1. Introduction

The objective of these notes is to present a few important results about complete ideals in two-dimensional regular local rings. The fundamental theorems about such ideals are due to Zariski found in appendix 5 of [16]. These results were proved by Zariski in [17] for two dimensional polynomial rings over an algebraically closed field of characteristic zero. Zariski states in [17], “It is the main purpose of the present investigation to develop an arithmetic theory parallel to the geometric theory of infinitely near points (in plane or on a surface without singularities).”

In order to state Zariski’s results, we recall the notion of integral closure of an ideal.

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Notes based on a course offered by C. Huneke in 1987 at Purdue, the Purdue thesis of V. Kodiyalam and other sources. I plan to add material on the structure of simple complete ideals, Lipman’s reciprocity law, blow-up algebras of complete ideals, Johnston-Kodiyalam formula for the multiplicity of complete ideals, Huneke-Swanson formula for cores of complete ideals.
Definition 1.1. Let \( I \) be an ideal of a commutative ring \( R \). An element \( x \in R \) is called integral over \( I \), if
\[
x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,
\]
for some elements \( a_i \in I^i \) for \( i = 1, 2, \ldots, n \). The set \( \overline{I} \) of elements of \( R \) which are integral over \( I \) is an ideal called the integral closure of \( I \). The ideal \( I \) is called a complete ideal if \( \overline{I} = I \). An ideal is called a simple ideal if it cannot be written as a product of proper ideals of \( R \).

We now state the two main theorems of Zariski.

Theorem 1.2 (Zariski’s Product Theorem). Let \((R, m)\) be a two dimensional regular local ring. Then product of complete ideals in \( R \) is complete.

Theorem 1.3 (Zariski’s Unique Factorization Theorem). Every complete ideal in a two dimensional regular local ring factors uniquely, up to order, as a product of simple complete ideals.

2. Reductions and integral closures of ideals

In this section we present some basic properties of integral closures and reductions of ideals. Prüfer defined integral closure of an ideal as we have defined above. However, Zariski defined complete ideals in terms of discrete valuation rings in the quotient field. We will present Lipman’s theorem that connects the two definitions.

D.G. Northcott and D. Rees [13] introduced the concept of reduction of an ideal. An ideal \( J \) contained in an ideal \( I \) of a commutative ring \( R \) is called a reduction of \( I \) if \( JI^n = I^{n+1} \) for some \( n \in \mathbb{N} \). This relationship is preserved under ring homomorphisms and ring extensions. If \( J \) is a zero dimensional ideal of a local ring then the reduction process simplifies \( I \) without changing its multiplicity.

Definition 2.1. A reduction \( J \) of \( I \) is called a minimal reduction of \( I \) if no ideal properly contained in \( J \) is a reduction of \( I \).

Proposition 2.2. Let \( J \subseteq I \) be \( m \)-primary ideals of a local ring \((R, m)\).

1. If \( J \) is a reduction of \( I \) then \( e(I) = e(J) \).
2. If \( K \) is a reduction of \( J \) and \( J \) is a reduction of \( I \) then \( K \) is a reduction of \( I \).
3. An ideal \( J \) is a reduction of \( I \) if and only if \( J + Im \) is a reduction of \( I \).

Proof. (1) If \( JJ^n = I^{n+1} \), then for all \( m \),
\[
\ell(R/I^{n+m}) \geq \lambda(R/J^m) \geq \lambda(R/I^m).
\]
Hence \( P_I(n + m) \geq P_I(m) \geq P_J(m) \) where \( P \) denotes the Hilbert polynomial. This shows that \( P_I(m) \) and \( P_J(m) \) have equal degrees and leading coefficients.
(2) Let $KJ^m = J^{m+1}$ and $JI^n = I^{n+1}$. Then $KI^{m+n} = KJ^mI^n = I^{m+n+1}$. 
(3) Let $JI^n = I^{n+1}$. Then $JI^n + mI^{n+1} = I^{n+1}$, hence $(J + mI)I^n = I^{n+1}$. Conversely let $(J + mI)I^n = I^{n+1}$. By Nakayama’s lemma, $JI^n = I^{n+1}$. □

**Definition 2.3.** For an ideal $I$ of a local ring $(R, m)$, the fiber cone of $I$ is the graded ring $F = \bigoplus_{n=0}^{\infty} I^n/m^n$. The Krull dimension of $F(I)$ is called the analytic spread of $I$. This will be denoted by $t(I)$.

**Proposition 2.4.** Let $I$ be an ideal of a local ring $(R, m)$ with residue field $k$. For $a \in I$, let $a^*$ be the residue class of $a$ in $I/mI$. Let $a_1, a_2, \ldots, a_s \in I$. Then $(a_1^*, a_2^*, \ldots, a_s^*)$ is a zero-dimensional ideal of $F(I)$ if and only if $J = (a_1, \ldots, a_s)$ is a reduction of $I$.

**Proof.** The $n^{th}$ homogeneous component of $K := (a_1^*, \ldots, a_s^*)$ is $(JI^{n-1} + mI^n)/mI^n$. Thus $K$ is zero dimensional if and only if for all $n$ large, $JI^{n-1} + mI^n = I^n$. This holds if and only if $J$ is a reduction of $I$. □

**Corollary 2.5.** Every reduction $J$ of $I$ contains a minimal reduction of $I$.

Let $a_1, a_2, \ldots, a_s$ be chosen from $J$ such that

(a) $a_1^*, \ldots, a_s^*$ are $k$-linearly independent,

(b) $\dim F(I)/(a_1^*, \ldots, a_s^*) = 0$ and

(c) The integer $s$ in (b) is minimal with respect to (b).

Then $a_1, a_2, \ldots, a_s$ is a minimal basis of a minimal reduction of $I$ contained in $J$.

**Proof.** Put $K = (a_1, a_2, \ldots, a_s)$. Observe that $K \cap mI = mK$ if and only if $\ker(K/mK \to I/mI) = 0$. This is a consequence of (a). Now (b) implies that $K$ is a reduction of $I$. Suppose that $K' \subset K$ is a reduction of $I$. Then $K' + mI = K + mI$ by (c). Hence $K \subset (K' + mI) \cap K = K' + mI \cap K = K' + mK$. By Nakayama’s lemma $K = K'$. It is clear that $a_1, \ldots, a_s$ minimally generate $K$. In fact $a_1, \ldots, a_s$ are part of a minimal basis of $I$. □

**Proposition 2.6.** Let $(R, m)$ be a local ring with infinite residue field $k$. Let $a_1, \ldots, a_t \in I$, an ideal of $R$. Then $a_1^*, \ldots, a_t^*$ form a homogeneous system of parameters of $F(I)$ if and only if $J = (a_1, \ldots, a_s)$ is a minimal reduction of $I$.

**Proof.** Since $k$ is infinite, it is possible to choose a homogeneous system of parameters of $F(I)$ from the degree one component of $F(I)$. Hence every minimal reduction of $I$ is minimally generated by $\dim F(I) = t(I)$ elements. If $a_1^*, \ldots, a_t^*$ form a homogeneous system of parameters of $F(I)$ then $s = \dim F(I)$ and $F(I)/(a_1^*, \ldots, a_t^*)$ is zero dimensional. Hence $a_1, \ldots, a_s$ generate a minimal reduction of $I$. Conversely if $J = (a_1, \ldots, a_s)$ is a minimal reduction of $I$ then $\dim F(I)/(a_1^*, \ldots, a_t^*) = 0$ and $s$ is minimal with respect to this property. Hence $a_1^*, \ldots, a_t^*$ constitute a homogeneous system of parameters. □
Proposition 2.7. For ideal $I$ of a local ring $(R, m)$ we have
$$\text{alt } I := \sup \{ht p : p \text{ is a minimal prime of } I \} \leq \ell(I) \leq \mu(I).$$

Proof. We may assume that $R/m$ is infinite. Let $J$ be a minimal reduction of $I$. Since $JI^n = I^{n+1}$ for some $n$, $V(I) = V(J)$. Therefore by the Krull's altitude theorem $\text{alt } I = \text{alt } J \leq \mu(J) = \ell(I)$. Since $\dim F(I) \leq \dim I/Im$, we get $\ell(I) \leq \mu(I)$. \qed

Proposition 2.8. The set of integral elements, $\bar{I}$, over $I$ is an ideal of $R$.

Proof. Consider the Rees algebra $R(I) = \bigoplus_{n=0}^{\infty} I^n t^n$ of $I$, where $t$ is an indeterminate. Let $x \in \bar{I}$ satisfy the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, for some $a_i \in I$, $i = 1, 2, \ldots, n$. Then
$$(xt)^n + (a_1 t)(xt)^{n-1} + \cdots + (a_n t^n)(xt)^{n-i} + \cdots + a_n t^n = 0.$$ Hence $xt$ is integral over $R(I)$. If $x, y \in \bar{I}$ then $xt, yt$ are integral over $R(I)$. Thus $xt + yt$ is integral over $R(I)$. Let $u \in R$ and $ut$ be integral over $R(I)$. Then there exist $b_1, b_2, \ldots, b_n \in R(I)$ such that $(ut)^n + b_1(ut)^{n-1} + \cdots + b_n = 0$. Equating coefficient of $t^n$ we obtain $u^n + b_1 u^{n-1} + \cdots + b_n = 0$ where $b_1, b_2, \ldots, b_n$ are defined by $b_i = \sum b_{ij} t^j$ where $b_{ij} \in I^i$ for $i = 1, 2, \ldots, n$. This shows that $u \in \bar{I}$. In particular $x + y \in \bar{I}$. If $x \in \bar{I}$ and $c \in R$, it is easy to see that $cx \in \bar{I}$. Hence $\bar{I}$ is an ideal. \qed

Proposition 2.9. Let $I$ be an ideal of a commutative ring $R$. Then $x \in \bar{I}$ if and only if $I$ is a reduction of $(I, x)$.

Proof. Suppose $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in I$, $i = 1, 2, \ldots, n$. Then $x^n \in \bar{I}(I, x)^{n-1}$ which yields $I(I, x)^n = (I, x)^n$. Conversely suppose that $I$ is a reduction of $(I, x)$ and $I(I, x)^{n-1} = (I, x)^n$. Then $x^n = \sum_{i=1}^{m} a_i b_i$ where $a_i \in I$ and $b_i \in (I, x)^{n-1}$. Thus $b_i = \sum_{j=0}^{n-1} a_{ij} x^{n-1-j}$ for some $a_{ij} \in I^j, j = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, m$. Hence $x^n = \sum_{i=1}^{m} \sum_{j=0}^{n-1} a_{ij} x^{n-1-i} = 0$. Thus $x \in \bar{I}$. \qed

Proposition 2.10. Let $I \subseteq J$ be ideals of a commutative ring $R$ such that $J$ is finitely generated. Then $I$ is a reduction of $J$ if and only if $J \subseteq \bar{I}$.

Proof. Let $J = (I, x_1, x_2, \ldots, x_m)$. Let $J \subseteq \bar{I}$, then $x_1$ is integral over $I$, hence $I$ is a reduction of $(I, x_1)$. Now apply induction on $m$ to see that $I$ is a reduction of $J$. Conversely let $I$ be a reduction of $J$. Then for an indeterminate $t$, $(It)(Jt)^{n-1} = (Jt)^n$ for some $n$. Therefore $R[It]$ is a finite $R[It]$-module. Hence $xt$ is integral over $R[It]$ for any $x \in J$. Therefore $x \in \bar{I}$. \qed

Proposition 2.11. Let $R$ be a commutative ring and let $S$ be a multiplicatively closed subset of $R$. Let $I$ be an ideal of $R$. Then $\bar{T}R_S = \bar{TR}_S$. In particular localization of a complete ideal is complete.
Proof. Let \( x \in \mathcal{I} \) and \( x^n + a_1 x^{n-1} + \cdots + a_n = 0 \) be an equation of integral dependence where \( a_i \in \mathcal{I}^i, i = 1, 2, \ldots, n \). Then
\[
(x/1)^n + (a_1/1)(x/1)^{n-1} + \cdots + a_n/1 = 0.
\]
Hence \( x/1 \in \overline{\mathcal{I}} \mathcal{R}_S \). Conversely, let \( x/s \in \overline{\mathcal{I}} \mathcal{R}_S \). Then \( x/1 \in \overline{\mathcal{I}} \mathcal{R}_S \). Hence there exist \( b_i \in \mathcal{I}^i \), for \( i = 1, 2, \ldots, m \) and \( t \in S \) such that
\[
(x/1)^m + (b_1/t)(x/1)^{m-1} + \cdots + b_n/t = 0.
\]
Multiply this equation by \( t^m \) to get
\[
(tx)^m + (b_1)(tx)^{m-1} + (tb_2)(tx)^{m-2} + \cdots + b_m t^{m-1} = 0,
\]
which implies \( tx \in \mathcal{I} \). Thus \( x \in \overline{\mathcal{I}} \mathcal{R}_S \).

\[
\Box
\]

Proposition 2.12. Let \( I \) be an ideal in a Noetherian ring \( R \). Suppose the associated graded ring \( G(I) = \oplus_{n=0}^\infty I^n/I^{n+1} \) of \( I \) is reduced. Then \( \overline{I^n} = I^n \) for all \( n \geq 1 \).

Proof. Let there be an \( n \geq 1 \), such that \( I^n \neq \overline{I^n} \) and pick an \( r \in \overline{I^n} \setminus I^n \). Then there is a \( k \) and elements \( a_i \in I^{n_i}, i = 1, 2, \ldots, k \) such that
\[
r^k + a_1 r^{k-1} + \cdots + a_k = 0. \tag{1}
\]
We can find a \( p \leq n - 1 \) such that \( a \in I^p \setminus I^{p+1} \). Let \( r^* \) denote the initial form of \( r \) in the \( p \)-th graded component of \( G(I) \). Then the equation (1) gives \( r^k \in I^{p+1} \). Hence \( r^* \) is nilpotent. This is a contradiction. \( \Box \)

Corollary 2.13. Let \((R, \mathfrak{m})\) be a regular local ring. Then \( \mathfrak{m}^n = \overline{\mathfrak{m}^n} \) for all \( n \geq 1 \).

Proposition 2.14. Let \( I \) and \( J \) be ideals of a Noetherian ring \( R \). Let \( M \) be a finitely generated \( R \)-module with nilpotent annihilator \( \text{ann}(M) \). Suppose \( IM = JM \). Then \( \overline{I} = \overline{J} \).

Proof. Since \( IM = JM \), we have \( IM = (I+J)M \). Thus we may assume that \( I \subseteq J \). We only need to show that \( J \subseteq \overline{I} \). Let \( b \in J \). Pick \( u_1, u_2, \ldots, u_n \in M \), such that \( M = Ru_1 + Ru_2 + \cdots + Ru_n \). Then for \( i,j = 1, 2, \ldots, n \), there exist \( a_{ij} \in I \) such that \( bu_i = \sum_{j=1}^n a_{ij} u_j \). Put \( A = (a_{ij}) \). Then \( (bI_n - A)u = 0 \) where \( I \) denotes the \( n \times n \) identity matrix and \( u = (u_1, u_2, \ldots, u_n)^t \). Thus \( \det(bI_n - A)u_i = 0 \) for all \( i = 1, 2, \ldots, n \). Hence there exists an \( r \) so that \( (\det(bI_n - A))^r = 0 \). This yields an equation of integral dependence over \( I \) for \( b \). \( \Box \)
3. Complete ideals and discrete valuation rings

**Definition 3.1.** A local domain \( (S, n) \) is said to dominate a local domain \((R, m)\) birationally if \(R \subset S \subset K\) where \(K\) is the fraction field of \(R\) and \(n \cap R = m\). If \(S\) birationally dominates \(R\), we write \(S \geq R\) or \(R \leq S\).

**Proposition 3.2.** Let \((R, m)\) be a local domain of positive dimension. Then there is a discrete valuation ring \((V, n)\) birationally dominating \((R, m)\).

**Proof.** First we show that there exists an \(x \in m\) such that \(x^k \notin m^{k+1}\) for all \(k \geq 1\). Let \(m = (x_1, x_2, \ldots, x_n)\) and assume by way of contradiction that \(x_i^k \in m^{k+1}\) for some \(k\) and for all \(i = 1, 2, \ldots, n\). Since \(x[i] := (x_1^k, x_2^k, \ldots, x_n^k)\) is a reduction of \(m^k\), there exists an \(s\) such that \(x[i]m^k = m^{k+s}\). Hence \(m^{k+s} \supset m^{k+k+1}\) which yields \(m^{k+s+k} = 0\) This is a contradiction as \(\dim R \geq 1\). Thus we may assume without loss of generality that for \(x_1 = x\), \(x^k \notin m^{k+1}\) for all \(k\).

The ring \(S = R[x/x] = R[x_2/x, x_3/x, \ldots, x_n/x]\) is called a monoidal transform of \(R\). It is easy to see that \(S = \{b/x^k : b \in m^k\text{ for some } k\}\). The ideal \(xS = mS\) is a proper ideal. Indeed, if \(1 \in xS\) then \(1 = bx/x^d\) for some \(d \geq 1\) and \(b \in m^d\). Hence \(x^d \in m^{d+1}\) contradicting the choice of \(x\). Thus \(xS\) is a height one ideal of \(S\). Let \(Q\) be a minimal prime of \(xS\). By Krull-Akizuki theorem, the integral closure \(T\) of \(S_Q\) in its fraction field \(K\) is a one dimensional Noetherian domain. Let \(N\) be a maximal ideal of \(T\) contracting to the maximal ideal of \(S_Q\) then \(NT \cap R = m\). Hence \(T\) is the desired discrete valuation ring birationally dominating \(R\).

**Theorem 3.3 (Lipman’s theorem).** Let \(S\) be a Noetherian domain with fraction field \(K\) and let \(I\) be a proper ideal of \(S\). Then \(T = \bigcap_V IV \cap S\) where the intersection is over all discrete valuation rings \(V\) in \(K\) such that \(V \supset R\).

**Proof.** Since principal ideals in integrally closed domains are complete and intersections of complete ideals are complete, the ideal \(J\) on the right hand side of the above equation is complete. Hence \(T \subseteq J\). Conversely let \(x \notin T\). Then we find a discrete valuation ring \(V \supset S\) in \(K\) such that \(x \notin IV\). Put \(T = S[1/x]\). Then \(x^{-1}IT\) is a proper ideal of \(T\). Indeed, if \(x^{-1}IT = T\) then \(1 = a_1/x + a_2/x^2 + \cdots + a_n/x^n\), where \(a_i \in I^2\) for \(i = 1, 2, \ldots, n\). Hence \(x^n = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n\) which shows that \(x \notin T\). This is a contradiction.

Pick a minimal prime \(Q\) of \(x^{-1}IT\). By Proposition 3.2, there exists a discrete valuation ring \((V, n)\) such that \(V \geq T_Q\). Hence \(x^{-1}IT \subset Q \subset QT_Q = n \cap T_Q\) and \(x^{-1}IV \subseteq n\). Thus \(x \notin IV\).
Theorem 3.4. Let \((R, \mathfrak{m})\) be a local domain and let \(I\) be an \(\mathfrak{m}\)-primary ideal. Then there exist discrete valuation rings \(V_1, V_2, \ldots, V_n\), birationally dominating \(R\) such that
\[
T = \bigcap_{i=1}^{n} IV_i \cap R.
\]

Proof. Let \(K\) be a fraction field of \(R\). By Theorem (3.3) we have \(T = \cap (IV \cap R)\) where the intersection varies over discrete valuation rings in \(K\) containing \(R\). If \(IV = V\), then we may remove \(V\) from this intersection. Thus we may assume that \(IV < V\) for all the discrete valuation rings appearing in (3.3). If \(I\) is \(\mathfrak{m}\)-primary then it follows that \(V \geq R\). Since \(R/I\) is Artinian, the descending chain of ideals \(\{\cap_{i=1}^{r} IV_i \cap R\}_{r \geq 1}\) terminates. \(\square\)

4. Quadratic transforms

In this section we introduce quadratic transform of a two dimensional regular local ring. We will prove that a quadratic transform of a two-dimensional regular local ring is again a two dimensional regular local ring. This construction facilitates inductive arguments for the proofs of main theorems about complete ideals.

Lemma 4.1. Let \(S\) be a commutative ring and \(a, b\) be a regular sequence in \(S\). Let \(x\) be an indeterminate. Then
\[
S[x]/(ax - b) \simeq S[b/a].
\]

Proof. Consider the map \(\phi : S[x] \to S[b/a]\) defined by \(\phi(x) = b/a\). We show that the \(\ker(\phi) = (ax - b)\). Let \(f(x) = r_n x^n + \cdots + r_0 \in \ker(\phi)\). Apply induction on the degree \(\deg(f(x))\) of \(f(x)\). Since \(r_n(b/a)^n + \cdots + r_0 = 0\), \(r_n b^n + \cdots + r_0 a^n = 0\). Hence \(r_n b^n \in (a)\). Since \(a, b\) is a regular sequence, \(r_n \in (a)\). Write \(r_n = as_n\) for some \(s_n \in S\). Then
\[
g(x) = f(x) - (ax - b)s_n x^{n-1} \in \ker(\phi)
\]
and \(\deg g(x) < \deg f(x)\). By induction \(g(x) \in (ax - b)\), and hence so does \(f(x)\). \(\square\)

Proposition 4.2. Let \((R, \mathfrak{m})\) be a two dimensional regular local ring. Let \(m = (x, y)\) and \(S = R[y/x]\). Then
(1) The ideal \(xS = \mathfrak{m}S\) is a prime ideal.
(2) The maximal ideals of \(S\) containing \(\mathfrak{m}S\) are in one-to-one correspondence with irreducible polynomials of the polynomial ring \(k[t]\) over \(k = R/\mathfrak{m}\).
(3) If \(N\) is any maximal ideal of \(S\) containing \(\mathfrak{m}S\) then \(S_N\) is a two dimensional regular local ring.
(4) We have \(\text{Spec}(S) = \text{Spec}(R_x) \cup \text{Spec}(k[t])\).
(5) The ring $S$ is a unique factorization domain.
(6) The valuation ring of the $\mathfrak{m}$-adic order valuation is $S_{\mathfrak{m}S}$.

Proof. (1) Since $x, y$ is a regular sequence, $S/\mathfrak{m}S \simeq R[t]/(xt - y, \mathfrak{m}[t]) \simeq k[t]$. Hence $xS = \mathfrak{m}S$ is a prime ideal.
(2) The maximal ideals of $S$ containing $\mathfrak{m}S$ are therefore in 1-1 correspondence with maximal ideals of $S/\mathfrak{m}S \simeq k[t]$. But the maximal ideals $k[t]$ are principal and generated by irreducible polynomials in $k[t]$.
(3) Let $N$ be a maximal ideal of $S$ containing $xS$. Then $N/xS$ is generated by an irreducible polynomial $g(t) \in k[t]$. Let $g(y/x)$ be any lift of $g(t)$ to $S$. Then $N = (x, g(y/x))$. Thus $\mu(N) = 2$, and $S_N$ is a 2-dimensional regular local ring.
(4) Notice that $R_x = S_x$. Hence prime ideals of $S$ not containing $x$ are in 1-1 correspondence with prime ideals of $R$ not containing $x$. The remaining primes of $S$ are in 1-1 correspondence with primes in $k[t]$.
(5) Since $x$ is a prime element of $S$, by Nagata's theorem, or from bare hands, it is enough to see that $S_x$ is a UFD. But $R_x = S_x$ and $R$ is a UFD, we see that $S_x$ is a UFD. Hence $S$ is a UFD.
(6) Let $V$ be the valuation ring of the $\mathfrak{m}$-adic order valuation. Since the $\mathfrak{m}$-adic value $o(y/x)$ of $y/x$ is 0, $S \subset V$. It is easy to see that $S_{\mathfrak{m}S} \subseteq V$. Since $S_{\mathfrak{m}S}$ is a discrete valuation ring it follows that $V = S_{\mathfrak{m}S}$.

Definition 4.3. The regular local ring $S_N$ constructed above is called a first quadratic transform of $R$.

Let $\mathfrak{m} = (x, y)$. Any first quadratic transform of $R$ is a localization of either $R[\mathfrak{m}/x]$ or $R[\mathfrak{m}/y]$. Any first quadratic transform of $R$ is a 2-dimensional regular local ring birationally dominating $R$. The $n^{th}$ quadratic transform of $R$ is defined to be the first quadratic transform of an $(n - 1)^{st}$ quadratic transform of $R$. The family of all the quadratic transforms of $R$ has a tree like structure emanating from the base $R$. This is evident from the next theorem.

Theorem 4.4. Let $T$ be a quadratic transform of $R$. Then there is a unique sequence $R = T_0 \subset T_1 \subset \cdots \subset T_n = T$ of quadratic transforms where $T_{i+1}$ is a first quadratic transform of $T_i$ for $i = 1, 2, \ldots, n$.

Proof. It is enough to show, by way of induction on $n$, that $T$ contains a unique first quadratic transform of $R$. Suppose there are two first quadratic transforms of $R$ contained in $T$. Then we can pick a minimal generator $x$ of $\mathfrak{m}$ and two distinct height two maximal ideals $P$ and $Q$ of $S = R[\mathfrak{m}/x]$ containing $\mathfrak{m}S$ such that $S_P, S_Q \subset T$. Let $n$ be the maximal ideal of $T$. Then $\mathfrak{n} \cap S \subset P \cap Q$. Since $\mathfrak{n} \cap S$ is a prime ideal of $S$ contained in two distinct prime ideals of $S$, it must be height one. But then $S_{\mathfrak{n} \cap S}$ is a discrete valuation ring contained in $T$. This is a contradiction. □
5. Ideals contracted from quadratic transforms

Throughout this section \((R, \mathfrak{m})\) is assumed to be a two-dimensional regular local ring with residue field \(k\).

**Definition 5.1.** An ideal \(I\) of \(R\) is called a contracted ideal if there is an \(x \in \mathfrak{m} \setminus \mathfrak{m}^2\) and an ideal \(K\) of \(S = R[\mathfrak{m}/x]\) such that \(K \cap R = I\), equivalently \(IS \cap R = I\).

**Proposition 5.2.** An ideal \(I\) of \(R\) is contracted from \(S = R[\mathfrak{m}/x]\) if and only if \(I : \mathfrak{m} = I : x\).

**Proof.** Let \(\mathfrak{m} = (x, y)\). Suppose \(I\) is contracted from \(S\). It is clear that \(I : \mathfrak{m} \subset I : x\). Let \(r \in I : x\). Then \(ry = rx(y/x) \in IS \cap R = I\). Hence \(r \in I : y\). Thus \(I : \mathfrak{m} = I : x\).

Conversely let \(I : \mathfrak{m} = I : x\). Let \(r \in IS \cap R\). We may write

\[ r = a_0 + a_1(y/x) + \cdots + a_n(y/x)^n \]

where \(a_i \in I\) for \(i = 0, 1, \ldots, n\). We induct on \(n\). If \(n = 0\) then \(r \in I\). Let \(n \geq 1\). Then

\[ x^nr = x^na_0 + \cdots + a_{n-1}xy^{n-1} + a_ny^n. \]

Thus \(x|a_n\). Write \(a_n = xb_n\), for some \(b_n \in R\). Hence

\[ rx^{n-1} = x^{n-1}a_0 + \cdots + a_{n-1}y^{n-1} + b_ny^n. \]

Therefore

\[ r = a_0 + a_1(y/x) + \cdots + (y/x)^{n-1}(a_{n-1} + b_ny). \]

Since \(xb_n \in I\), we have \(yb_n \in I\). Thus \(a_{n-1} + yb_n \in I\). By induction \(r \in I\). \(\Box\)

**Proposition 5.3.** An \(\mathfrak{m}\)-primary ideal \(I\) is contracted from \(S = R[\mathfrak{m}/x]\) for some \(x \in \mathfrak{m} \setminus \mathfrak{m}^2\) if and only if there exists an \(a \in I\) such that \(\mathfrak{m}I = a\mathfrak{m} + xI\).

**Proof.** Suppose that \(I\) is contracted from \(S = R[\mathfrak{m}/x]\). Then \(I : \mathfrak{m} = I : x\). Since \(R/\langle x \rangle\) is a discrete valuation ring, there is an \(a \in I\) so that \((a, x)/\langle x \rangle = (I, x)/\langle x \rangle\). Hence \((I, x) = (a, x)\). Let \(b \in I\). Then \(b = ap + xq\) for some \(p, q \in R\). Hence \(q \in I : x = I : \mathfrak{m}\). Hence \(I \subseteq (a) + x(I : \mathfrak{m})\) and thus \(\mathfrak{m}I = a\mathfrak{m} + xI\). Conversely suppose that there is an \(a \in I\) such that \(\mathfrak{m}I = a\mathfrak{m} + xI\).

Let \(r \in I : x\). Let \(\mathfrak{m} = (x, y)\). Then \(rxy = ap + xq\), for some \(p \in \mathfrak{m}\) and \(q \in I\). Hence \(x(ry - q) = ap\). Since \(a, x\) is a regular sequence, \(ry - q = as\) for some \(s \in R\). Hence \(ry \in I\). Therefore \(I : x = I : \mathfrak{m}\). \(\Box\)

**Proposition 5.4.** The product of two \(\mathfrak{m}\)-primary ideals \(I\) and \(J\) contracted from \(S = R[\mathfrak{m}/x]\) is also contracted from \(S\).

**Proof.** Let \(a \in I\) and \(b \in J\) such that \(\mathfrak{m}I = a\mathfrak{m} + xI\) and \(\mathfrak{m}J = b\mathfrak{m} + yJ\).

Hence \(IJ\mathfrak{m} = I(b\mathfrak{m} + xJ) = b(a\mathfrak{m} + xI) + xIJ = ab\mathfrak{m} + xIJ\). Therefore \(IJ\) is contracted from \(S\). \(\Box\)
We will now prove a very useful numerical criterion due to Lipman and Rees for \( m \)-primary ideals that are contracted from some quadratic transform of \( R \).

**Definition 5.5.** The \( m \)-adic order of \( a \in R \), denoted by \( o(a) \), is the number \( o(a) = \max\{n : a \in m^n\} \). If \( I \) is an \( R \)-ideal then the \( m \)-adic order of \( I \) is the number \( o(I) = \min\{o(a) : a \in I\} \).

We will need the Hilbert-Burch theorem which identifies the structure of ideals of projective dimension one in regular local rings. We state the following special version useful to us.

**Theorem 5.6 (Hilbert-Burch Theorem).** Let \( I \) be an \( m \)-primary ideal of a 2-dimensional regular local ring \((R, m)\) with \( \mu(I) = n \). Then there is an \( n - 1 \times n \) matrix \( A \) with entries from \( m \) such that \( I \) is generated by the maximal minors of \( A \). Furthermore, there is an exact sequence

\[
0 \to R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \to R/I \to 0,
\]

where the maps \( \phi_1 \) and \( \phi_2 \) are defined as follows: Let \( \Delta_i = (-1)^{i+1} \det A_i \) where \( A_i \) is the submatrix of \( A \) obtained by deleting the \( i \)th column of \( A \). The map \( \phi_2 \) is the matrix multiplication by \( A \) and the map \( \phi_1 \) is the matrix multiplication by \((\Delta_1, \Delta_2, \ldots, \Delta_n)^t\).

**Example 5.7.** Let \( I = (x^2, xy, y^3) \). Then \( I \) is generated by 2 \times 2-minors of the matrix

\[
A = \begin{bmatrix}
y & -x & 0 \\
0 & y^2 & -x
\end{bmatrix}
\]

and we have the following minimal resolution of \( R/I \):

\[
0 \to R^2 \xrightarrow{A} R^3 \xrightarrow{B} R \to R/I \to 0.
\]

where \( B = (x^2, xy, y^3)^t \).

**Example 5.8.** Let \( I = (y^5, y^4x, y^3x^3, x^6) \). Then \( I \) is generated by the maximal minors of the matrix

\[
A = \begin{bmatrix}
x & -y & 0 & 0 \\
0 & x^2 & -y & 0 \\
0 & 0 & -x^3 & y^3
\end{bmatrix}.
\]

Put \( B = (y^5, y^4x, y^3x^3, x^6)^t \). Then we have the following minimal resolution of \( R/I \):

\[
0 \to R^3 \xrightarrow{A} R^4 \xrightarrow{B} R \to R/I \to 0.
\]

**Lemma 5.9.** Let \( I \) be an \( m \)-primary ideal of \( R \). Then

\[
\lambda \left( \frac{I : m}{I} \right) = \mu(I) - 1.
\]
Proof. Let $\mu(I) = n$. We compute $\text{Tor}_2^R(R/I, k)$ in two ways. By Hilbert-Burch theorem, we have the following minimal resolution of $R/I$,

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R \longrightarrow 0.$$ 

Tensor with $k$ to get the complex:

$$0 \longrightarrow k^{n-1} \xrightarrow{\phi_2} k^n \xrightarrow{\phi_1} k \longrightarrow 0.$$ 

Since the maps in the above complex are zero maps, $\text{Tor}_2^R(R/I, k) = \ker \phi_2 = k^{n-1}$. Hence $\lambda(\text{Tor}_2^R(R/I, k)) = \mu(I) - 1$.

We can calculate $\text{Tor}_2^R(R/I, k)$ as a homology module of the Koszul complex resolving $k$ as an $R$-module tensored with $R/I$:

$$0 \longrightarrow R/I \xrightarrow{\alpha} R/I \oplus R/I \xrightarrow{\beta} R/I \longrightarrow 0$$

where $\alpha = (\bar{y}, -\bar{x})$ and $\beta = (\bar{x}, \bar{y})$. Hence

$$\text{Tor}_2^R(R/I, k) \cong \{ \bar{r} \in R/I : ry, rx \in I \} = \frac{I : m}{I}.$$ 

\square

**Theorem 5.10 (Lipman-Rees theorem).** Let $(R, m)$ be a two-dimensional regular local ring with infinite residue field $k$. Then an $m$-primary ideal is contracted from $S = R[\frac{m}{x}]$ for some $x \in m \setminus m^2$ if and only if $\mu(I) = o(I) + 1$.

Proof. Let $x \in m \setminus m^2$. Pick an $a \in I$ such that $(I, x) = (a, x)$. It follows that $o(a) = o(I) = \lambda(R/(I, x))$. Consider the exact sequence

$$0 \longrightarrow I : x/I \longrightarrow R/I \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0.$$ 

Hence $\lambda(I : x/I) = \lambda(R/(I, x)) = o(I)$. Thus $I$ is contracted

$$\iff I : m = I : x \iff \lambda \left( \frac{I : m}{I} \right) = \lambda \left( \frac{I : x}{I} \right) \iff \mu(I) - 1 = o(I).$$ 

\square

6. The transform of a complete ideal

Throughout this section $(R, m)$ denotes a 2-dimensional regular local ring and $I$ an $m$-primary ideal. Let $x$ be a minimal generator of $m$ and put $S = R[\frac{m}{x}]$. Let $o(I) = r$ and $I^S = x^{-r}IS$. The ideal $I^S$ is either a height two ideal or $S$. We say $I^S$ is the transform of $I$ in $S$. Let $N$ be a height two maximal ideal of $S$ and $T = S_N$. The ideal $I^ST$ is called the transform of $I$ in $T$. An important step in the proofs of the fundamental theorems of Zariski for complete ideals in $R$ is the fact that transforms of complete ideals are complete. This section is devoted to the proof of this result. We begin with
Proposition 6.1. Let $I$ be an $m$ primary complete ideal. Then $Im^i$ is complete for all $i \geq 1$.

Proof. Let $I = IV_1 \cap IV_2 \cap \cdots \cap IV_g \cap R$ where $V_1, V_2, \ldots, V_g$ are certain discrete valuations domains birationally dominating $R$. Choose an $x \in m \setminus m^2$ such that $xV_i = mV_i$ for all $i = 1, 2, \ldots, g$. It is enough to show that $Im$ is complete. Put $J = \overline{mI}$. Since $R/xR$ is a discrete valuation ring $(J, x) = (mI, x)$. Hence $J = mI + (J : x)x$. We show that $J : x = I$ to finish the proof. If $rx \in J$ then $rxV_i \subseteq JV_i = ImV_i = xIV_i$ for all $i$. Hence $r \in \cap_{i=1}^g IV_i \cap R = I$.

Theorem 6.2. Let $I$ be an $m$-primary complete ideal contracted from $S = R[m/x]$. Then $IS$ is complete.

Proof. Since $S$ is a UFD, it is enough to show that $x^rIS = IS$ is complete. Let $s \in TS$ and

$$s^n + a_1s^{n-1} + \cdots + a_n = 0$$

where $a_i \in IS$ for $i = 1, 2, \ldots, n$. Since $IS = \cup_{n \geq 0}I^nm^n/x^n$, we may write $s = t/x^l$ and $a_i = b_i/x^l$ where $l \geq 1$ and $b_i \in I^nm^l$ for $i = 1, 2, \ldots, n$. Substitute these in the above equation and multiply by $x^{ln}$ to get

$$t^n + b_1t^{n-1} + b_2x^lt^{n-2} + \cdots + b_nx^{ln-1} = 0.$$  

Hence $t \in \overline{Im^l} = Im^l$ by Proposition 6.1. Therefore $s = t/x^l \in IS$ and $IS$ is complete.

Example 6.3. Let $(R, m)$ be a 2-dimensional regular local ring and $m = (x, y)$. Put $I = (x^2 + y^3, x^3, x^2y)$. Observe that $I$ is contracted from $S = R[x/y]$. Moreover $o(I) = 2$ and $\mu(I) = 1 + o(I) = 3$. We write $IS = y^2((x/y)^2 + y(x/y)^3, y(x/y)^2)$. Put $x_1 = x/y$. Then $IS = y^2IS$ where $IS = (x_1^2 + y, yx_1^2)$. The only maximal ideal of $S$ containing $IS$ is $N = (x_1, y)$. Now consider the ideal $I^2S_N$ in the regular local ring $S_N$. Then $I^2S_N$ is contracted from $S_N[y/x_1]$. Moreover $I^2S_N$ is a complete ideal. Since $IS = I^2S_N \cap S$, we see that $IS$ and hence $I$ is a complete ideal.

7. The Hoskin-Deligne length formula

In this section we prove a formula originally due to Hoskin, and reproved several times thereafter by various authors, for the co-length of an $m$-primary complete ideal $I$ of a 2-dimensional regular local ring $R$. A number of fundamental properties of complete ideals follow from this formula. It will provide us with an inductive tool, viz, $\ell(R/I)$. No analogue of this formula is known for complete ideals in higher dimensional regular local rings.
**Proposition 7.1.** Let $I$ be an $m$-primary ideal of order $r$. Let $x$ be a minimal generator of $m$ and $S = R[m/x]$. Suppose that $I$ is contracted form $S$. Then the natural map $\phi : m^r/I \rightarrow m^rS/IS$ is an $R$-module isomorphism.

**Proof.** First we note that $m^r = I + xm^{r-1}$. By the exact sequence

$$0 \rightarrow (I : x)/I \rightarrow R/I \xrightarrow{x} R/I \rightarrow R/(I, x) \rightarrow 0,$$

we get

$$\ell(R/(I, x)) = \ell((I : x)/I) = \ell((I : m)/I) = \mu(I) - 1 = r.$$

Since $\ell(R/(x, m^r)) = r = \ell(R/(I, x))$, we have $(I, x) = (m^r, x)$. Thus $m^r \subseteq (I, x)$, and hence $m^r = I + x(m^r : x) = I + xm^{r-1}$. Therefore for all $n \geq 0$, by using induction on $n$ we get $m^{r+n} = Im^n + xn^m^r$.

An element of $m^rS$ is of the form $a/x^n$ where $a \in m^{r+n}$. Since $m^{r+n} = Im^n + xn^m^r$, we can write $a = b + cx^n$ where $b \in Im^n$ and $c \in m^r$. Thus $a/x^n = b/x^n + c$. Since $b/x^n \in Im^n/x^n \subseteq IS$, it follows that $c$ and $a/x^n$ have the same image in $m^rS/IS$. Thus $\phi$ is surjective.

For injectivity of $\phi$, note that since $I$ is contracted from $S$ and $I \subset m^r$,

$$IS \cap m^r = IS \cap R \cap m^r = I \cap m^r = I.$$

$\square$

**Proposition 7.2.** Let $I$ be an $m$-primary ideal of order $r$ contracted from $S = R[m/x]$ where $x$ is a minimal generator of $m$. Then

$$m^r/I \simeq \bigoplus_T T/I^T,$$

where the direct sum extends over all the first quadratic transforms of $R$.

**Proof.** By the above result, $m^r/I \simeq m^rS/IS$. Since $I^S = x^{-r}IS$ we have $m^rS/IS \simeq S/I^S$. Since $I^S$ is a height two ideal of $S$, there are finitely many maximal ideals $N_1, N_2, \ldots, N_g$ of $S$ containing $I^S$. Put $T_i = S_{N_i}$ for $i = 1, 2, \ldots, g$. Thus, by Chinese remainder theorem, we have the isomorphism

$$S/I^S \simeq \bigoplus_{i=1}^g T_i/I^ST_i = \bigoplus_{i=1}^g T_i/I^T_i.$$

If $T$ is a localization of $S$ at a height two maximal ideal different from $N_i$ for $i = 1, 2, \ldots, g$, $I^T = T$.

It remains to show that $T$ is a first quadratic transform of $R$ which is not a localization of $S$ then $I^T = T$. Let $m = (x, y)$. Then there exists a height two maximal ideal $P$ of $S = R[m/y]$ such that $T = S_P$. Thus $x/y \in P$. Since $m^r = I + xm^{r-1}$, we get $y^r \in I + xm^{r-1}$. Therefore $1 \in y^{-r}IT + mT$. Since $mT \subset P$, and $y^{-r}IT = I^T$, we obtain $I^T = T$. $\square$
If $T$ is a quadratic transform of $R$, then the residue field of $T$ is a finite algebraic extension of the residue field of $R$. We denote this field degree by $[T : R]$. The order of $I^T$ with respect to the maximal ideal of $T$ will be denoted by $o(I^T)$. By the symbol $R \leq T$ we mean $T$ is either $R$ or a quadratic transform of $R$.

**Theorem 7.3 (Hoskin-Deligne).** Let $(R, m)$ be two dimensional regular local ring. Let $I$ be a complete $m$-primary ideal of $R$. Then

$$
\lambda(R/I) = \sum_{R \leq T} \left( \frac{o(I^T) + 1}{2} \right) [T : R]
$$

**Proof.** Induct on $l = \lambda(R/I)$. If $l = 1$ then $I = m$. In this case $m^T = T$ for all quadratic transforms of $R$ other than $R$. Hence the formula holds. Now let $l \geq 2$. Since $I$ is complete, it is contracted from some quadratic transform $S = R[m/x]$ where $x$ is a minimal generator of $m$. Let $o(I) = r$. Then $m^r/I \simeq \oplus T/I^T$ where the direct sum is over all the first quadratic transforms of $R$. Since $I \subseteq m^r$,

$$
\lambda(R/I) = \lambda(R/m^r) + \lambda(m^r/I) = \left( \frac{r + 1}{2} \right) + \sum_{R \leq T} \lambda_T(T/I^T) [T : R]
$$

where the sum extends over all the first quadratic transforms $T$ of $R$. Hence $\ell(T/I^T) < \ell(R/I)$ for all $T > R$ in the above sum. Since $I^T$ is complete for all $T$, by induction hypothesis, for all the first quadratic transforms $T$ of $R$,

$$
\lambda_T(T/I^T) = \sum_{U \leq T} \left( \frac{o(U^T) + 1}{2} \right) [U : T]
$$

where the sum extends over all the quadratic transforms $U$ of $T$. Since $[U : R] = [U : T][T : R]$, substituting this in the equation (2), yields the formula. □

8. **Some consequences of the Hoskin-Deligne formula**

The proof of the Hoskin-Deligne formula shows that $\lambda(R/I) > \lambda_T(T/I^T)$ where $I$ is a contracted ideal in $R$ and $T$ is a first quadratic transform of $R$. This gives us an inductive tool needed to prove several results in the Zariski-Lipman theory of complete ideals. We prove the product theorem of Zariski and the Lipman-Rees formula for the Hilbert function of a complete $m$-primary ideal in a 2-dimensional regular local ring.

**Theorem 8.1.** The product of complete ideals ideals in a 2-dimensional regular local ring is complete.

**Proof.** Let $I$ and $J$ be complete ideals in a 2-dimensional regular local ring $(R, m)$. Since $R$ is a UFD, it is enough to prove the theorem when $I$ and $J$ are $m$-primary and complete. We apply induction on $l = \lambda(R/I) + \lambda(R/J)$. 
When \( l = 2 \), \( I = J = m \). In this case since \( G(m) \) is a polynomial ring, \( m^n = \overline{m^n} \) for all \( n \geq 1 \). Now suppose that \( l \geq 3 \). We can find a minimal generator \( x \) of \( m \) such that both \( I \) and \( J \) are contracted from \( S = R[m/x] \). Hence \( IJ \) is also contracted from \( S \). Since \( IJ = IJS \cap R \), it is enough to show that \( IJS \) is complete. Let \( o(I) = r \) and \( o(J) = s \). Then \( IJS = x^{r+s}I^SJ^S \).

Hence it is enough to show that \( I^SJ^S \) is complete. Let \( N_1, N_2, \ldots, N_g \) be all the maximal ideals containing \( I^SJ^S \). Then \( I^SJ^S = \cap_{i=1}^g I^SJ^S S_{N_i} \). Thus it is enough to show that the product of the complete ideals \( I^S S_{N_i} \) and \( J^S S_{N_i} \) is complete. Since the co-lengths of these ideals is smaller than the co-lengths of \( I \) and \( J \) respectively, by induction hypothesis, we are done. \( \square \)

For any \( m \)-primary ideal in a local ring \( (R, m) \) of dimension \( d \), the Hilbert function \( H(I, n) = \lambda(R/I^n) \) is given by the Hilbert polynomial \( P(I, n) \) for all large \( n \). This polynomial is written in the form

\[
P(I, n) = e_0(I) \binom{n + d - 1}{d} - e_1(I) \binom{n + d - 2}{d - 1} + \cdots + (-1)^d e_d(I),
\]

for some integers \( e_0(I), e_1(I), \ldots, e_d(I) \) called the Hilbert coefficients of \( I \).

As a consequence of the H-D formula, we derive a formula for \( P(I, n) \) where \( I \) is a complete \( m \)-primary ideal of a 2-dimensional regular local ring.

**Corollary 8.2.** Let \( I \) be an \( m \)-primary complete ideal of a 2-dimensional regular local ring \( (R, m) \). Then

1. \( H(I, n) = P(I, n) \) for all \( n \geq 1 \).
2. \( P(I, n) = e_0(I) \binom{n + 1}{2} - (e_0(I) - \lambda(R/I))n \).
3. If \( R/m \) is infinite then for any minimal reduction \( J \) of \( I \), \( JJ = I^2 \).

**Proof.** Since \( I^n \) is complete for all \( n \geq 1 \), and \( o((I^n)^T) = n o(I^T) \), we have for all \( n \geq 1 \),

\[
\lambda(R/I^n) = \sum_{R \leq T} \binom{n o(I^T) + 1}{2} [T : R].
\]

Hence \( H(I, n) = P(I, n) \) for all \( n \geq 1 \). Writing this formula in the standard form we get

\[
e_0(I) = \sum_{R \leq T} o(I^T)^2 [T : R] \quad \text{and} \quad e_1(I) = \sum_{R \leq T} \left( o(I^T)^2 \right) [T : R].
\]

Hence \( e_0(I) - e_1(I) = \lambda(R/I) \). It is well known that this condition implies (3), however we present a short proof. The H-D formula gives

\[
e_0(I) = \lambda(R/I^2) - 2\lambda(R/I).
\]

Let \( J = (a, b) \) be a minimal reduction of \( I \). Then we have \( R/I \oplus R/I \simeq J/ JJ \). Hence \( 2\lambda(R/I) = \lambda(R/ JJ) - e_0(I) \). The last two formulae yield \( I^2 = JJ \). \( \square \)
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