\omega)\big],$$
$$b_{pq}(\mathsf{M}\,|\,R\Omega) = \sum_{\omega\in\Omega} a_p\left(\frac{R}{R(\Omega\cup\Pi^\omega)}\right)\big[a_q(F_\omega) - a_{q+1}(F_\omega)\big].$$

Note the formal analogy between the special case of the above formulas when Ω is linearly ordered and the formulas for a d-sequence of Corollary 3.1.1: if $\Omega = \{1,\ldots,n\}$ is linearly ordered, then F_l is the polynomial ring $k[T_1,\ldots,T_l]$, so $a_q(F_l)$ is 1 if $q = l-1$ and 0 otherwise.

We now recover, from the above corollary, expressions for the multiplicities of $\mathrm{gr}_{(\mathsf{M},(R\Omega)t)}\, R[(R\Omega)t]$ and $\mathrm{gr}_{(\mathsf{M}/R\Omega,\,R\Omega/(R\Omega)^2)}\,\mathrm{gr}_{R\Omega}(R)$. The following statement contains within it Theorem 2.2 of [RS].

COROLLARY 3.2.2. *Suppose further that Ω is nonempty and that*

$$\textit{all maximal chains in } \Pi \textit{ have the same length}. \qquad (*)$$

Then the multiplicity of $\mathrm{gr}_{(\mathsf{M},(R\Omega)t)}\, R[(R\Omega)t]$ *is given by*

$$\mathrm{e}\big(\mathrm{gr}_{(\mathsf{M},(R\Omega)t)}\, R[(R\Omega)t]\big) = \sum_{\omega\in\Omega}\mathrm{e}\left(\frac{R}{R\Pi^\omega}\right)\mathrm{e}(\omega), \qquad (7)$$

where $\mathrm{e}(R/R\Pi^\omega)$ *is the multiplicity of the graded algebra* $R/R\Pi^\omega$ *and* $\mathrm{e}(\omega)$ *is the number of maximal chains in* Ω *of the type* $\omega_1 \lneq \cdots \lneq \omega_p$. *The mixed multiplicities of* $(\mathsf{M}\,|\,R\Omega)$ *are given by*

$$a_{pq}(\mathsf{M}\,|\,R\Omega) = \sum_{\omega\in\Omega,\ \mathrm{rk}(\omega),=q+1}\mathrm{e}\left(\frac{R}{R\Pi^\omega}\right)\mathrm{e}(\omega),$$
$$\textit{where } p+q = \dim R - 1. \qquad (8)$$

The multiplicity of $\mathrm{gr}_{(\mathsf{M}/R\Omega,\,R\Omega/(R\Omega)^2)}\,\mathrm{gr}_{R\Omega}(R)$ *is given by*

$$\mathrm{e}\big(\mathrm{gr}_{(\mathsf{M}/R\Omega,\,R\Omega/(R\Omega)^2)}\,\mathrm{gr}_{R\Omega}(R)\big) = \sum_{\omega\in\partial\Omega}\mathrm{e}\left(\frac{R}{R(\Omega\cup\Pi^\omega)}\right)\mathrm{e}(\omega), \qquad (9)$$

where $\partial\Omega = \{\omega\in\Omega\cup\{-\infty\}\,|\,\omega$ *has an upper neighbor in* $\Pi\cup\{\infty\}$ *not in* $\Omega\}$, *where* $-\infty$ *and* ∞ *are the smallest and largest elements added to* Π.

Proof. Since ASLs are reduced and $(*)$ implies that all minimal primes of R have maximal dimension, it is clear that $\dim \mathrm{gr}_{(M, R\Omega)} R[(R\Omega)t] = d + 1$, where $d = \dim R$. Thus $\mathrm{e}(\mathrm{gr}_{(M, R\Omega)} R[(R\Omega)t]) = a_d(\mathrm{gr}_{(M, R\Omega)} \times R[(R\Omega)t])$ and $a_d(R) = 0$. Since $R/R\Pi^\omega$ is an ASL on $\Pi \setminus \Pi^\omega$, it follows that dimension of $R/R\Pi^\omega$ is $\mathrm{rk}(\Pi \setminus \Pi^\omega)$, so $a_j(R/R\Pi^\omega) = 0$ for $j \geq \mathrm{rk}(\Pi \setminus \Pi^\omega)$ and $a_j(R/R\Pi^\omega) = \mathrm{e}(R/R\Pi^\omega)$ for $j = \mathrm{rk}(\Pi \setminus \Pi^\omega) - 1$. The dimension of F_ω is $\mathrm{rk}(\omega)$ and its multiplicity is $\mathrm{e}(\omega)$, so $a_k(F_\omega) = 0$ for $k \geq \mathrm{rk}(\omega)$ and $a_k(F_\omega) = \mathrm{e}(\omega)$ for $k = \mathrm{rk}(\omega) - 1$. It follows from $(*)$ that $\mathrm{rk}(\Pi \setminus \Pi^\omega) + \mathrm{rk}(\omega) = \mathrm{rk}(\Pi) + 1 = d + 1$ for every ω. Using the formula for $a_p(\mathrm{gr}_{(M, (R\Omega)t)} R[(R\Omega)t])$ in Corollary 3.2.1 to calculate a_d, we get eq. (7). From the above considerations we also get eq. (8). The proof of eq. (9) is similar: note that $\mathrm{rk}(\Pi \setminus (\Omega \cup \Pi^\omega)) + \mathrm{rk}(\omega) \leq \mathrm{rk}(\Pi) + 1$ and that equality holds if and only if $\omega \in \partial\Omega$. ∎

EXAMPLE 3.2.3 (generic maximal minors). Let $R = k[X]$ be the polynomial ring obtained by adjoining to a field k the entries of an $m \times n$ matrix X of indeterminates (we assume $m \leq n$). It is well known that R is a graded ASL over k on the poset Π of all q-minors of X, $1 \leq q \leq m$. A q-minor is denoted $[r_1 \cdots r_q \,|\, c_1 \cdots c_q]$, where $r_1, \ldots, r_q$ and $c_1, \ldots, c_q$ indicate row and column indices. The partial order on Π is given by the following rule:

$$\left[r_1, \ldots, r_q \,|\, c_1 \cdots c_q\right] \leq \left[r'_1, \ldots, r'_{q'} \,|\, c'_1, \ldots, c'_{q'}\right]$$

if and only if $q \geq q'$ and $r_i \leq r'_i$, $c_i \leq c'_i$ for $1 \leq i \leq q'$. The ideal Ω of all m-minors is a straightening-closed ideal ([H2, item 1.19], [BST, Example 2.1.3]). Since all elements of Ω have the same degree, Theorem 2.4 applies.

The Hilbert series of the rings $R/R\Pi^\omega$ have been calculated by Abhyankar [A] and by several authors after him, e.g., [G], [CH], and [Gh]. An element ω of Ω is characterised by its column indices $c_1, \ldots, c_m$. From eq. (2) of [CH], we see that

$$H\left(\frac{R}{R\Pi^\omega}; X\right) = \frac{\det\left(\sum_k \binom{m-i}{k}\binom{n - c_i + i - j}{k + i - j}(1 - X)^k\right)_{i, j = 1, \ldots, m}}{X^d},$$

where $d = m(2n + m + 1)/2 - \sum c_i$ is the dimension of $R/R\Pi^\omega$. Substituting the above equation and information about the face rings F_ω into the formulas of Theorem 2.4, we can calculate the bigraded Hilbert series of $\mathrm{gr}_{(M, R\Omega)} R[(R\Omega)t]$ and $\mathrm{gr}_{(M/R\Omega, R\Omega/(R\Omega)^2)} \mathrm{gr}_{R\Omega}(R)$.

We will now describe a way of looking at this example which leads naturally to a question. Our reference for this paragraph is [S]. Let $G_{m, m+n}$ be the Grassmannian of m-planes in $(m + n)$-space. The opposite

big cell of $G_{m,m+n}$ is an affine space of dimension mn. The Schubert varieties in $G_{m,m+n}$ are indexed by the poset Π of the above example. If we intersect the Schubert variety corresponding to $[1,\ldots,m-1 \mid 1,\ldots,m-1]$ with the opposite big cell, the resulting affine variety has ideal $R\Omega$ (see Proposition 6.10 of Chapter I of [S]). This leads us to ask the following question: let V be some other Schubert variety and I be the ideal of the intersection of V with the opposite big cell; then what is the Hilbert series of $\mathrm{gr}_{(\mathcal{M},It)} R[It]$? Note that the Grassmannian $G_{r,n}$ is SL_n/P for a maximal parabolic subgroup of SL_n. The question can also be asked for Schubert varieties in the quotient spaces of $SO(n)$ and $SP(2n)$ by maximal parabolic subgroups. Except for the situation of the above example and for some other rather special situations which can be reduced easily to the situation of the above example, we do not know the answer.

3.3. *Defining Ideals of Projective Monomial Space Curves Lying on the Quadric Surface $xy - zw = 0$*

Let $R = k[x,y,z,w]$ be the polynomial ring in four variables over a field k, and let I denote the homogeneous ideal of the projective monomial curve

$$(x:y:z:w) = (u^{b+c} : u^b v^c : u^c v^b : v^{b+c}),$$

where $b > c$ and $\mathrm{bcd}(b,c) = 1$. It is proved in Propositions 2.1 and 2.2 of [MS] that the ideal I is generated by a particular set of $b-c+2$ elements which form a quadratic sequence in a particular linear ordering. Let us use this ordered set of generators of I to present $R[It]$ and $\mathrm{gr}_I(R)$ as quotients of the polynomial ring $S := R[T_1,\ldots,T_{b-c+2}]$. Let J and K be the kernels, respectively, of the maps $S \twoheadrightarrow R[It]$ and $S \twoheadrightarrow \mathrm{gr}_I(R)$. Let $\mathcal{F}$ be the degree-lexicographic filtration on S defined as in [RS, Sect. 1]. If $\mathcal{F}$ denotes also the induced filtration of $R[It]$, the associated graded ring $\mathrm{gr}_{\mathcal{F}}(R[It])$ has the same bigraded Hilbert series as $R[It]$. Since $\mathrm{gr}_{\mathcal{F}}(S) \cong S$, we have $\mathrm{gr}_{\mathcal{F}}(R[It]) \cong \mathrm{gr}_{\mathcal{F}}(S/J) \cong S/J_*$, where J_* is the ideal generated by initial forms of elements of J with respect of $\mathcal{F}$. Similar statements are true for $\mathrm{gr}_I(R)$. Theorems of [RS, Sect. 1], apply in this situation (see [RS] for details) and we get

$$\begin{aligned} J_* &= (T_2,\ldots,T_{b-c+2}) \cap (xw - yz, T_3,\ldots,T_{b-c+2}) \\ &\quad \cap \left(x, y, (T_3,\ldots,T_{b-c+1})^2\right), \\ K_* &= (I, T_3,\ldots,T_{b-c+2}) \cap \left(x, y, z^b, (T_3,\ldots,T_{b-c+1})^2\right). \end{aligned}$$

Using this we can calculate, as in the proof of Theorem 2.4, the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ and $\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R))$. Among the ideals $\mathfrak{U}$ occurring in the right-hand side of the above display, the only one for which the computation of Hilbert series of $S/\mathfrak{U}$ may perhaps take some time is $(I, T_3, \ldots, T_{b-c+2})$. So we compute the Hilbert series of R/I below. Since y is a regular element of degree 1 in R/I, we have $H(R/I; X) = (1/X)H(R/(I, y); X)$. Now $(I, y) = (y, xw, x^{b-c}z^c, \ldots, xz^{b-1}, z^b)$ is a monomial ideal. For $k < b$, the graded piece of degree k of $R/(I, y)$ is spanned by the $(2k+1)$ monomials $x^k, x^{k-1}z, \ldots, xz^{k-1}, z^k$ and $w^k, w^{k-1}z, \ldots, wz^{k-1}$; for $k \geq b$, it is spanned by the $b + c$ monomials $x^k, x^{k-1}z, \ldots, x^{k-c+1}z^{c-1}$ and $w^k, w^{k-1}z, \ldots, w^{k-b+1}z^{b-1}$. Thus

$$H\left(\frac{R}{I}; X\right) = \frac{b+c}{X^2} - \frac{1}{X}\left(\sum_{k=0}^{b-1} (b + c - 2k - 1)(1 - X)^k\right).$$

Calculating the bigraded Hilbert series of $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ and $\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R))$ as in the proof of Theorem 2.4, we obtain

$$\begin{aligned}\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}, It)} R[It]; X, Y\big) &= \left[\frac{1}{X^4Y} + \frac{2}{X^3Y^2} + \frac{b-c}{X^2Y^3}\right] \\ &\quad + \left[\frac{-(b-c+1)}{X^2Y^2} + \frac{-2}{X^3Y}\right] + \frac{1}{X^2Y}, \\ \mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}/I^2, I/I^2)}(\mathrm{gr}_I(R))\big) &= \frac{b+c}{X^2Y^2} + \frac{b-c}{X^2Y^3}(1-Y)\big(1-(1-X)^b\big) \\ &\quad + \frac{1}{XY^2}\left(\sum_{j=0}^{b-1} (2j + 1 - b - c)(1 - X)^j\right).\end{aligned}$$

3.4. *Huckaba–Huneke Ideals of Analytic Deviation* 1 *and* 2

As shown in [R2, Sect. 4], the ideals of analytic deviation 1 and 2 studied by Huckaba and Huneke in [HH1] and [HH2] are generated by quadratic sequences. While the ambient ring in [HH1] and [HH2] is always local, the structure theorems established in those papers and in [R2] for these ideals hold good for certain homogeneous ideals; a particular example is given below. So it seems worthwhile to derive formulas for $\mathrm{gr}_{(\mathfrak{M}, It)} R[It]$ and $\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R))$ for homogeneous ideals having this structure. We proceed to do this.

Let R be a standard graded algebra over a field k and let $\mathfrak{M}$ be the irrelevant maximal ideal of R. Let $x_1, \ldots, x_m, y_{m+1}, \ldots, y_n$ be homoge-

neous elements of R which are indexed by the poset Ω specified as follows. We denote the elements of Ω also by $x_1, \ldots, x_m, y_{m+1}, \ldots, y_n$. The relations in Ω are $x_i \le x_j$ for $1 \le i \le j \le m$ and $x_i \le y_j$ for every pair (i, j) with $1 \le i \le m$ and $m + 1 \le j \le n$. Let I be the ideal $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_n)$ and J the ideal $(x_1, \ldots, x_m)$. Suppose that $x_1, \ldots, x_m$ form a d-sequence with respect to I, that is,

$$\big((x_1, \ldots, x_{j-1}) : x_j\big) \cap I = (x_1, \ldots, x_{j-1}) \qquad \forall\, 1 \le j \le m \tag{10}$$

and that $JI = I^2$. Further suppose that I is not generated by any proper subset of $\{x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\}$. Then $x_1, \ldots, x_m, y_{m+1}, \ldots, y_n$ from a quadratic sequence on Ω [R2, Proposition 4.7]. For any linearization $\#$, we have $\Delta = \{(i, j) \mid m + 1 \le i, j \le n\}$ and thus $\#$ is stable. Assuming that Ω admits a linearization with nondecreasing degrees, we may apply Theorem 2.4 to this situation. For $1 \le l \le m$, we have $F_l = k[T_1, \ldots, T_l]$. For $m + 1 \le l \le n$, we have $F_l = k[T_1, \ldots, T_m, T_l]/(T_l^2)$. Thus $H(F_l; Y)$ is $1/Y^l$ for $1 \le l \le m$ and $(2 - Y)/Y^m$ for $m + 1 \le l \le n$. Substituting these into the formulas of Theorem 2.4, we obtain

COROLLARY 3.4.1. *In the situation just described,*

$$\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}, It)} R[It]; X, Y\big)$$
$$= H(R; X) + \frac{1 - Y}{Y^m}\left[\sum_{l=1}^{m} H\left(\frac{R}{I_l}; X\right) Y^{m-l} + \sum_{l=m+1}^{n} H\left(\frac{R}{I_l}; X\right)\right]$$

$$\mathbb{H}\big(\mathrm{gr}_{(\mathfrak{M}/I, I/I^2)}(\mathrm{gr}_I(R)); X, Y\big)$$
$$= H\left(\frac{R}{I}; X\right)$$
$$+ \frac{1 - Y}{Y^m}\left[\sum_{l=1}^{m} H\left(\frac{R}{I + I_l}; X\right) Y^{m-l} + \sum_{l=m+1}^{n} H\left(\frac{R}{I + I_l}; X\right)\right].$$

EXAMPLE 3.4.2. This example is discussed in [HH1] (see [HH1, Example 4.7]). Let $R = k[a, b, c, d, e]$ be the polynomial ring in five variables over a field k, char $k \ne 2$, and let $\mathfrak{M}$ denote the irrelevant maximal ideal of R. Let I be the homogeneous ideal of R generated by the eight elements $f_1, \ldots, f_8$ displayed below (this ideal is a prime of codimension 2 and defines the base locus of a section of the Horrocks–Mumford bundle):

$$f_1 = 5abcde - a^5 - b^5 - c^5 - d^5 - e^5,$$

$$f_2 = ab^3c + bc^3d + a^3be + cd^3e + ade^3,$$

$$f_3 = a^2bc^2 + b^2cd^2 + a^2d^2e + ab^2e^2 + c^2de^2,$$

$$f_4 = abc^5 - b^4c^2d - 2a^2b^2cde + ac^3d^2e - a^4de^2 + bcd^2e^3 + abe^5,$$

$$f_5 = ab^2c^4 - b^5cd - a^2b^3de + 2abc^2d^2e + ad^4e^2 - a^2bce^3 - cde^5,$$

$$f_6 = a^3b^2cd - bc^2d^4 + ab^2c^3e - b^5de - d^6e + 3abcd^2e^2 - a^2be^4 - de^6,$$

$$f_7 = a^4b^2c - abc^2d^3 - ab^5e - b^3c^2de - ad^5e + 2a^2bcde^2 + cd^2e^4,$$

$$f_8 = b^6c + bc^6 + a^2b^4e - 3ab^2c^2de + c^4d^2e - a^3cde^2 - abd^3e^2 + bce^5.$$

As can be verified on a computer (see also [HH1]), this example satisfies the hypothesis of Corollary 3.4.1—the integer m in this case is 3. Fixing the linearization $f_1, \ldots, f_8$, we can calculate using a computer the Hilbert series $H(R/I_l; X)$. Plugging these series into the formula for $\mathbb{H}(\mathrm{gr}_{(M, It)}(R[It]; X, Y)$, we get

$$\begin{aligned}
&\mathbb{H}\big(\mathrm{gr}_{(M, It)} R[It]; X, Y\big) \\
&\quad = \left[\frac{1}{X^5Y} + \frac{5}{X^4Y^2} + \frac{10}{X^3Y^3}\right] - \left[\frac{5}{X^4Y} + \frac{20}{X^3Y^2} + \frac{15}{X^2Y^3}\right] \\
&\qquad + \left[\frac{10}{X^3Y} + \frac{25}{X^2Y^2} + \frac{5}{XY^3}\right] - \left[\frac{10}{X^2Y} + \frac{10}{XY^2} - \frac{1}{Y^3}\right] \\
&\qquad + \left[\frac{5}{XY} + \frac{54X}{Y^3}\right] - \left[\frac{1}{Y} + \frac{54X}{Y^2} + \frac{99X^2}{Y^3}\right] + \left[\frac{99X^2}{Y^2} + \frac{68X^3}{Y^3}\right] \\
&\qquad - \left[\frac{68X^3}{Y^2} + \frac{20X^4}{Y^3}\right] + \left[\frac{20X^4}{Y^2} + \frac{2X^5}{Y^3}\right] - \left[\frac{2X^5}{Y^2}\right].
\end{aligned}$$

4. REMARKS ABOUT THE HILBERT SERIES OF $\mathrm{gr}_{(M, It, t^{-1})} R[It, t^{-1}]$ AND $\mathrm{gr}_{(M/I, I/I^2)} \mathrm{gr}_I(R)$

In Theorem 2.4 and in the examples of Sect. 3, an expression for the bigraded Hilbert series of $\mathrm{gr}_{(M, It, t^{-1})} R[It, t^{-1}]$ can also be derived similarly. But we have not bothered to do this for two reasons. First, as can be seen from the proposition below, the Hilbert series of $\mathrm{gr}_{(M, It, t^{-1})} R[It, t^{-1}]$ can in most cases be recovered from those of $\mathrm{gr}_{(M, It)} R[It]$ and $\mathrm{gr}_{(M/I, I/I^2)} \mathrm{gr}_I(R)$. Second, it is not clear what the bigraded pieces of $\mathrm{gr}_{(M, It, t^{-1})} R[It, t^{-1}]$ are and whether they are of interest.

PROPOSITION 4.1. 1. *Under the hypothesis of Theorem* 2.4,

$$\mathbb{H}\left(\mathrm{gr}_{(\mathsf{M}, It, t^{-1})} R[It, t^{-1}]; X, Y\right)$$
$$= \begin{cases} \dfrac{1-X}{X} H(R; X) + \mathbb{H}\left(\mathrm{gr}_{(\mathsf{M}, It)} R[It]; X, Y\right) \\ \qquad \textit{if } \deg(x_i) \geq 3\ \forall i, 1 \leq i \leq n, \\ \dfrac{1}{Y} \mathbb{H}\left(\mathrm{gr}_{(\mathsf{M}/I, I/I^2)} \mathrm{gr}_I(R); X, Y\right) \\ \qquad \textit{if } \deg(x_i) \leq 2\ \forall_i, 1 \leq i \leq n. \end{cases}$$

2. *In any Noetherian ring R, if I is an ideal and M is a maximal ideal such that $I \subseteq \mathsf{M}^2$, then*

$$H\left(\mathrm{gr}_{(\mathsf{M}, It, t^{-1})} R[It, t^{-1}]; X\right)$$
$$= \frac{1-X}{X} H(\mathrm{gr}_{\mathsf{M}}(R); X) + H\left(\mathrm{gr}_{(\mathsf{M}, It)} R[It]; X\right).$$

Proof. 1. Fix notation as in [RS, Sect. 1]. If $\deg(x_i) \leq 2$ for all i, then from Theorems 1.2 and 1.3 of [RS] it is clear that L_* is the extension of K_* to $R[T_1, \ldots, T_n, U]$. This proves the second half of the statement. Now suppose $\deg(x_i) \geq 3$ for all i. Then by Theorems 1.1 and 1.3 of [RS], we have $L_* = (UT_1, \ldots, UT_n, J_*)$. Thus $L_* = (U, J_*) \cap (T_1, \ldots, T_n)$. Letting $\mathsf{A} = R[T_1, \ldots, T_n, U]$, we have $\mathrm{A}/(U, J_*) \cong R[T_1, \ldots, T_n]/J_*$, $\mathsf{A}/(T_1, \ldots, T_n) \cong R[U]$, and $\mathsf{A}/(U, J_*, T_1, \ldots, T_n) \cong R$. The first part of the statement follows.

2. As a routine calculation shows,

$$\frac{(\mathsf{M}, It, t^{-1})^n}{(\mathsf{M}, It, t^{-1})^{n+1}} = \frac{R}{\mathsf{M}} t^{-n} \oplus \frac{\mathsf{M}}{\mathsf{M}^2} t^{-n+1} \oplus \cdots$$
$$\oplus \frac{\mathsf{M}^n}{\mathsf{M}^{n+1}} \oplus \frac{\mathsf{M}^{n-1} I}{\mathsf{M}^n I} t \oplus \cdots \oplus \frac{I^n}{\mathsf{M} I^n} t^n$$
$$= \left(\sum_{j=0}^{n-1} \oplus [\mathrm{gr}_{\mathsf{M}} R]_j\right) \oplus \frac{(\mathsf{M}, It)^n}{(\mathsf{M}, It)^{n+1}}.$$

Thus

$$\dim \frac{(M, It, t^{-1})^n}{(M, It, t^{-1})^{n+1}} = \dim\left(\sum_{j=0}^{n} \oplus [\mathrm{gr}_M R]_j\right)$$
$$- \dim[\mathrm{gr}_M(R)]_n + \dim \frac{(M, It)^n}{(M, It)^{n+1}}.$$

Note that the first term on the right-hand side is just the dimension of the nth piece of a polynomial ring in one variable over $\mathrm{gr}_M(R)$ which has Hilbert series $H(\mathrm{gr}_M(R); X)/X$. ∎

Remark 4.2. A routine calculation shows that the graded piece of degree (r, s) of $\mathrm{gr}_{(M/I, I/I^2)} \mathrm{gr}_I(R)$ is $(M^r I^s + I^{s+1})/(M^{r+1} I^s + I^{s+1})$. From this bigraded Hilbert series we can therefore read off the polynomial in r and s whose value for large r and s equals $\dim(M^r I^s + I^{s+1})/(M^{r+1} I^s + I^{s+1})$.

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