EDUCATIVE COMMENTARY ON JEE 2005 MATHEMATICS PAPERS

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EDUCATIVE COMMENTARY ON

JEE 2005 MATHEMATICS PAPERS

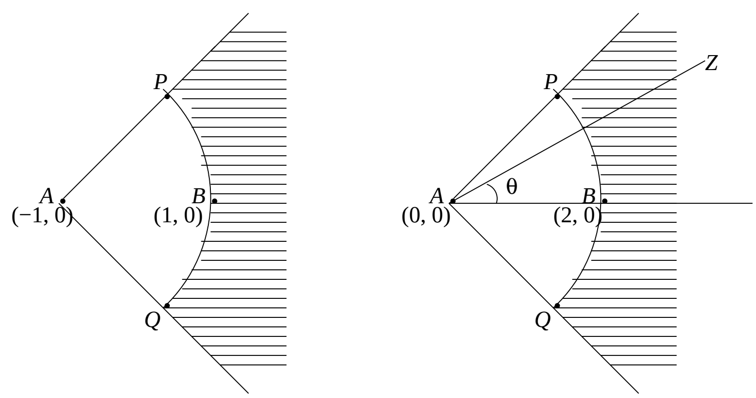
It is a pity that the IIT's have still not adopted the practice of putting the questions and the answers of the screening paper on the Internet immediately after the examination. No conceivable harm could arise if they do so. So, as in the past, these comments are based on memorised versions of the questions. The opinions expressed are the author's personal ones and all references are to the book 'Educative JEE Mathematics' by the author.

SCREENING PAPER

Q. 1 In the diagram (a) below, the points A, B, P, Q are respectively, $(-1, 0)$, $(1, 0), (\sqrt{2} - 1, \sqrt{2})$ and $(\sqrt{2} - 1, -\sqrt{2})$. The locus of z which lies in the shaded region is best described by

(A) $z : |z + 1| > 2, \arg(z + 1) < \frac{\pi}{4}$ (B) $z : |z - 1| > 2, \arg(z - 1) < \frac{\pi}{4}$

(C) $z : |z + 1| > 2, \arg(z + 1) < \frac{\pi}{2}$ (D) $z : |z - 1| > 2, \arg(z - 1) < \frac{\pi}{2}$



(a) (b)

Answer and Comments: (A) The word 'locus' is a little misleading. Normally, it is used to denote a curve and hence is represented by an equation or by two parametric equations. In the present problem, it means a certain region and is described by inequalities. It would have been better to use the word 'set', which is applicable everywhere.

The key idea in the problem is that of shifting the origin. If we shift the origin to the point $A = (-1, 0)$, then the new coordinates of the points

A, B, C, D will be $(0,0), (2,0), (\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ respectively as shown in Figure (b). Let $z = x + iy$ and $Z = X + iY$ denote the complex numbers associated with the same point in the old and the new system respectively. Then we have

$$z = Z - 1 \tag{1}$$

It is easier to identify the region in terms of the new coordinates. It is bounded by three curves, of which two are straight lines and one is the arc of a circle of radius 2 centred at the origin A. As the region lies outside this arc, we have $|Z| > 2$ for all points Z in it. Also, the two straight line boundaries are inclined at angles $\pm\frac{\pi}{4}$ with the X-axis. Hence the argument, say θ of every point Z in this region lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. (Note that here by argument we mean the principal argument which takes values in the semi-open interval $(-\pi, \pi]$. If we take some other definition of the argument, the description of the shaded region would be different.) Therefore in the new coordinates, the shaded region is precisely the set

$$\{Z : |Z| > 2,\ -\frac{\pi}{4} < \ \arg Z < \frac{\pi}{4}\} \tag{2}$$

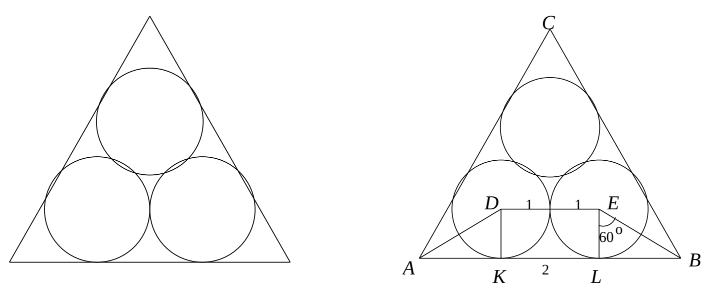
From (1), we have $Z = z + 1$. Substituting this into (2), we get (A) as the correct answer. Note that from the problem, it is not clear whether the shaded region is to include the points on its boundaries. If they are to be included then the strict inequalities will have to be replaced so as to allow possible equalities. In that case the correct answer would have been

$$z : |z+1| \geq 2, \arg(z+1) \leq \frac{\pi}{4}$$

As this is not given to be one of the alternatives, we assume that the boundaries are not to be included in the region.

Q. 2 Three coins of radii 1 unit each touch each other externally and also the sides of an equilateral triangle as shown in Figure (a). Then the area of the equilateral triangle is

(A) $4 + 2\sqrt{3}$ (B) $6 + 4\sqrt{3}$ (C) $12 + \dfrac{7\sqrt{3}}{4}$ (D) $3 + \dfrac{7\sqrt{3}}{4}$



(a) (b)

Answer and Comments: (B) In the last question, a diagram was an integral part of the problem. This is not so with the present question. In fact, in the past, a problem like this would have been asked without a diagram, so that one of the things tested would be a candidate's ability to correctly interpret and visualise the data given by a verbal description. This ability goes hand-in-hand with the dual ability to express oneself in words, concisely and precisely. Although somewhat non-mathematical in nature, both the abilities are as desirable in mathematics as in real life.

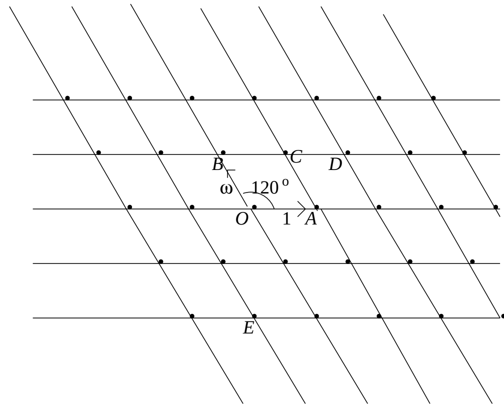
Now, coming to the problem itself, call the triangle as ABC and let a be its side. Then its area is $\frac{\sqrt{3}}{4}a^2$. So the problem is as good as solved when we find a. For this, let D and E be the centres of the circles touching the side AB. Drop perpendiculars DK and EL to AB as shown in Figure (b) above. Then $KL = DE = 2$ units. By symmetry, $AK = LB$. To find LB, note that ΔLEB is a right-angled triangle with $\angle LEB = 60°$. Since $EL = 1$ unit, we have $LB = \tan 60° = \sqrt{3}$ units. Hence $a = AK + KL + LB = KL + 2LB = 2 + 2\sqrt{3}$ units. Therefore the area equals $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 = \sqrt{3}(1+\sqrt{3})^2 = 4\sqrt{3} + 6$ sq. units.

Q. 3 If a, b, c are integers not all equal and ω is a cube root of unity ($\omega \neq 1$), then the minimum value of $|a + b\omega + c\omega^2|$ is

(A) 0 (B) 1 (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$

Answer and Comments: (B). The complex number ω satisfies the equation $\omega^2 + \omega + 1 = 0$. Hence we can write $a + b\omega + c\omega^2$ as $(a - c) + (b - c)\omega$. This is a linear combination of the two complex numbers 1 and ω with integer coefficients, viz., $a - c$ and $b - c$. As a, b, c are not all equal, this is a non-zero linear combination. The question thus reduces to finding which non-zero linear combination of 1 and ω with integer coefficients has the least possible absolute value.

The best way to answer this is geometric. We represent the complex numbers 1 and ω by their position vectors, say $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Complete the parallelogram $OACB$. Then it is clear that the various integral linear combinations of 1 and ω correspond to the vertices of various parallelograms whose sides are integral multiples of $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The vertices of such parallelograms are shown as dotted points in the diagram below. Thus, for example, A, B, C, D, E correspond, respectively, to the complex numbers $1, \omega, 1 + \omega, 2 + \omega$ and $-1 - 2\omega$. Therefore the question amounts to asking what is the shortest non-zero distance (from the origin) of the marked points. The parallelogram $OACB$ is actually a rhombus since $|\omega| = 1$. Also $\angle AOB = 120°$. Therefore A, B, C are at distance 1 each from O. These three points along with their reflections in O form a regular hexagon centred at O. These are the closest points to the origin. Hence the answer is 1.



The problem can also be done in a purely algebraic manner. We take ω as $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. (The other possibility is $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. But that does not affect the answer.) Then a straightforward computation gives

$$\begin{aligned} |a + b\omega + c\omega^2| &= |(a-c) + (b-c)\omega| \\ &= (|(a - c - \frac{b-c}{2}) + i\frac{(b-c)\sqrt{3}}{2}| \\ &= \frac{1}{2}\sqrt{(2a-b-c)^2 + 3(b-c)^2} \\ &= \sqrt{a^2+b^2+c^2-ab-bc-ca} \\ &= \frac{1}{\sqrt{2}}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2} \qquad (1) \end{aligned}$$

We are given that a, b, c are all integers and further that not of all of them are equal. So at most one of the three differences $a-b, b-c$ and $c-a$ can be 0. From (1), the minimum is attained when one of these differences is 0 and the other two are ± 1 each (e.g. when $a = b = 1$ and $c = 0$). The minimum value then is $\frac{1}{\sqrt{2}}\sqrt{1+0+1} = 1$, the same answer as before.

In effect, the second solution is obtained by converting a problem about complex numbers to one which is purely about real numbers. In the present case, the conversion is simple enough. But generally, it is a better idea to look at complex numbers as complex numbers or as vectors (as was done in the first solution). This can inspire new ideas. For example, the set of dotted points shown in the figure above is an example of what is called a **lattice**. More generally, a plane lattice is the set of all integral linear combinations of two (linearly independent) vectors, say $\mathbf{u}$ and $\mathbf{v}$. The parallelogram spanned by these two vectors is called the fundamental parallelogram of the lattice and its area is called the lattice constant of that lattice. (For the lattice in the figure above, $OACB$ is the fundamental parallelogram and the lattice constant is $\frac{\sqrt{3}}{2}$.) Note that the replicas of this parallelogram give us a tiling of the plane. This simple fact, used ingeniously, gives surprisingly elementary proofs of some results in number theory (for example, Fermat's theorem that every prime of the form $4k+1$ can be expressed as a sum of two perfect squares or Lagrange's theorem that every positive integer is a sum of four perfect squares). All these things are beyond the JEE level. But surely it does not hurt to at least know that they exist.

Q. 4 Suppose α and β are the roots of the quadratic equation $ax^2+bx+c=0$. If $\Delta = b^2-4ac$ and $\alpha+\beta, \alpha^2+\beta^2, \gamma^3+\gamma^3$ are in G.P., then

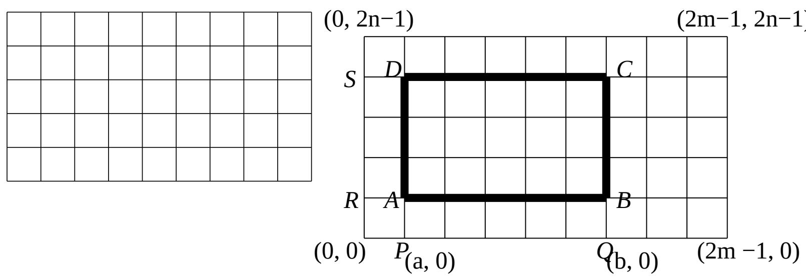
(A) $\Delta \neq 0$ (B) b$\Delta = 0$ (C) $c\Delta = 0$ (D) $\Delta = 0$.

Answer and Comments: (C). This is a straightforward problem. The given condition about α, β translates into $(\alpha^2+\beta^2)^2 = (\alpha+\beta)(\alpha^3+\beta^3)$, which upon simplification, becomes $\alpha\beta(\alpha-\beta)^2 = 0$. It is well-known that $\alpha\beta = \frac{c}{a}$ while $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \frac{b^2}{a^2} - 4\frac{c}{a} == \frac{b^2-4ac}{a^2} = \frac{\Delta}{a^2}$. So the condition means $\frac{c\Delta}{a^3} = 0$. This implies $c\Delta = 0$. (Note that we have assumed that $a \neq 0$ as otherwise the quadratic degenerates into a first degree equation.)

Although the computation involved in this solution is quite simple and straightforward, it is instructive to see if there is some way to bypass it and simply 'smell' the answer. Such a pursuit is not always successful. But it is worth a try, as it sharpens your reasoning. In the present problem, no matter what α is, $\alpha, \alpha^2, \alpha^3$ are in a G.P. by very definition, the common ratio being α. Similarly, β, β^2, β^3 are in a G.P. with C.R. β. We are further given that $\alpha+\beta, \alpha^2+\beta^2$ and $\alpha^3+\beta^3$ are in G.P. Now, normally, when two G.P.'s are added term-by-term, we do *not* get a G.P., unless they have the same C.R. This means $\alpha = \beta$, which means $\Delta = 0$ (the condition for equality of roots). But we must guard against hastily ticking off (D) as the correct answer. There is yet another way the sum of two G.P.'s is a G.P., viz. when one of the G.P.'s is a degenerate one, i.e. consists only of zeros. In such a case, its C.R. can be taken to be any real number and so its sum with any G.P. will give a G.P. This degeneracy can occur when α or β or both vanish. Together this is equivalent to saying that $\alpha\beta = 0$. But this means $c = 0$. So all the possibilities are summed up together by $c\Delta = 0$. Note that all the work in this method can be done without a paper and a pencil! The only danger is that of hastily ticking off (D) as the correct answer. But an alert student can protect himself against it by observing that if Δ vanishes then so will $b\Delta$ and $c\Delta$. But then (B) and (C) will also be correct answers. As it is given that only one answer is correct, it can't be (D).

Q. 5 There are $2n$ horizontal lines equispaced at unit distance and $2m$ vertical lines equispaced at unit distance as shown in Figure (a). How many rectangles can be formed from these lines such that the lengths of their sides are odd?

(A) 4^{m+n-2} (B) $(m+n-1)^2$ (C) m^2n^2 (D) $m(m+1)n(n+1)$



(a) (b)

Answer and Comments: (C). This problem is very similar to the first part of Exercise (1.20). But there is an additional restriction that the sides of the rectangles are to be of odd lengths. Put coordinates and take a typical rectangle $ABCD$ as shown in Figure (b). Let P and Q be the projections of A and B on the x-axis and R, S be the projections of A and D on the y-axis respectively. Then the rectangle $ABCD$ is uniquely determined by these four points P, Q, R, S. Let the x-coordinates of A and B be a and b respectively. Then the length of the side AB is odd if and only if a and b have the opposite parities, i.e. one of them is even and the other odd. This splits into two cases : (i) a even, b odd and (ii) a odd and b even. We further want $a < b$. It is easier to do the counting by temporarily dropping this additional restriction. In that case, in (i), a and b can be chosen in m ways each. So there are m^2 possibilities. Similarly, there are m^2 possibilities in (ii). So together there are $2m^2$ ordered pairs of the form (a, b) where a, b are integers of opposite parities varying from 0 to $2m-1$. Note that if (a, b) is one such pair, so is (b, a). But we want only those pairs in which $a < b$. So we take half of these, i.e. m^2. Thus we see that the projection PQ of the side AB of the rectangle can be formed in m^2 ways. By a similar reasoning, the projection RS of the side AD can be formed in n^2 ways. As these two are independent of each other, the total number of rectangles is m^2n^2.

There is a less tricky way to count the number of ways in which the projection PQ can be chosen by classifying the cases according to the length of the segment PQ which is given to be an odd integer varying from 1 to $2m-1$. Suppose PQ has length 1. Then a can be anything from 0 to $2m-2$ and then b has to be $a+1$. So there are $2m-1$ possibilities. If PQ has length 3, then $0 \leq a \leq 2m-4$ and $b = a+3$. This gives $2m-3$ possibilities. Thus the total number of possibilities is $(2m-1) + (2m-3) + (2m-5) + \ldots + 3 + 1$. This is the sum of the first m odd positive integers and can be easily shown to be m^2. So there are m^2 possible choices of the projection PQ. By a similar reasoning, the projection RS can be chosen in n^2 ways. Hence the rectangle can be formed in m^2n^2 possible ways.

Although the reasoning above looks longish when written down in full, basically it is simple and does not take much time when you do not have to show your work. This problem is, therefore, well-suited to be asked as an objective type question. As in the case of Q. 2, in the present question too, the diagram could have been omitted. Apparently, the papersetters desired that a candidate whose medium of instruction is not Hindi or English should not be at a disadvantage in understanding the problem.

Q. 6 Let $\binom{n}{r} = {}^nC_r$. Then the value of

$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$$

is

(A) $\binom{30}{10}$ (B) $\binom{30}{15}$ (C) $\binom{60}{30}$ (D) $\binom{31}{10}$

Answer and Comments: (A). The given sum cannot be readily equated with some standard sum. But if we use the identity $\binom{n}{r} = \binom{n}{n-r}$ to convert the second factor in each term, then the sum can be rewritten as

$$\binom{30}{0}\binom{30}{20} - \binom{30}{1}\binom{30}{19} + \binom{30}{2}\binom{30}{18} - \ldots + \binom{30}{20}\binom{30}{0}$$

which is readily seen as the coefficient of x^{20} in the expansion of $(1+x)^{30}(1-x)^{30}$. Rewriting this as $(1-x^2)^{30}$, we see that the expansion will consist of only even powers of x and the coefficient of x^{2r} will be $(-1)^r\binom{30}{r}$. Taking $r = 10$ gives the answer.

This is a typical single idea question. If you get the key idea, the rest of the work is simple. Otherwise there is not much progress you can make. Note that the value of the sum coincides with the first term. This means that the sum of the remaining terms is 0. If the problem had been given in the form of evaluating the sum

$$\binom{30}{1}\binom{30}{11} - \binom{30}{2}\binom{30}{12} + \ldots - \binom{30}{20}\binom{30}{30}$$

it would be a tricky problem because to solve it we would have to add and subtract the term $\binom{30}{0}\binom{30}{20}$. Such tricky questions are not suitable as multiple choice questions because when one of the given answers is 0, there is a tendency to select it blindly. It then becomes impossible to decide whether the candidate has arrived at it honestly or not. However, such a question would be fine for the Main Paper where a candidate has to show his work.

Q. 7 Which of the following relationships is true in every triangle ABC ?

(A) $(b-c)\sin(\frac{B-C}{2}) = a\cos\frac{A}{2}$ (B) $(b-c)\cos\frac{A}{2} = a\sin(\frac{B-C}{2})$

(C) $(b+c)\sin(\frac{B+C}{2}) = a\cos\frac{A}{2}$ (D) $(b+c)\cos\frac{A}{2} = a\sin(\frac{B+C}{2})$

Answer and Comments: (B). Trigonometry abounds in numerous formulas involving the sides and the angles of a triangle. The key to success is to pick the right one that will do the trick in a given problem. In the present problem, note that all the alternatives can be recast so that ratios of the form $\dfrac{b \pm c}{a}$ are equated with certain trigonometric functions of the angles A, B, C. This suggests that the key idea is to convert ratios of sides to ratios of trigonometric functions of the angles. The easiest way to do this is the sine rule, by which the sides a, b, c are proportional to $\sin A, \sin B, \sin C$. (The constant of proportionality is $2R$, where R is the circumradius of the triangle. But that is not vital here.)

With standard manipulations (and the fact that $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$), the expressions $\frac{b-c}{a}$ and $\frac{b+c}{a}$ can be reduced as follows.

$$\begin{aligned}
\frac{b-c}{a} &= \frac{\sin B - \sin C}{\sin A} \\
&= \frac{2\cos(\frac{B+C}{2})\sin(\frac{B-C}{2})}{2\sin\frac{A}{2}\cos\frac{A}{2}} \\
&= \frac{\sin(\frac{B-C}{2})}{\cos\frac{A}{2}} \quad (\text{since } \cos(\frac{B+C}{2}) = \sin\frac{A}{2}) \qquad (1)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{b+c}{a} &= \frac{\sin B + \sin C}{\sin A} \\
&= \frac{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}{2\sin\frac{A}{2}\cos\frac{A}{2}} \\
&= \frac{\cos(\frac{B-C}{2})}{\sin\frac{A}{2}} \quad (\text{since } \sin(\frac{B+C}{2}) = \cos\frac{A}{2}) \qquad (2)
\end{aligned}$$

From (1) we see that (B) is the right answer. As it is given that only one alternative is correct, we need not scan further. This makes (2) redundant. Time can be saved by first proving (1) and seeing if it gives the answer and then proving (2) only if necessary. But this is a matter of luck. Since the first two alternatives involve $b-c$, it is natural to begin by proving (1). If the wrong alternatives involving $b+c$ were given first, normally one would first prove (2) and seeing that it does not give the answer would then be forced to prove (1) too. In some examinations, to avoid copying, not only are the questions permuted in different question papers, but for the same question, the order of the alternatives is also changed. If that is done to a question like this, a minor complaint can be made that those candidates in whose question papers the correct alternatives appeared later had to do more work than those in whose answerbooks they appeared earlier!

Q. 8 The number of pairs (α, β), where $\alpha, \beta \in [-\pi, \pi]$ and satisfy the equations $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is

(A) 0 (B) 1 (C) 2 (D) 4

Answer and Comments: (D). The first equation implies that $\alpha - \beta$ is an integral multiple of 2π. Since α, β both lie in the interval $[-\pi, \pi]$ whose length is 2π, the only way this can happen is when (i) $\alpha = \beta$, (ii) $\alpha = \pi, \beta = -\pi$ and (iii) $\alpha = -\pi, \beta = \pi$. In cases (ii) and (iii), we have $\alpha + \beta = 0$ and so the second equation, viz., $\cos(\alpha + \beta) = \frac{1}{e}$ does not hold. That leaves us with only (i), i.e. with $\alpha = \beta$. This reduces the second equation to $\cos 2\alpha = \frac{1}{e}$. Here 2α varies over $[-2\pi, 2\pi]$ which is an interval of length 4π. As $-1 < \frac{1}{e} < 1$, the equation $\cos\theta = \frac{1}{e}$ has two solutions in every interval of length 2π. So there are four values of α in $[-\pi, \pi]$ which satisfy $\cos 2\alpha = \frac{1}{e}$. The problem does not ask us to identify these values, but only their number. Since $\alpha = \beta$, each of these four values gives an ordered pair satisfying the given conditions.

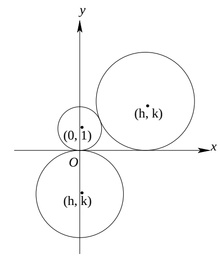
The problem is simple, but not suitable, as it stands, for a multiple choice question. To arrive at the answer honestly, one has to eliminate the cases (ii) and (iii) listed above. But a candidate who does not even think of these two possibilities and considers only (i) will also get the correct answer, with much less time than a careful student. If the interval over which α, β vary was given to be larger than $[-\pi, \pi]$ so that some ordered pairs falling under (ii) or (iii) would also satisfy the second equation (viz., $\cos(\alpha + \beta) = \frac{1}{e}$), then it would have been possible to distinguish between candidates who get the answer by hard work and those who get it sloppily. But then the problem would probably require more time than would be justified by the credit it gets. As it stands, the problem rewards sloppy students.

Q. 9 The locus of the centre of a circle which touches the x-axis and also externally touches the circle $x^2 + (y-1)^2 = 1$ is

(A) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \leq 0\}$ (B) $\{(x, y) : (x-1)^2 + y^2 = 1\}$

(C) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \leq 0\}$ (D) $\{(x, y) : x^2 = 4y\}$

Answer and Comments: (A). A straightforward problem on finding a locus. Although a diagram is not mandatory, it is always a good idea to draw at least a rough one. The centre and the radius of the given circle are $(0, 1)$ and 1 respectively. To get the locus, we calculate the radius of the moving circle in two ways.

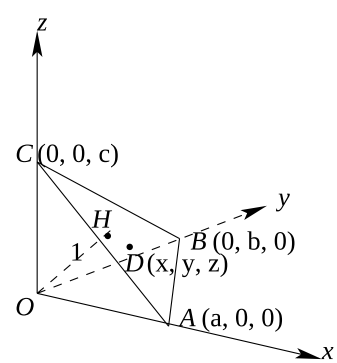


Let (h, k) be the centre of the moving circle. Then its distance from the centre of the given circle is $\sqrt{h^2 + (k-1)^2}$ while its distance from the x-axis is $|k|$. So its radius is $\sqrt{h^2 + (k-1)^2} - 1$ on one hand and $|k|$ on the other. Equating and squaring the two, we get $h^2 + k^2 - 2k + 1 = |k|^2 + 2|k| + 1$ and hence $h^2 - 2k = 2|k|$. When $k \geq 0$, this gives $h^2 = 4k$, while for $k < 0$ we get $h^2 = 0$, i.e. $h = 0$. So the locus is the union of the two sets $\{(x, y) : y \geq 0, x^2 = 4y\}$ and $\{(0, y) : y < 0\}$. The second component of the locus is likely to be missed if calculated hastily. In such a case, the diagram acts like a deterrent because from the diagram one sees readily that the condition for simultaneous tangency can also be satisfied by a circle with its centre on the negative y-axis.

Q. 10 A plane at a distance of 1 unit from the origin cuts the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of the triangle ABC satisfies the relation $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = k$, the value of k is

(A) 3 (B) 1 (C) 1/3 (D) 9

Answer and Comments: (D). Solid geometry has been introduced recently into the JEE syllabus. So, those candidates who have studied under HSC boards that have not included it in their syllabi find it a little unfamiliar. But the analogy with plane coordinate geometry is a big help. In the present problem, for example, the equation of the plane can be written down in terms of its intercepts on the three coordinate axes, in complete analogy with the intercepts form of the equation of a straight line in a plane.



So, if we let $A = (a, 0, 0), B = (0, b, 0)$ and $C = (0, 0, c)$, then the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{1}$$

The formula for the perpendicular distance of a point from a plane is the exact analogue of the formula for the distance of a point from a straight line in a plane. As we are given that the perpendicular distance of the origin from the plane given by (1) is 1 unit, we get

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = 1 \tag{2}$$

The point $D = (x, y, z)$ is the centroid of the triangle ABC. This gives us three relations :

$$x = a/3, \quad y = b/3 \quad \text{and} \quad z = c/3 \tag{3}$$

Finally, we are given

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = k \tag{4}$$

(3) and (4) together give

$$\frac{9}{a^2}+\frac{9}{b^2}+\frac{9}{c^2} = k \tag{5}$$

But by (2), $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = 1$. Hence from (5), we get $k = 9$.

This is a straightforward problem. The idea is probably only to test the knowledge of some basic formulas in solid coordinate geometry which are direct analogues of the corresponding formulas in plane coordinate geometry. It would have been better to denote the coordinates of the point D by some other symbols such as (x_0, y_0, z_0) rather than by (x, y, z), because x, y, z are variables and should not denote the coordinates of a specific point. Although this is a minor distinction, ignoring it could confuse students who are not well familiar with three dimensional coordinates.

Q. 11 A six faced fair dice is thrown until 1 shows. The probability that 1 comes at an even numbered trial is

(A) 5/11 (B) 5/6 (C) 6/11 (D) 1/6

Answer and Comments: (A). This problem is conceptually very similar to the Main Problem in Chapter 23 (or rather, its extension in Comment No. 1.) The similarity becomes more apparent if we replace the die by a biased coin which has probability 1/6 of showing a head. Let us also introduce two players A and B who toss the coin (or rather, the die) alternately, i.e. A tosses it on all odd-numbered rounds and B on all even numbered rounds. Let us say that whoever first gets a head wins. Then the problem is equivalent to finding the probability that B wins.

The solution, too, can be paraphrased in this new format. To win on any round, all the previous tosses must have produced tails and the last toss must be a head. Thus, B can win on the 2nd, the 4th, the 6th , ... rounds with probabilities $\frac{5}{6} \times \frac{1}{6}, (\frac{5}{6})^3 \times \frac{1}{6}, (\frac{5}{6})^5 \times \frac{1}{6}, \ldots$. These are terms of a geometric series with first term $\frac{5}{36}$ and common ratio $\frac{25}{36}$. The probability of B's win is the sum of this infinite series. It equals $\dfrac{5/36}{1-(25/36)} = \dfrac{5}{11}$.

An analogy such as the above is not an absolute must to be able to solve the problem. The solution can as well be obtained and worded directly because the essential ideas are identical. But the ability to see the conceptual similarity (and, at the same time, the differences) between a given problem and a familiar problem is an invaluable asset. It also suggests alternate ways of attacking the problem. In the present problem, for example, let us see how the alternate solution to the Main problem in Chapter 23 can be modified. Let α and β be, respectively, the winning probabilities of A and B. (If we want to stick to the original problem, then α is the probability that the face marked 1 will first show on an odd numbered round and β is the probability that it will show on an even numbered round. Our interest is to find β.) Clearly, α and β are probabilities of complementary events and so we have

$$\alpha + \beta = 1 \tag{1}$$

We need one more equation in α and β to get the values of α and β. Consider the outcome of the first toss. The probability of its being a head is $\frac{1}{6}$ while that of its being a tail is $\frac{5}{6}$. If it is a head then A wins the game and the probability of this is $\frac{1}{6}$. If the first toss is a tail, then after that, in effect we begin a new game, in which B plays first and which will be won by A with probability β (and not α, because now A is the second and not the first player). Thus we get

$$\alpha = \frac{1}{6} + \frac{5}{6}\beta \tag{2}$$

If we solve (1) and (2) we get $\alpha = \frac{6}{11}$ and $\beta = \frac{5}{11}$. Of course, we are interested only in the latter.

Because of the infinite number of cases in which the desired event gets split, this problem falls under what is called infinitistic probability. So it is not surprising that a tool like the infinite series is needed. The first solution uses infinite series directly. It is tempting to think that the second solution is free of it. But it is not quite so. When we wrote down (1), we *assumed* that α and β add up to 1. This amounts to saying that if we wait long enough then the face 1 is sure to occur sooner or later. A rigorous justification of this fact again involves a limiting process. So, no matter which way we get the solution, the use of some limiting process is unavoidable.

Q. 12 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$. Suppose $A^{-1} = \frac{1}{6}(A^2 + cA + dI)$ where I is the unit matrix. Then the values of c and d are

(A) $-6, -11$ (B) $6, 11$ (C) $-6, 11$ (D) $6, -11$

Answer and Comments: (C). There are various ways to do this problem. The most straightforward method is to calculate both the sides of the given matrix equation

$$A^{-1} = \frac{1}{6}(A^2 + cA + dI) \tag{1}$$

separately so as to get an equality of two 3×3 matrices and then equate the corresponding pairs of entries.

There is a formula for computing the inverse of a (non-singular) $n \times n$ matrix, say A, in terms of the determinant of A and the determinants of the various minors of A. The (i, j)-th minor M_{ij} of A is defined as the submatrix of A obtained by deleting the i-th row and the j-th column of A. For example, in the given matrix A, $M_{23} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. It can be shown that the (i, j)-th entry of A^{-1} equals $(-1)^{i+j} \frac{\det M_{ji}}{\det A}$. (Note that we are taking M_{ji} and not M_{ij}.)

A direct proof of this result is not difficult. It is based on the expansion of a determinant by a particular row or column. But this method of finding the inverse is rarely applied in practice for matrices of order bigger than 3 because of the large number of determinants to be evaluated. For matrices of order 3, however, the situation is not so bad, because the minors are 2×2 matrices whose determinants can be written down by inspection. In the present problem this becomes especially easy because the matrix has many zero entries. So, if we do this exercise, then it is not hard to show that

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} \tag{2}$$

The R.H.S. of (1) is relatively less laborious to compute. A direct multiplication of A with itself gives

$$A^2 = A \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} \tag{3}$$

Multiplying both the sides of (1) by 6 we now get

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + c \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} + d \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -1+c+d & 5+c \\ 0 & -10-2c & 14+4c+d \end{bmatrix} \quad (4)$$

Comparison of the corresponding entries of the two sides now gives us a host of 9 equations. Some of these are of the $0 = 0$ type. Out of the remaining five we can choose any two we like so as to get the values of c and d as quickly as possible. Equating the entries in the second row third column we get $-1 = 5+c$ which gives $c = -6$. Equating the entries in the first row and first column, we get $6 = 1+c+d = -5+d$ which gives $d = 11$. Hence (C) is the correct answer. (The remaining three equations give us no new information about c and d, because they are consistent with these values of c and d. They better be. Otherwise it would mean that the matrix A^{-1} *cannot* be expressed in the form given in the statement of the question.)

Obviously, this solution is far too long to be the expected solution. We have, in fact, given it only so that on its background the brevity of other solutions is easier to appreciate. One such solution is to multiply both the sides of the matrix equation (1) by A (or rather, by $6A$) and get

$$6I = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad = \quad A^3 + cA^2 + dA \quad (5)$$

We now need to compute A^3. We already have A^2 from (3). A straightforward multiplication gives

$$\begin{aligned} A^3 &= A \times A^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} \end{aligned} \quad (6)$$

Substituting (3) and (6) into (5), we get

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -11-c+d & 19+5c+d \\ 0 & -38-10c-2d & 46+14c+4d \end{bmatrix} \quad (7)$$

As before, this is equivalent to a set of 9 equations in c and d of which four are redundant. Out of the remaining five, we can choose any two. The simplest choice is to take $1+c+d = 6$ and $-11-c+d = 6$. Subtracting the second from the first gives $c = -6$. Substitution into either one gives $d = 11$, the same answer as before.

Although somewhat shorter than the first solution, this solution is still too long. Taking the cube of a 3×3 matrix is a torture. In the present problem, the fact that 4 out of the 9 entries of A are 0 does make the torture a little bearable. In fact, if we further note the location of these zero entries, it is possible to do the calculations a little more quickly. Let us introduce two matrices B and C by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \tag{9}$$

It is trivial to check that

$$A = B + C \tag{10}$$

If A, B, C stood for real (or complex) numbers then an equation like this, coupled with the binomial theorem will enable us to express the powers of A in terms of those of B and C. This is so because the multiplication of real numbers is commutative. This is not the case with matrix multiplication in general. That is why the binomial theorem is not used so frequently with matrices. But if the two matrices whose sum is taken commute with each other, then the binomial theorem does apply. In the present case, the matrices B and C do commute with each other. In fact, it is trivial to show by direct computation that

$$BC = CB = O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Here we are crucially using the fact not only four of the entries of the matrix A vanish, but also that these vanishing entries make the first row and the first column of A identically zero except for one single entry at their intersection.

Now, let m be a positive integer. The relations (1) and (11), along with the binomial theorem enable us to write the power A^m as $\sum\limits_{k=0}^{m} \binom{m}{k} B^k C^{m-k}$. Note that for $k = 1, 2, \ldots, m-1$, the k-th term will contain the product BC and will therefore vanish. That leaves us only with two terms, so that we get

$$A^m = B^m + C^m \tag{12}$$

for every positive integer m. What makes this equation powerful is that the powers of B and C are very easy to calculate. In fact, one can check directly that $B^2 = B$ and hence that all powers of B equal B. The picture is not so rosy with the matrix C, but not entirely hopeless either. Let us consider a 2×2 matrix E defined by

$$E = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \tag{13}$$

Clearly, E is a 2×2 submatrix of the 3×3 matrix C. But it is much more than that. All other entries of C are 0. In fact, we can consider C as having been obtained from E by adding a 'dummy' row and an equally dummy column consisting of all zeros. The effect of these dummies is that if C is multiplied (whether on the right or on the left) by any 3×3 matrix, say F, the product will depend only on the 2×2 submatrix, say G, of F obtained by deleting the first row and the first column of F. It is as if we multiply the two matrices E and G and add a dummy row and a dummy column of zeros to get the product of C and F. So in this sense, E is not just a submatrix of C. We may, figuratively, say that it is the soul or the core of C.

In particular we see that the powers of C can be obtained from the corresponding powers of the submatrix E and then adding a dummy row and a dummy column consisting of all zeros. Being a 2×2 matrix, it is much easier to form the various powers of E. Indeed, we have

$$E^2 = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix} \quad \text{and} \quad E^3 = \begin{bmatrix} -11 & 19 \\ -38 & 46 \end{bmatrix} \tag{14}$$

from which we get

$$C^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} \quad \text{and} \quad C^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} \tag{15}$$

As we already know that $B^m = B$ for all m, (12) now provides less laborious derivations of (3) and (6). So, the second solution given above can be shortened. It is possible that the paper-setters originally designed the problem for the 2×2 matrix E and then extended it to the matrix A.

The first solution given above can also be shortened considerably by working in terms of the matrix E. The inverse of a 2×2 matrix can be written down almost by inspection because the minors are 1×1 matrices, i.e. mere numbers so that no work is needed in finding their determinants. So, by the same method that we applied to calculate A^{-1}, but with much less work, we get

$$E^{-1} = \frac{1}{6} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \tag{16}$$

If we rewrite (2) and (16) by taking the coefficient $\frac{1}{6}$ inside the matrices (i.e. by multiplying every entry by it), we see that A^{-1} is obtained from

E^{-1} by adding one row and one column, their common entry being 1 and all entries 0.

The reason such a shortening of our earlier solutions was possible is the peculiar feature of the matrix A. The method is applicable to more general situations. Let $A = (a_{ij})$ be an $n \times n$ matrix. Assume that there is some positive integer $k < n$ such that (i) for every $i = 1, 2, \ldots, k$, we have $a_{ij} = 0$ for every $j > k$ and (ii) for every $i = k+1, k+2, \ldots, n$, we have $a_{ij} = 0$ for every $j \leq k$. Verbally, (i) means that in the first k rows of A, all the entries from the $k+1$-th column onwards vanish while (ii) means that in the las $n-k$ rows of A, all the entries in the first k columns vanish. These conditions can be expressed in a visual form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2k} & 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & a_{k+1,k+1} & a_{k+1,k+2} & \ldots & a_{k+1,n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & a_{n-1,k+1} & a_{n-1,k+2} & \ldots & a_{n-1,n} \\ 0 & 0 & \ldots & 0 & a_{n,k+1} & a_{n,k+2} & \ldots & a_{n,n} \end{bmatrix} \tag{17}$$

Let P be the $k \times k$ submatrix of A consisting of the first k rows and the first k columns. Similarly, let Q be the $(n-k) \times (n-k)$ submatrix consisting of the last $n-k$ rows and the last $n-k$ columns of A. Then it is customary to write A in a more compact form as

$$A = \begin{bmatrix} P & \vdots & O_1 \\ \ldots & \ldots & \ldots \\ O_2 & \vdots & Q \end{bmatrix} \tag{18}$$

where O_1 is a $k \times (n-k)$ matrix with all zero entries and O_2 is an $(n-k) \times k$ matrix with all zero entries. (Often O_1 and O_2 are denoted by the same symbol O.) The 3×3 matrix A given in the statement of the problem is a matrix of this type, with $k = 1$, $Q = E$, the matrix defined by (13) and P consisting of just one number, viz. 1.

The work we have done can be easily adapted to prove the following more general result.

Theorem: Suppose the matrix A can be written as in (18). Then for every positive integer m we have

$$A^m = \begin{bmatrix} P^m & \vdots & O_1 \\ \ldots & \ldots & \ldots \\ O_2 & \vdots & Q^m \end{bmatrix} \tag{19}$$

Moreover,

$$\det A = \det P \times \det Q \tag{20}$$

If, further, the matrices P and Q are invertible then so is A and A^{-1} is given by

$$A^{-1} = \begin{bmatrix} P^{-1} & \vdots & O_1 \\ \dots & \dots & \dots \\ O_2 & \vdots & Q^{-1} \end{bmatrix} \tag{21}$$

We now give yet another solution to the problem. It is a sophisticated solution because it is based on a well-known, non-trivial theorem, called **Cayley-Hamilton theorem**. To state it we need the concept of the characteristic polynomial of an $n \times n$ matrix A (see Exercise (2.37)). This is a polynomial, say $p(\lambda)$ of degree n in an indeterminate which is often denoted by λ rather than by the more usual x. It is defined as the determinant of the matrix $A - \lambda I$. For example, for the matrix A given in the statement of the question, $p(\lambda)$ is given by

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)[(1-\lambda)(4-\lambda)+2] \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \qquad (22) \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \qquad (23) \end{aligned}$$

(Although it is not relevant here, by way of general information, we remark that the roots of the characteristic polynomial of a matrix A are called the **characteristic roots** or the **eigenvalues** of the matrix A. They are extremely important in the study of matrices. It can be shown that the trace and the determinant of a matrix equal, respectively, the sum and the product of its eigenvalues. From (23) we easily see that this is indeed the case for the matrix A we are dealing with.)

In any polynomial $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_{n-1}\lambda^{n-1} + a_n\lambda^n$, where $a_0, a_1, \ldots, a_n$ are some fixed real (or complex) numbers, if we replace the indeterminate λ by a real or a complex number, say α, then the expression will be another real or complex number, which we generally denote by $p(\alpha)$. This is too familiar to require much elaboration. What is not so commonly done is the replacement of the indeterminate by a square matrix, say B. We then get the expression $p(B) = a_0 I + a_1 B + a_2 B^2 + \ldots + a_{n-1}B^{n-1} + a_n B^n$ where I is the identity matrix of the same order as B. Since all powers, sums and scalar multiples of square matrices of a given order are square matrices of the same order, the expression $p(B)$ is again a square matrix. (Note that it is vital that B is a square matrix. If not, it would be meaningless to talk of B^2 and of higher powers of B.)

Now suppose that $p(\lambda)$ is the characteristic polynomial of a square matrix A. What happens if we replace the indeterminate λ by the matrix A itself and form the matrix $p(A)$? The celebrated Cayley-Hamilton theorem says that the answer is surprisingly simple. In fact, it is the identically zero matrix of the appropriate order! In other words, $p(A) = O$. Stated verbally, every square matrix satisfies its own characteristic equation.

The proof of Cayley-Hamilton theorem is far from trivial and well beyond our scope. But we can verify it for square matrices of orders 1 and 2. The case of a matrix of order 1 is trivial because every such matrix can be identified with some number, say c and its characteristic polynomial is simply $c - \lambda$ which indeed has c as a root (in fact the only root). he next case is that of a 2×2 matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then by a direct computation, $p(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$. Also, $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & bc+d^2 \end{bmatrix}$. A direct substitution gives

$$\begin{aligned} p(A) &= A^2 - (a+d)A + (ad-bc)I \\ &= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc-(a+d)a+(ad-bc) & ab+bd-(a+d)b+0 \\ ca+dc-(a+d)c+0 & bc+d^2-(a+d)d+(ad-bc) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{24}$$

For matrices of order 3 or more, a direct verification of the Cayley-Hamilton theorem is too cumbersome. There is, in fact, no easy proof. (There is no dearth of proofs. But there is no proof whose simplicity will match that of the statement of the theorem.)

If we use the Cayley-Hamilton theorem, we get a quick solution to our problem. In (23), we already calculated the characteristic polynomial of the 3×3 matrix A given in the question. So, by Cayley-Hamilton theorem, we have

$$A^3 - 6A^2 + 11A - 6I = O \tag{25}$$

On the other hand, (5) can be rewritten as

$$A^3 + cA^2 + dA - 6I = O \tag{26}$$

A direct comparison between these two equations gives $c = -6, d = 11$ at least as one possible solution. To prove that this is the only solution is not so easy. But in a multiple choice question, one need not bother about it. In fact, the present question gives an unfair advantage to those who know at least the statement of Cayley-Hamilton theorem. This theorem is not a part of the JEE syllabus. But considering that many students appear for JEE when they are in their first or even second year of a degree program, it is not unthinkable that some of them know it. In a conventional examination where a candidate has to show his work, a solution based on an advanced technique such as the Cayley-Hamilton theorem can be specifically disallowed. But in an objective type test what matters is the answer and not how you arrived at it. The situation is analogous to the use of Lagrange's multipliers (see Comment No. 4 of Chapter 14) which often make a mincemeat of problems of trigonometric optimisation.

On the other hand, for those who do not know the Cayley-Hamilton theorem, the problem is far too laborious (even with the short cuts indicated) compared to the time available to solve it.

We caution the reader against using the Cayley-Hamilton theorem indiscriminately by analogy with polynomial equations of real or complex numbers. Suppose for example, that a polynomial $p(\lambda)$ factors as $q(\lambda)r(\lambda)$. If α is a real or complex root of $p(\lambda)$, then it must be a root of either $q(\lambda)$ or $r(\lambda)$. This is so because, if both $q(\alpha)$ and $r(\alpha)$ are non-zero numbers, then so is their product. With matrices, the situation is different. The product of two nonzero matrices can be the zero matrix as we see in (11). As a result, even if we have $p(A) = O$ where A is a square matrix, we cannot conclude that either $q(A) = O$ or $r(A) = O$. Indeed, (22) gives us a factorisation of the characteristic polynomial $p(\lambda)$ of the matrix A in our question. By Cayley-Hamilton theorem, we have $p(A) = O$. But neither $I = A$ nor is $A^2 - 5A + 6I = O$.

Q. 13 If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^TQ^{2005}P$ equals

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Answer and Comments: (A). Another problem about matrices. This time we are dealing with 2×2 and not 3×3 matrices. So, matrix multiplication is not so laborious as in the last question. Still, nobody can compute the 2005-th power of a 2×2 matrix by brute force! There has to be some other way. The trick lies in recognising a special feature of the matrix P, viz. that its transpose is also its inverse, i.e.

$$P^T = P^{-1} \tag{1}$$

This can be *verified* by directly computing P^TP. But how does one think of it in the first place? One method is to note that the column vectors of P form an orthonormal system in the plane. In fact, this argument would be valid for any real $n \times n$ matrix P. If its columns form an orthonormal system of vectors in $\mathbb{R}^n$, then the transpose of that matrix is also its inverse. This follows because the (i,j)-th entry of the product matrix P^TP is simply the dot product of the i-th and the j-th columns of the matrix P. Such matrices are called **orthogonal matrices**. They seem to be rather favourite with JEE papersetters. The Main Paper of JEE 2003 also contained a problem about such matrices. (Unfortunately, the problem had a serious mistake. Besides being given that A is a real 3×3 matrix with $A^TA = I$, it was also given that the entries of A are all positive. These two conditions are mutually contradictory.)

For 2×2 matrices, there is alternate geometric argument. Note that we can rewrite P as $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where $\theta = -\frac{\pi}{6}$. Therefore, from the discussion in Comment No. 12 of Chapter 8, the matrix P represents a (counterclockwise) rotation of the plane through $-\frac{\pi}{6}$, or equivalently, a clockwise rotation of the plane through an angle $\frac{\pi}{6}$. Since P and P^{-1} cancel each other's effect, it is then clear geometrically, that P^{-1} ought to represent the clockwise rotation through $-\frac{\pi}{6}$, or equivalently, a counterclockwise rotation through $\frac{\pi}{6}$. Either way, P^{-1} will come out to be $\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$, i.e. $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$. But this is nothing but the transpose of the matrix P. So we get (1).

Equation (1) enables us to express the otherwise formidable power Q^{2005} quite succinctly. By definition, $Q = PAP^T$. So, $Q^2 = (PAP^T)(PAP^T)$, which, using the associativity of matrix multiplication, can be written as $PA(P^TP)AP^T$. But by (1), P^TP is simply the identity matrix I of order 2. So, Q^2 is the same as PA^2P^T. Multiplying by Q again, we get $Q^3 = QQ^2 = (PAP^T)(PA^2P^T) = PA(P^TP)A^2P^T = PAIA^2P^T = PA^3P^T$. Continuing in this manner (or, more formally, by induction on n), we can show that for every positive integer n,

$$Q^n = PA^nP^T \tag{2}$$

In particular, $Q^{2005} = PA^{2005}P^T$. Our interest lies in $P^TQ^{2005}P$. We have

$$\begin{aligned} P^TQ^{2005}P &= P^T(PA^{2005}P^T)P \\ &= (P^TP)A^{2005}(P^TP) \\ &= IA^{2005}I \\ &= A^{2005} \end{aligned} \tag{3}$$

We are not quite done yet. We still have to calculate the 2005-th power of the matrix A. Again, we begin by doing some experimentation. A direct calculation gives

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} && (4) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} && (5) \\ \text{and} \quad A^3 &= AA^2 \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} && (6) \end{aligned}$$

There is a clear pattern now. By induction on n it can be shown that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \tag{7}$$

for every positive integer n. (Of course, in a multiple choice test, you do not actually prove such things by induction. You simply guess the pattern and use it. In a conventional examination, you may have to give a proof. But sometimes this too can be bypassed if the focal point of the solution is somewhere else and the result you have guessed is only one step in the solution.)

So, finally, from (3) and (7), we get that $P^T Q^{2005} P = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$.

An alternate derivation of (7) is instructive. We write A as $I + B$ where I is the identity matrix and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since the identity matrix I commutes with every matrix, we can apply the binomial theorem to this sum and get

$$\begin{aligned} A^n &= (I+B)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} I^k B^{n-k} && (8) \end{aligned}$$

All powers of the identity matrix equal I itself. As for the powers of B, a direct computation gives that B^2 is the zero matrix. Hence all powers of B from B^2 onwards vanish. The 0-th power B^0 is to be taken as the identity matrix I. Therefore, the sum in (8) consists of only two terms, viz., $I^n + nB$. On expansion, this is simply the R.H.S. of (7).

In essence, this question is a combination of three independent ideas. The first is to get (7). The next idea is that if P is any invertible matrix, A is any matrix and we let Q be the matrix PAP^{-1}, then for every positive integer n, $Q^n = PA^nP^{-1}$ and hence $P^{-1}Q^nP = A^n$. The work demanded would have been reasonable if only these two ideas were involved. But apparently the papersetters wanted to involve their favorite orthogonal matrices too. So instead of giving the problem in terms of P^{-1}, they gave a matrix P for which P^T is the same as P^{-1} and gave the problem in terms of P^T rather than P^{-1}. That increases the work needed beyond reasonable.

Finally, a word about the figure 2005 occurring in the problem. It has no special significance as far as the problem is concerned. Instead of 2005, the question could have been asked with any positive integer and that would have made no difference in the method of solution. So one may wonder why the particular number 2005 is chosen. The answer lies in a tradition followed at the International Mathematical Olympiads, held annually. In every olympiad, one of the questions asked involves the number of the calendar year in which that particular olympiad is held. Usually, that particular number has no special significance. The other lower level olympiads such as the Indian National Mathematics Olympiad or the various Regional Mathematics Olympiads in India have also adopted this practice. It appears that this fashion has now caught on with JEE. Probably, some of the papersetters are fans of Mathematics Olympiads!

Q. 14 The tangent to the curve $y = x^2 + 6$ at the point $P(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) $(-6, -11)$ (B) $(-9, -13)$ (C) $(-10, -15)$ (D) $(-6, -7)$

Answer and Comments: (D). The problem may appear confusing because the value of c is not specified and therefore the given circle is not unique. But maybe the data does determine it uniquely. Or even if it is not uniquely determined, maybe the answer to the question is not affected. So let us not worry about it right now. Instead, let us attack the problem with the information we already have. The starting point, obviously, is to find the equation of the tangent to the curve $y = x^2 + 6$ at the point $P(1, 7)$. Call this line L. The equation of L can be written down from the equation of a tangent to a parabola at a given point. But the trouble is that this parabola is not in the standard form $y^2 = 4ax$. We can, of course, modify the formula suitably. But it is much better to use calculus to get the slope of L by taking the derivative of $x^2 + 7$ at $x = 1$. (This is a point to be noted. We often classify problems as belonging to a particular area such as trigonometry, coordinate geometry, inequalities and so on. But there is no reason why we cannot use something outside the conventional scope of that particular area.)

Thus the slope of L is 2 and hence its equation is

$$y = 2x + 5 \tag{1}$$

We are given a circle, say C, with equation

$$x^2 + y^2 + 16x + 12y + c = 0 \tag{2}$$

As c is not given, we cannot find the radius of C. But (2) does determine the centre, say M, of C as $M = (-8, -6)$. We are further given that the line L touches the circle C at the point Q. This is equivalent to saying that Q is the foot of the perpendicular from M to the line L. This is, in fact, the key idea. Once it sinks in, it is clear that we need not know the radius of the circle C. In fact, we can find it once we know the point Q. So the data does determine c uniquely, although it is not needed in the solution.

The perpendicular from M to the line L will have slope $-\frac{1}{2}$ and hence its equation will be $y + 6 = -\frac{1}{2}(x + 8)$, i.e.

$$y = -\frac{1}{2}x - 10 \tag{3}$$

The coordinates of Q are obtained by solving (1) and (3) simultaneously. Q comes out as $(-6, -7)$.

As noted before, we can now determine the radius of C as the distance MQ which is $\sqrt{5}$. So the equation of C is $x^2 + y^2 + 16x + 12y + 95 = 0$. Thus the value of c is 95. Of course, this is not asked, nor is it needed in the solution. Had the problem asked us to find c rather than Q, then it would have been a straightforward problem. By asking to find Q rather than c, the papersetters have apparently tried to test a candidate's ability to weed out the inessential details of a problem and to focus on its crux. This is a very desirable quality and makes the problem a good one. But once again, the papersetters have made it more complicated (and quite unnecessarily so) by specifying the line L in a twisted form instead of giving it directly by (1). The fact that L is the tangent to a given curve at some point on it has absolutely no bearing with the main part of the problem. It only makes the problem more laborious and the work more prone to numerical mistakes, without achieving anything.

Q. 15 Let S be the set of all polynomials $P(x)$ of degree less than or equal to 2 which satisfy the conditions $P(1) = 1, P(0) = 0$ and $P'(x) > 0$ for all $x \in [0, 1]$. Then

(A) $S = \emptyset$

(B) $S = \{(1-a)x^2 + ax : 0 < a < 2\}$

(C) $S = \{(1-a)x^2 + ax : a \in (0, \infty)\}$

(D) $S = \{(1-a)x^2 + ax : 0 < a < 1\}$

Answer and Comments: (B). Write $P(x)$ as $a_0 + a_1x + a_2x^2$ where a_0, a_1, a_2 are some real numbers. Each of the two conditions $P(1) = 1$ and $P(0) = 0$ gives us one equation in a_0, a_1, a_2, viz.

$$\begin{aligned} a_0 &= 0 & (1)\\ \text{and} \quad a_0 + a_1 + a_2 &= 1 & (2) \end{aligned}$$

As there are more unknowns than equations, this system does not determine all the unknowns. It only gives us

$$a_1 + a_2 = 1 \tag{3}$$

From (1) and (3), if we call a_1 as a, then we can write $P(x)$ in the form

$$P(x) = (1-a)x^2 + ax \tag{4}$$

So far we have not used the third condition on $P(x)$, viz., that $P'(x) > 0$ for all $x \in [0,1]$. In view of (3), this condition becomes

$$2(1-a)x + a > 0, \quad \text{for all } x \in [0,1] \tag{5}$$

As this condition is not in the form of an equality, it cannot determine a uniquely. It is in the form of an inequality and the problem asks us to identify those values of a for which it holds true.

Note that $2(1-a)x+1$ is a linear function of x. Its graph is a straight line. Whether it is increasing or decreasing will depend on the sign of $(1-a)$. But no matter what it is, the condition that the graph will lie above the x-axis for all $x \in [0,1]$ will be satisfied if and only if it is above the x-axis at the two end points, $x = 0$ and $x = 1$. So, condition (5) is equivalent to the simultaneous holding of the two inequalities

$$\begin{aligned} a &> 0 && (6) \\ \text{and} \quad 2(1-a)+a &> 0 && (7) \end{aligned}$$

Rewriting (7) as $2 > a$ and combining it with (6) we get $0 < a < 2$ as the necessary and sufficient condition for (5) to hold.

Essentially the same argument can be given purely algebraically, i.e. without geometric appeal as follows. We make two cases depending upon whether the coefficient of x in (5) is non-negative or negative. First assume that $1-a \geq 0$, i.e. $a \leq 1$. In this case the minimum of $2(1-a)x+a$ for $x \in [0,1]$ occurs at $x = 0$ and so this expression is positive for all $x \in [0,1]$ if and only if it is so at $x = 0$, which gives $a > 0$. So we get $0 < a \leq 1$. On the other hand, suppose $1-a < 0$, i.e. $a > 1$. Then the expression $2(1-a)x+1$ has its minimum for $x \in [0,1]$ at $x = 1$. Hence it is positive for all $x \in [0,1]$ if and only if it is so at $x = 1$, i.e. $2(1-a)+a > 0$ which gives $a < 2$ and hence $1 < a < 2$. Putting together (5) holds for all $a \in (0,2)$.

Although the reasoning takes a long time to write down fully, taking stock of all possible cases, it is very easy to conceive. So this is another problem well suited to be asked as an objective type question.

Q. 16 A tangent at a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ cuts the coordinate axes at P and Q. The minimum area of the triangle OPQ is

(A) ab (B) $\dfrac{a^2+b^2}{2}$ (C) $\dfrac{(a+b)^2}{2}$ (D) $\dfrac{(a^2+ab+b^2)}{3}$

Answer and Comments: (A). This is a straightforward problem which is a combination of coordinate geometry and maxima/minima. Since at the JEE level we study maxima and minima of functions of only one variable, the first task is to express the given area as a function of a single variable. This suggests that parametric equations of the ellipse are a natural choice. So, let $(a\cos\theta, b\sin\theta)$ be a typical point on the given ellipse. The equation of the tangent at this point is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \tag{1}$$

To find the area of the triangle OPQ we do not have to identify the points P and Q. Since the triangle is rightangled, and the sides are the intercepts on the axis, we can read the area directly from (1) by first recasting it in the two intercepts form, viz.,

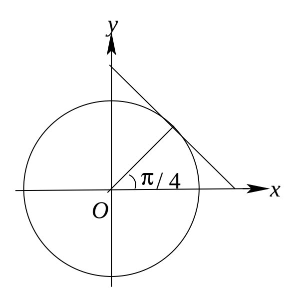
$$\frac{x}{a\sec\theta} + \frac{y}{b\ \mathrm{cosec}\ \theta} = 1 \tag{2}$$

So the area of the triangle OPQ is $f(\theta) = \frac{ab}{2|\sin\theta\cos\theta|}$

(In an objective type test, you do not have to show your work. Even where you do, you can bypass some of the intermediate steps such as (2) and directly write $f(\theta) = \frac{ab}{2|\sin\theta\cos\theta|}$. Also, the absolute value sign may be omitted by restricting the point on the ellipse to lie in the first quadrant because of symmetry.)

Minimising $f(\theta)$ as a function of θ can be done with calculus or simply by observing that the denominator equals $|\sin 2\theta|$ whose maximum value is 1 (occurring when $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, although this is not asked). Hence the minimum area of the triangle OPQ is ab.

Although the problem is simple enough as it stands, it is worth seeing if any shortcuts are possible. The case $a = b$ corresponds to a circle, which can be thought as a limiting case of an ellipse. Let us take a point R on the arc of the circle in the first quadrant. It is clear that as R moves towards either the end $(a, 0)$ or the end $(0, a)$, one of the sides of the triangle OPQ and hence its area tend to ∞. So the minimum must occur when R is midway, i.e. at $(a/\sqrt{2}, a/\sqrt{2})$. This corresponds to $\theta = \frac{\pi}{4}$ with the standard parametrisation of the circle. So it is a reasonable guess to suppose that it would hold for the ellipse too. (This guess can be substantiated rigorously if we use the sophisticated technique of converting an ellipse into a circle by a suitable linear transformation, given at the end of Comment No. 11 in Chapter 8.)



An alert reader will hardly fail to notice the similarity of this question and Q. 21 of the JEE 2003 Screening Paper in Mathematics. There the problem was to minimise the sum of the intercepts on the axes, while in the present problem we are essentially minimising their product. But the answers are different. The minimum of the product of the intercepts occurs at $\theta = \pi/4$ as we just saw. But the minimum of their sum occurs at $\theta = \tan^{-1} \frac{b^{1/3}}{a^{1/3}}$. This means that the sophisticated technique of converting the problem to a circle does not always work. The explanation for this rather subtle. It is due to the fact that the linear transformation (given by $u = x/a, v = y/b$) does not stretch all the lengths uniformly. For example, a segment of length a along the x-axis is taken to a segment of length 1 on the u-axis, so that the stretching factor is $\frac{1}{a}$ for such segments. But a segment of length b parallel to the y-axis is taken to a segment of length 1 parallel to the v-axis so that the stretching factor is $\frac{1}{b}$. Segments in other directions are stretched with different stretching factors. But the areas are all stretched by the same factor, viz., $\frac{1}{ab}$. As a result, the minimum area for the original problem will correspond to the minimum area in the new problem. But a minimum sum of two lengths in different directions need not correspond to the minimum sum of the corresponding lengths in the new problem.

Q. 17 If $y = y(x)$ satisfies the relation $x \cos y + y \cos x = \pi$, then $y''(0)$ equals

(A) 1 (B) -1 (C) π (D) $-\pi$

Answer and Comments: (C). This is a simple problem on implicit differentiation. A common mistake is to try to solve the given relation so as to express y as an explicit function of x. But this is impossible to do. Nor is it needed. Our interest is only in the second derivative of y w.r.t. x at $x = 0$. Therefore, we find y only when $x = 0$. This comes out as π.

Differentiating the given relation implicitly w.r.t. x we get,

$$\cos y - x \sin y \, y' + y' \cos x - y \sin x = 0 \tag{1}$$

Putting $x = 0$ and using $y(0) = \pi$, we get

$$\cos \pi + y'(0) \cos 0 = 0 \tag{2}$$

i.e. $y'(0) = 1$. To get $y''(o)$, we have a choice. Even though the original equation in the question could bot be solved explicitly for y, we can solve (1) for y' and get

$$y' = \frac{dy}{dx} = \frac{y \sin x - \cos y}{\cos x - x \sin y} \tag{3}$$

We can now differentiate this w.r.t. x and get $\frac{d^2y}{dx^2}$ using the quotient rule. Then putting $x = 0, y = \pi$ and $y' = 0$ we can get $y''(0)$. But the expression for $\dfrac{d^2y}{dx^2}$ will be quite complicated since there are many terms which have to be differentiated using the product rule. A better approach is to differentiate (1) as it is and then put $x = 0$ etc. to get $y''(0)$. Differentiation gives

$$\begin{aligned} -2\sin y\, y' &- x\cos y\,(y')^2 - x\sin y\, y'' \\ &+ y''\cos x - y'\sin x - y'\sin x - y\cos x = 0 \end{aligned} \tag{4}$$

We already know that $y(0) = \pi$ and $y'(0) = 1$. So evaluating (3) at $x = 0$ we get

$$y''(0) - \pi = 0 \tag{5}$$

which gives $y''(0) = \pi$. Note that in getting (5), those terms in (4) which vanished at $x = 0$ contributed nothing. In fact, had we taken cognizance of this while writing down (4), we could have saved some time. In differentiating a product of the type $f(x)g(x)$ we apply the product rule for derivatives. But if our interest is only in finding the derivative of this product at some point a at which we know that one of the functions, say f vanishes (i.e. $f(a) = 0$), then it is redundant to take the term $f(x)g'(x)$ and the work in forming $g'(x)$ is essentially a waste. So when we differentiated (1), then from the second term, viz., $x\sin y\, y'$ we need consider only $\sin y\, y'$ as the effective derivative. Similarly, as far as the derivative of $y\sin x$ at $x = 0$ is concerned, we need take only the term $y\cos x$ and drop $y'\sin x$.

Shortcuts like this can save you precious time. But, as always, they have to be applied with care. For example, a similar shortcut is not possible while getting (1). Here we got $\cos y - x\sin y\; y'$ by differentiating the product $x\cos y$ w.r.t. x. If our interest was only in $y'(0)$, then we could have taken cognizance of the fact that in the product $x\sin y$, the factor x vanishes at $x = 0$ and hence we could have dispensed with the derivative of $\cos y$ (viz., $-\sin y\, y'$). But our interest is *not just* in the first derivative at $x = 0$, but *also* in the first derivative at *all nearby points*. This is because the second derivative of a function at a point a depends not only on its first derivative at a, but on its first derivative at all points in a neighbourhood of a. Even if some of the terms in the derivative vanished at a, they may not vanish in a neighbourhood of a and dropping them will give us a wrong second derivative. Apparently, the papersetters wanted to test this. If the idea was only to test the knowledge of implicit differentiation, they could have stopped at asking $y'(0)$. Asking $y''(0)$ would then only add to the drudgery. Asking for the second derivative also tests whether a candidate is able to come over the temptation to write down (3) and differentiate.

Q. 18 If $f(x)$ is a continuous and differentiable function and $f(\frac{1}{n}) = 0$ for every positive integer n, then

(A) $f(x) = 0,\ x \in [0, 1]$
(B) $f(0) = 0, f'(0) = 0$
(C) $f'(x) = 0,\ x \in [0, 1]$
(D) $f(0) = 0$ but $f'(0)$ need not be zero

Answer and Comments: (B). There is some redundancy in the question. Differentiability always implies continuity and so it would have been enough to give that f is differentiable.

Continuity of f at 0 means

$$\lim_{x \to 0} f(x) = f(0) \tag{1}$$

Verbally, as x tends to 0 through all possible values of x (other than the value 0), $f(x)$ tends to $f(0)$. So, if we let x tend to 0 through some restricted values, then also $f(x)$ will tend to $f(0)$. We have total freedom in choosing these values. All that matters is that they should approach 0. So we take these values as $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$. For these values of x, the function vanishes. So

$$f(0) = \lim_{x \to 0} f(x) = \lim_{\frac{1}{n} \to 0} f(\frac{1}{n}) = \lim_{\frac{1}{n} \to 0} 0 = 0 \tag{2}$$

which proves that $f(0) = 0$. An exactly similar argument works for $f'(0)$. By definition, this is the limit of the ratio $\dfrac{f(x) - f(0)}{x - o}$ as x tends to 0. We are given that this limit exists. To find it we are free to let x tend to 0 through any set of values. Choosing these values as $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ and using $f(0) = 0$ which we proved already, we get,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{\frac{1}{n} \to 0} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0} = \lim_{\frac{1}{n} \to 0} 0 = 0 \tag{3}$$

The key idea in this problem is that of a **restricted limit**. If $\lim\limits_{x \to c} f(x)$ exists and equals L (say), and if you let the variable x approach c through some values restricted to some set S, then no matter what this set S is, this new limit will also exist and equal L. The converse is false. It may happen that a restricted limit exists but the limit does not. For example, if you get two different limits by restricting the variable to two different subsets, then the (unrestricted) limit does not exist. Indeed, this is a common method for proving the non-existence of some limits. Consider, for example, the limit $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$. Here, if you let x approach 0 through values of the form $\frac{1}{n\pi}$, the limit is 0. But if you let x approach 0 through values of the form $\frac{1}{(2n+\frac{1}{2})\pi}$, the limit is 1. As you get two different limits depending upon how x approaches 0, the original limit, viz., $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist.

The problem can be looked at in a somewhat different way too. The sequence $\{\frac{1}{n}\}$ converges to 0 as n tends to ∞. Since f is given to be continuous at 0 we can apply Theorem 1 in Comment No. 3 of Chapter 16 and get that the sequence $\{f(\frac{1}{n})\}$ tends to $f(0)$ as $n \to \infty$. But by hypothesis, $\{f(\frac{1}{n})\}$ is a constant sequence with every term 0. So its limit has to be 0. Hence $f(0) = 0$, because no sequence can have two different limits.

To get $f'(0)$ we apply the same theorem but to a different function $g(x)$ defined by

$$g(x) = \begin{cases} \dfrac{f(x) - f(0)}{x - 0} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases} \tag{4}$$

Then, by very definition of the derivative of a function,

$$\begin{aligned} \lim_{x \to 0} g(x) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= f'(0) \\ &= g(0) \end{aligned} \tag{5}$$

which shows that g is continuous at 0. Therefore by the same theorem as above, $g(\frac{1}{n}) \to g(0)$ as $n \to \infty$. But from the hypothesis, $g(\frac{1}{n}) = 0$ for every positive integer n. So we get

$$\begin{aligned} f'(0) &= \lim_{n \to \infty} g(\frac{1}{n}) \\ &= \lim_{n \to \infty} 0 = 0 \end{aligned} \tag{6}$$

Q. 19 The function given by $y = ||x| - 1|$ is differentiable for all real numbers except the points

(A) $0, 1, -1$ (B) $1, -1$ (C) 1 (D) -1

Answer and Comments: (A). This is a problem on the differentiability of the composite of two functions. The given function can be written as the composite function $g \circ f$ where f and g are the functions defined by

$$f(x) = |x| \tag{1}$$

$$\text{and} \quad g(x) = |x - 1| \tag{2}$$

By the chain rule, the composite $g(f(x))$ is differentiable at all points a satisfying the two conditions, viz., (i) f is differentiable at a and (ii) g is differentiable at $f(a)$. In the present case, the function $f(x) = |x|$ is differentiable everywhere except at $x = 0$. So condition (i) is satisfied at all x except $x = 0$. As for condition (ii), g is non-differentiable only at $x = 1$. So the second condition is satisfied whenever $f(x) \neq 1$, i.e. $|x| \neq 1$ which gives $x = \pm 1$.

Put together, f is differentiable everywhere except possibly at $0, 1$ and -1. A hasty student may then mark (A) as the correct answer. But further analysis is necessary. The chain rule gives only a *sufficient* condition for differentiability of the composite function. Even when one of the two conditions (i) and (ii) given above is not satisfied for some point a, the composite $g \circ f$ *may be* differentiable at a. As an extreme example, suppose g is an identically constant function. Then so is the composite $g \circ f$. So it is differentiable everywhere, regardless of where the function f is differentiable. As a less extreme example, consider $\cos|x|$ which is the composite of the absolute value function followed by the cosine function. Here the absolute value function $|x|$ is not differentiable at $x = 0$. But the composite $\cos|x|$ is the same as $\cos x$ which is differentiable everywhere, including $x = 0$. As an example where condition (ii) fails but the composite is differentiable, consider $|x^3|$ which is the composite of the function $f(x) = x^3$ and the function $g(x) = |x|$.

So, the point to note is that after getting $0, 1$ and -1 as *possible* points of non-differentiability, we have to analyse further to see if each one of these is indeed a point of non-differentiability. For this there is no golden method. One has to study the behaviour of the given function $||x| - 1|$ in a neighbourhood of each of these bad points. Near $x = 0$, we have

$$||x| - 1| = \begin{cases} 1 - x & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases} \tag{3}$$

which shows that the right and the left handed derivatives of the function at 0 are -1 and 1 respectively. Hence the given function is not differentiable at 0.

Near $x = 1$ we have

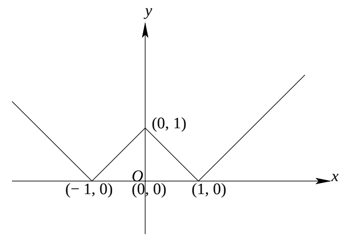
$$||x| - 1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases} \tag{4}$$

So again we see that the left and the right handed derivatives differ. So 1 is also a point of non-differentiability. A similar analysis can be made at -1 too. But that is hardly necessary. The given function $||x| - 1|$ is an even function of x. So, its non-differentiability at 1 implies that it is non-differentiable at -1. We can *now* say legitimately that (A) is the right answer.

An alternate approach is to draw a graph of the function. Although this is no substitute for an analytic proof, a good graph inspires ideas and often it is a routine matter to back them up by an analytical argument. In the present case, the fact that the given function is even saves us half the work in drawing its graph. We focus only on $x \geq 0$. For these values, $|x| = x$ and so we have

$$\begin{aligned} ||x| - 1| &= |x - 1| \\ &= \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } 0 \leq x < 1 \end{cases} \end{aligned} \tag{5}$$

It is now an easy matter to draw the graph for $x \geq 0$. As noted above, the entire graph is obtained by taking its reflection in the y-axis. From the graph we see at once that the given function is differentiable everywhere except at $0, 1$ and -1.



This question is hardly suited to be asked as an objective type question. A candidate who hastily marks (A) as the correct answer stands to gain in terms of precious time over a candidate who indulges in a further analysis.

Q. 20 If $f(x)$ is twice differentiable and $f(1) = 1, f(2) = 4, f(3) = 9$, then

(A) $f''(x) = 2$ for every $x \in (1, 3)$ (B) $f''(x) = 5$ for some $x \in (2, 3)$

(C) $f''(x) = 3$ for every $x \in (2, 3)$ (D) $f''(x) = 2$ for some $x \in (1, 3)$

Answer and Comments: (D). Evidently, this is a problem on the Mean Value Theorems. Further, since the second derivative of the function is involved, it is clear that we have to apply the MVT twice, once to some function and then again to the derivative of some function. The trick is to choose these functions cleverly. And the only clue is to look at the values which the function f assumes at certain given points.

The first thing that strikes is that the values assumed at $1, 2, 3$ are simply their squares respectively. In other words, the given function agrees with the function x^2 at these points. Put differently, if we consider a new function $g(x)$ defined as $g(x) = f(x) - x^2$, then g is also twice differentiable and vanishes at the points $1, 2$ and 3. This is just the thing we need to apply Rolle's theorem. In fact, we can apply it separately to the intervals $[1, 2]$ and $[2, 3]$ to get some points $c_1 \in (1, 2)$ and $c_2 \in (2, 3)$ such that

$$g'(c_1) = g'(c_2) = 0 \tag{1}$$

Clearly, we have $c_1 < c_2$. Because of (1), we can apply Rolle's theorem again, but this time to the function $g'(x)$ over the interval $[c_1, c_2]$ to get some $c_3 \in (c_1, c_2)$ such that

$$g''(c_3) = 0 \tag{2}$$

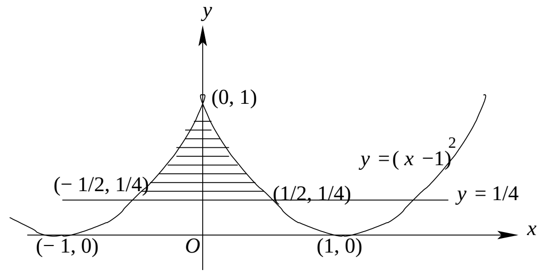
Since $g(x) = f(x) - x^2$, a direct computation gives $g''(x) = f''(x) - 2$ for all x. So from (2), $f''(c_3) = 2$. Hence (D) is correct.

The problem is a good combination of knowledge of a theorem and a little perceptivity. The implicit hint is just right.

Q. 21 The area bounded by the parabolas $y = (x + 1)^2$, $y = (x - 1)^2$ and the line $y = 1/4$ is

(A) 4 (B) 1/6 (C) 4/3 (D) 1/3

Answer and Comments: (D). This is a straightforward problem on finding the area using integrals. The first task is to sketch the given region. The two parabolas meet each other only at the point $(0,1)$. The line $y = 1/4$ meets the parabola $y = (x-1)^2$ at $(1/2, 1/4)$ and $3/2, 1/4)$ and the parabola $y = (x+1)^2$ at $(-1/2, 1/4)$ and $(-3/2, 1/4)$ so that the given region is the shaded region in the figure.



As the region is evidently symmetric about the y-axis, its area is

$$\begin{aligned} \text{area} &= 2\int_0^{1/2} (x-1)^2 - \frac{1}{4}dx & (1)\\ &= 2\int_0^{1/2} x^2 - 2x + \frac{3}{4}dx \\ &= 2 \times \left[\frac{x^3}{3} - x^2 + \frac{3x}{4}\Big|_0^{1/2}\right] \\ &= 2 \times \left[\frac{1}{24} - \frac{1}{4} + \frac{3}{8}\right] \\ &= 2 \times \frac{4}{24} = \frac{1}{3} & (2) \end{aligned}$$

Although quite simple, the problem is not suitable for a multiple choice question. The work involves several ideas such as solving equations, sketching curves and integration. That increases the chances of numerical mistakes. For example, suppose a candidate correctly identifies the region, realises its symmetry about the y-axis and also does the integration above correctly but forgets to multiply the answer by 2 at the end. His answer then would be 1/6. As this is also one of the given choices, there is nothing to alert him that he has made a silly mistake. In a conventional examination this would come to surface and the candidate could be given partial credit for doing the bulk of the work correctly. In the present set-up, he gets a negative credit! While there is some justification for negative marking to discourage wanton guesswork, in the present problem, there is hardly any scope to simply guess the answer. As a result, a candidate who does the work honestly and correctly except for one silly slip pays a heavy price both in terms of time and credit. In an examination where the time is severely limited, a student who does not even understand the question is better off because he can devote his time to other questions.

Q. 22 $\displaystyle\int_{-2}^{0}(x^3+3x^2+3x+3+(x+1)\cos(x+1))dx$ is equal to

(A) -4 (B) 0 (C) 4 (D) 6

Answer and Comments: (C). It would be foolish to integrate the expression as it stands and even more foolish to expand $\cos(x+1)$. The presence of $(x+1)$ at several places suggests that a far better idea is to call $x+1$ as some u. Once this strikes, it is easy to club together the various powers of x to form $(x+1)^3$. Then a straightforward calculation gives the integral, say I as

$$\begin{aligned} I &= \int_{-2}^{0}(x^3+3x^2+3x+3+(x+1)\cos(x+1))dx \\ &= \int_{-2}^{0}((x+1)^3+(x+1)\cos(x+1)+2)dx \\ &= \int_{-1}^{1} u^3+u\cos u+2du \quad \text{where } u=x+1 \qquad (1) \end{aligned}$$

Now the integral is amenable to evaluation by finding an antiderivative of the integrand. In fact this is very tempting for those who are equipped with huge collections of integration formulas waiting to be applied. But even now, some shortcuts are possible by noting that the interval of integration, viz. $[-1,1]$ is symmetric about 0. As a result, the integral of any odd function over it is 0. In the present case, u^3 and $u\cos u$ are odd functions of u. So we are left with $I=\int\limits_{-1}^{1}2du$ which is too simple to need a pen, being the integral of a constant function. Thus $I=2\times(1-(-1))=2\times 2=4$.

Unlike the last problem, the present problem involves more thought than computation. Once the key idea is conceived, there is little likelihood of a numerical mistake. So the problem is very well suited for a multiple choice question.

Q. 23 If $\displaystyle\int_{\sin x}^{1} t^2 f(t)\,dt = 1-\sin x$, then $f(\frac{1}{\sqrt{3}})$ is

(A) $1/3$ (B) $1/\sqrt{3}$ (C) 3 (D) $\sqrt{3}$

Answer and Comments: (C). The first task is to determine the function $f(x)$ from the data. Clearly, the right tool is the second form of the Fundamental Theorem of Calculus, which says that the derivative (w.r.t. x) of a function of the form $\int\limits_{a}^{x} g(t)dt$ is simply $g(x)$. In the present case, the upper limit of integration is constant and the lower one is a function of x, viz. $\sin x$. The former reverses the sign of the derivative while the latter makes it necessary to apply the chain rule along with the fundamental theorem. (More generally, one can apply (19) in Comment No. 12 of Chapter 17. But instead of relying on a huge collection of formulas, it is far better to rely on the underlying reasoning.)

So, differentiating both the sides of the given relation w.r.t. x we get

$$\begin{aligned} -\cos x &= \frac{d}{d\sin x}\left(\int_{\sin x}^{1} t^2 f(t)\, dt\right) \times \frac{d}{dx}(\sin x) \\ &= -\sin^2 x f(\sin x) \times \cos x \\ &= -f(\sin x)\sin^2 x \cos x \qquad (1) \end{aligned}$$

from which we get

$$f(\sin x) = \frac{1}{\sin^2 x} \qquad (2)$$

Here as x varies so does $\sin x$. So we might as well take $\sin x$ as an independent variable and replace it by any symbol, say t. Then (2) becomes

$$f(t) = \frac{1}{t^2} \qquad (3)$$

from which we get at once that $f(\frac{1}{\sqrt{3}}) = 3$. (Note, however, that $\sin x$ can take only values from -1 to 1. So, (3) is valid only for $t \in [-1, 1]$. If the problem had asked to find $f(10)$ (say), then the data of the problem is insufficient to give it.)

There is also a sneaky way to get (3) and hence the answer. In the integral given in the problem, the integration is over the interval $[\sin x, 1]$. The length of this interval is $1 - \sin x$. The value of the integral is also $1-\sin x$. One of the ways this could happen is if the integrand is identically equal to 1. Of course, if the integral equals the length of the interval of integration for a particular value of x, then that does not mean much. But if this happens for every x, then it can indeed be shown that the integrand must be identically equal to 1. In fact, that is basically what we did above. Now, in an objective type test, it is enough if you suspect something. So, if the observation that the integral equals the length of the interval of integration leads a candidate to think that (3) holds, he gets the answer with much less work.

Except for this unintended short cut, the problem is a good one.

Q. 24 $y = y(x)$ is a solution of the differential equation $(x^2 + y^2)dy = xydx$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is

(A) $\sqrt{2(e^2 - 1)}$ (B) $\sqrt{2(e^2 + 1)}$ (C) $\sqrt{3}\, e$ (D) $\sqrt{\frac{e^2+1}{2}}$

Answer and Comments: (C). The given differential equation is very similar to that in Exercise (19.16)(vi). The coefficients of dx and dy are homogeneous polynomials of degree 2 in x and y. Therefore the method given in Comment No. 15 of Chapter 19 is applicable. So we put $y = vx$ which gives $dy = vdx + xdv$. The differential equation changes to

$$(1 + v^2)(vdx + xdv) = vdx \tag{1}$$

which can be recast in the separate variable form as

$$\frac{dx}{x} = -\frac{1 + v^2}{v^3}dv \tag{2}$$

Integrating, the solution comes as

$$\ln x = \frac{1}{2v^2} - \ln v + c \tag{3}$$

or equivalently,

$$\ln vx = \frac{1}{2v^2} + c \quad \text{i.e.} \quad \ln y = \frac{x^2}{2y^2} + c \tag{4}$$

where c is a constant. The initial condition $y(1) = 1$ determines c as $-\dfrac{1}{2}$. Hence the particular solution is

$$\ln y = \frac{x^2}{2y^2} - \frac{1}{2} \tag{5}$$

We are given $y(x_0) = e$. So from (5),

$$1 = \frac{x_0^2}{2e^2} - \frac{1}{2} \tag{6}$$

which gives $x_0^2 = 3e^2$ and hence $x_0 = \sqrt{3}\ e$. (Here we are implicitly assuming that x_0 is positive. Without this assumption, (3) will not make sense. However, if we replace $\ln x$ by $\ln |x|$, then (3) makes sense even for $x < 0$. The substitution $y = vx$ makes sense for any $x \neq 0$. Actually, the d.e. has a general solution for all $x \neq 0$. This breaks into two intervals : $(-\infty, 0)$ and $(0, \infty)$. The solutions over these two portions are independent of each other. The initial condition given deals with a point in the second interval. It cannot possibly determine a unique solution over the other half, viz. $(-\infty, 0)$. So, we have to assume $x_0 > 0$. At the JEE level, such a hairsplitting can hardly be expected even when a candidate has to show the work.)

Although the problem is straightforward once the correct substitution is made, the work involved is too long and prone to errors. So, like Q. 21, this question is not well suited as a multiple choice question.

Q. 25 For the primitive integral equation $ydx + y^2dy = xdy$; $x \in I\!R$, $y > 0$, if $y(1) = 1$ then $y(-3)$ is

(A) 3 (B) 2 (C) 1 (D) 5

Answer and Comments: (A). The phrase 'primitive integral equation' is hardly standard and it is difficult to see what is gained by using it instead of the standard phrase 'differential equation'. The use of an unusual phrase or notation only serves to unsettle some of the candidates.

Coming to the question itself, the present question is of the same spirit as the last one. But the equation given is of a different type. Unlike in the last problem, the present d.e. does not fall under a standard type. It is certainly not in the separate variable form. It can be rewritten as $(y^2 - x)\frac{dy}{dx} + y = 0$ which is not linear because of the presence of y in the coefficient of $\frac{dy}{dx}$. But if we reverse the roles of the dependent and the independent variables, we can rewrite the equation as

$$y\frac{dx}{dy} - x = -y^2 \tag{1}$$

which is now a linear equation. To solve it we further rewrite it as

$$\frac{dx}{dy} - \frac{x}{y} = -y \tag{2}$$

The integrating factor is $e^{-\int \frac{dy}{y}} = e^{-\ln y} = 1/y$. Multiplying by it, the equation becomes

$$\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = -1 \tag{3}$$

The L. H. S. can be recognised as $\frac{d}{dy}(\frac{x}{y})$. So the general solution is

$$\frac{x}{y} = -y + c \tag{4}$$

where c is an arbitrary constant. (We could have bypassed the intermediate steps by using the readymade formula (38) of Comment No. 12, Chapter 19, taking care, of course, to reverse the roles of x and y. But, as remarked in the comments on Q. 23, it is far better to rely on reasoning than on readymade formulas.) Having obtained the general solution, the rest of the work is routine. The initial condition $y = 1$ when $x = 1$ determines the constant c as 2. Hence the particular solution is

$$\frac{x}{y} = 2 - y \tag{5}$$

If we now put $x = -3$ we get a quadratic in y, viz., $y^2 - 2y - 3 = 0$, which has two roots, viz. 3 and -1. But we are given that $y > 0$. So we get 3 as the value of y when $x = -3$.

The idea of interchanging the dependent and the independent variables is a rather tricky one. Let us see if there is some easier way of rewriting the given differential equation so that both the sides can be recognised as some differentials. As a starter we rewrite it as

$$ydx + (y^2 - x)dy = 0 \tag{6}$$

But this is not exact (see Comment No. 11 of Chapter 19 for a definition of an exact equation) because here the two partial derivatives $\frac{\partial}{\partial y}(y)$ and $\frac{\partial}{\partial x}(y^2 - x)$ are, respectively, 1 and -1 which are not equal.

But, we can look for an integrating factor. As mentioned at the end of Comment No. 11 of Chapter 19, there is no golden method for this. But we can make some intelligent observations. The term y^2dy is already a differential (of $\frac{y^3}{3}$) and so we can safely leave it aside. The remaining two terms give $ydx - xdy$. This is not the differential of anything as it stands. But if we merely divide it by y^2, it becomes the differential of $\frac{x}{y}$. Further, division by y^2 is not injurious to the term y^2dy. In fact, it will become even simpler. So dividing (6) by y^2 we get

$$\frac{ydx - xdy}{y^2} + dy = 0 \tag{7}$$

which is hardly different from (3), except that we can now recognise more readily that the L.H.S. is the differential of $\frac{x}{y} + y$ and therefore get the general solution as

$$\frac{x}{y} + y = c \tag{8}$$

where c is an arbitrary constant. This is, of course, identical to (4). It is just that we have obtained it more quickly. It is instructive to compare the two derivations. In getting (3) from (1), we first divided by y to get (2) and then again divided by y to get (3). On the other hand, we got (7) by directly dividing (1) by y^2. So, ultimately, we did the same work. But in the first approach, there were two natural steps in it, while in the second approach, it was most natural to do it directly.

In case you are unable to think of either of these two methods, everything is not lost. A suitable substitution may still pull you through. The substitution $y = vx$ was used in the last question. This substitution is sure to work for those d.e.'s where $\frac{dy}{dx}$ is a ratio of two homogeneous polynomials (in x and y) of the same degrees. And the d.e. in the last question fell in this category. But somehow, this substitution also works in many other problems too and is worth trying if you cannot think of any other way.

Let us see what happens if we put $y = vx$ in the given problem. As before, we now have $dy = xdv + vdx$ and the given equation reduces to

$$vxdx + v^2x^2(xdv + vdx) = x(xdv + vdx) \tag{9}$$

which upon simplification and cancellation of the factor x^2 becomes

$$v^2(xdv + vdx) = dv \tag{10}$$

We note that $xdv + vdx$ equals $d(xv)$. So if we take the factor v^2 to the other side (which is free of x), we get the general solution as

$$\begin{aligned} xv &= \int \frac{dv}{v^2} \\ &= -\frac{1}{v} + c \end{aligned} \tag{11}$$

When we put back $v = y/x$ this becomes (8). The rest of the work remains the same.

Many differential equations can be solved by a variety of methods. The present problem is a good example. Once the general solution of a differential equation is obtained, finding a particular solution using the given initial condition is a purely clerical matter. It is difficult to see the rationale behind testing this drudgery twice, once in the last question and then again in the present one. The drudgery is, in fact, increased by further asking to evaluate this particular solution at a particular point. Probably this is inevitable because, the problem of finding the general solution of a d.e. cannot be asked as a multiple choice question. For, if so asked, the correct answer can be obtained simply by verifying each of the alternatives given. That would kill the very essence of the problem, because, as remarked in Comment No. 11 of Chapter 19, obtaining the solution of a d.e. is an art, verifying it is pure labour.

Still, the duplication could have been avoided. For example, in the present problem, instead of asking the value of y for a given value of x, the question could have asked to identify the type of curves represented by the given differential equation. In that case, after getting the general solution (4), one can tell that it is a parabola regardless of the value of c. In fact, this would have also tested some knowledge of classifying the conics (which is not tested anywhere else in the question paper). That would have been far better than forcing the candidates to do the same type of clerical work twice.

Q. 26 Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-zero, non-coplanar vectors. Let $\mathbf{b}_1 = \mathbf{b} - \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}, \mathbf{b}_2 = \mathbf{b} + \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a},$

$\mathbf{c}_1 = \mathbf{c} - \dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} + \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{b}_1$, $\mathbf{c}_2 = \mathbf{c} - \dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \dfrac{\mathbf{b}_1\cdot\mathbf{c}}{|\mathbf{b}_1|^2}\mathbf{b}_1$,
$\mathbf{c}_3 = \mathbf{c} - \dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} + \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{b}_1$, $\mathbf{c}_4 = \mathbf{c} - \dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|^2}\mathbf{b}_1$,
Which of the following set of vectors is orthogonal?

(A) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_3)$ (B) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2)$
(C) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_1)$ (D) $(\mathbf{a}, \mathbf{b}_2, \mathbf{c}_2)$

Answer and Comments: (B). The very form of the question is clumsy unless you see the purpose behind the constructions of the vectors, $\mathbf{b}_1, \mathbf{b}_2$ etc. From the definitions of the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, it is clear that the problem has to do with the component of a vector along and perpendicular to some other vector. As proved at the end of Comment No. 13, Chapter 21, if $\mathbf{u}$ is any vector, then its components along and perpendicular to $\mathbf{a}$ are, respectively, $\dfrac{\mathbf{a}\cdot\mathbf{u}}{|\mathbf{a}|^2}\mathbf{a}$ and $\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{u}}{|\mathbf{a}|^2}\mathbf{a}$. So, we see that the given vector $\mathbf{b}_1$ is nothing but the component of $\mathbf{b}$ perpendicular to $\mathbf{a}$. Hence $\mathbf{a}\cdot\mathbf{b}_1 = 0$. (Of course, we also see this by direct verification. But if we know it beforehand, that makes us feel comfortable.) On the other hand, $\mathbf{b}_2$ is *not* orthogonal to $\mathbf{a}$ except when $\mathbf{b}$ is itself orthogonal to $\mathbf{a}$ (in which case, $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal). So the correct answer has to be from (A), (B) or (C). In all these three alternatives, the first two vectors are the same, viz., $\mathbf{a}$ and $\mathbf{b}_1$ and we already know that they are mutually orthogonal. So to get the correct answer, we have to check which of the three vectors $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}_1$.

Let us first consider orthogonality of these three vectors with the vector $\mathbf{a}$. This can be done by taking their dot products with th vector $\mathbf{a}$. Note that these three vectors differ only in the coefficients of the vector $\mathbf{b}_1$. But $\mathbf{b}_1$ is already known to be orthogonal to $\mathbf{a}$, so that $\mathbf{a}\cdot\mathbf{b}_1 = 0$. So, either all three or none of them will be orthogonal to $\mathbf{a}$. A direct computation gives

$$\begin{aligned} \mathbf{a}\cdot\mathbf{c}_1 &= \mathbf{a}\cdot\mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}\cdot\mathbf{a} + \frac{\mathbf{b}\cdot\mathbf{c}_1}{|\mathbf{c}|^2}(\mathbf{a}\cdot\mathbf{b}_1) \\ &= \mathbf{a}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{a} + 0 \\ &= 0 \end{aligned} \tag{1}$$

which proves that $\mathbf{a}$ is orthogonal to $\mathbf{c}_1$ and hence also to $\mathbf{c}_2$ and $\mathbf{c}_3$. So the problem is now reduced to checking which of the vectors $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ is orthogonal to the vector $\mathbf{b}_1$. Again, this is done by taking the dot products, $\mathbf{c}_1\cdot\mathbf{b}_1$, $\mathbf{c}_2\cdot\mathbf{b}_1$ and $\mathbf{c}_3\cdot\mathbf{b}_1$. In each of these dot products, the dot product of $\mathbf{b}_1$ with the middle term, viz., $\dfrac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$ vanishes because we already know that $\mathbf{b}_1$ is orthogonal to $\mathbf{a}$. Also the first term is $\mathbf{c}\cdot\mathbf{b}_1$ in all three cases. Hence the answer now depends only on the dot product of $\mathbf{b}_1$ with the third terms in the definitions of $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$. We see that only in the case of $\mathbf{c}_2$, will the dot product with the third term equal $\mathbf{c} \cdot \mathbf{b}_1$. Thus, only $\mathbf{c}_2$ is orthogonal to $\mathbf{b}_1$. Hence $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2)$ is an orthogonal set.

The question itself is simple as its solution involves absolutely nothing beyond elementary properties of the dot product of vectors. But a candidate is likely to be baffled by the somewhat arbitrary and clumsy manner in which the six new vectors (viz., $\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$) are defined from the three given vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ with which we start. Actually, the construction is not all that clumsy once you know its purpose. Note that if the original set $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is orthogonal, then $\mathbf{b}_1$ and $\mathbf{c}_2$ will coincide with $\mathbf{b}$ and $\mathbf{c}$ respectively and we shall get nothing new. But even when the original set is not orthogonal, the new set always is as we just proved. Further, the span of the the first two vectors, viz., $\mathbf{a}$ and $\mathbf{b}_1$ is the same as that of the first two vectors of the original set. The span of the three vectors $\mathbf{a}, \mathbf{b}_1$ and $\mathbf{c}_2$ is the same as that of the original set $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. In a three dimensional space this does not mean much. But the construction could have as well been carried for higher dimensional vectors and the spans are still equal.

In fact, suppose we have four linearly independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in $\mathbb{R}^n$. (For this to be possible, n will have to be at least 4.) These vectors need not be mutually orthogonal. But starting from them in the given order, we can generate a set of four vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ which is orthogonal and has the same span as the original set $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$. (Normally, this set would be denoted by $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. But in this problem, the order of the vectors matters in the construction. To emphases this, we write the set like an ordered quadruple of vectors.)

The present question already gives us what the first three of these new vectors will be, viz.,

$$\mathbf{a}' = \mathbf{a} \tag{2}$$

$$\mathbf{b}' = \mathbf{b} - \frac{\mathbf{b} \cdot \mathbf{a}'}{|\mathbf{a}'|^2}\mathbf{a}' \tag{3}$$

$$\mathbf{c}' = \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{a}'}{|\mathbf{a}'|^2}\mathbf{a}' - \frac{\mathbf{c} \cdot \mathbf{b}'}{|\mathbf{b}'|^2}\mathbf{b}' \tag{4}$$

The trend is now clear. So we expect that the fourth vector $\mathbf{d}'$ will be

$$\mathbf{d}' = \mathbf{d} - \frac{\mathbf{d} \cdot \mathbf{a}'}{|\mathbf{a}'|^2}\mathbf{a}' - \frac{\mathbf{d} \cdot \mathbf{b}'}{|\mathbf{b}'|^2}\mathbf{b}' - \frac{\mathbf{d} \cdot \mathbf{c}'}{|\mathbf{c}'|^2}\mathbf{c}' \tag{5}$$

It is a trivial matter to verify that the set $(\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}')$ is orthogonal. It is obvious how to generalise this method of converting a given basis to an orthogonal basis. This transition is very important because of the special advantages which orthogonal bases have over ordinary bases. The method given here is called **Gram-Schmidt orthogonalisation process**. It is not a part of the JEE syllabus. But those who happen to know it will have an easy time with this question because they will be able to mark (B) as correct simply by inspection. So, like Q. 12, in this question too an advanced knowledge helps. Of course, the advantage gained in this question is not so substantial as that gained by the knowledge of Cayley-Hamilton theorem in Q. 12. But when time is precious, even a little saving matters.

Q. 27 The functions $f(x)$ and $g(x)$ from $I\!R$ to $I\!R$ are defined by

$$f(x) = \begin{cases} 0, & x \text{ rational} \\ x, & x \text{ irrational} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & x \text{ irrational} \\ x, & x \text{ rational} \end{cases}$$

Then $(f - g)(x)$ is

(A) one-one and onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one

Answer and Comments: (A). The best thing is to write down $(f-g)(x)$ explicitly for $x \in I\!R$. Note that to define f, the real line has been decomposed into two parts, viz., the set of rationals and the set of irrationals. Similarly, to define g, $I\!R$ is partitioned into 2 parts. Normally, this would mean that to define $(f - g)(x)$ $I\!R$ will have to be decomposed into 4 (= 2×2) parts. But here we are lucky that the two decompositions of $I\!R$ are the same. So we have

$$(f - g)(x) = f(x) - g(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational} \end{cases} \tag{1}$$

Denote the sets of rational and irrationals by A and B respectively. They are complements of each other. Note that $(f - g)$ maps the set A to itself. In fact, when restricted to A, the function $f - g$ behaves exactly like the identity function on A and is therefore a bijection of A. Also it maps B to itself, because whenever x is irrational, so is $-x$. Restricted to B, this function is also a bijection. In effect, the function $f - g$ consists of two bijections, one of A to itself and another of B to itself. From this it is clear that $f - g$ is a bijection. (A formal proof is tedious. For example, to show that $f - g$ is one-one, we would have to take two distinct real numbers x and y and make four cases, viz., (i) $x \in A, y \in A$, (ii) $x \in A, y \in B$, (iii) $x \in B, y \in A$ and (iv) $x \in B, y \in B$ and show that in every case $(f - g)(x)$ differs from $(f - g)(y)$. Comparatively, showing that $(f - g)(x)$ is onto is easier because it breaks into only two case. Given $y \in I\!R$, if $y \in A$, then we have $(f - g)(y) = y$. But if $y \in B$, then $(f - g)(-y) = y$. In either case y has a preimage.)

This is one of those problems where the underlying reasoning is simple to conceive but takes long to write in detail. Such problems are ideal for multiple choice questions.

Q. 28 Suppose X and Y are two sets and $f : X \longrightarrow Y$ is a function. For a subset A of X, define $f(A)$ to be the subset $\{f(a) : a \in A\}$ of Y. For a subset B of Y, define $f^{-1}(B)$ to be the subset $\{x \in X : f(x) \in B\}$ of X. Then which of the following statements is true?

(A) $f^{-1}(f(A)) = A$ for every $A \subset X$.

(B) $f^{-1}(f(A)) = A$ for every $A \subset X$ if only if $f(X) = Y$.

(C) $f(f^{-1}(B)) = B$ for every $B \subset Y$.

(D) $f(f^{-1}(B)) = B$ for every $B \subset Y$ if only if $f(X) = Y$.

Answer and Comments: (D). The condition that $f(X) = Y$ is just a paraphrase of saying that f is onto. Now, if Y is a singleton set then the function f is a constant function and hence is trivially onto (unless $X = \emptyset$). But in such a case, even if A consists of just one point, $f(A)$ is the entire set Y and so $f^{-1}(f(A))$ is the entire set X, which could be much bigger than A. So neither of the first two statements is true even if we assume that $f(X) = Y$. As for statement (C), if $f(X)$ is a proper subset of Y (so that f is not onto), then we see that (C) fails for $B = Y$, because $f^{-1}(Y) = X$ but $f(f^{-1}(Y)) = f(X) \neq Y$. So, by elimination, statement (D) is correct. For an honest answer, one must, of course, prove it. As we just saw, if we put $B = Y$, then $f(f^{-1}(Y))$ is simply the range of the function f. If this is to equal Y, then the function must be onto. Thus the given condition is surely necessary. Conversely, assume that $f(X) = Y$, i.e. that f is onto. Now let B be any subset of Y. For any $x \in f^{-1}(B)$, we have $f(x) \in B$ by definition. So, the inequality $f(f^{-1}(B)) \subset B$ always holds whether f is onto or not. For the other way inclusion, we need that f is onto. So suppose $b \in B$. As f is onto, there exists some $x \in X$ such that $f(x) = b$. Also, by definition, this x lies in the subset $f^{-1}(B)$. Therefore, again by definition, $f(x) \in f(f^{-1}(B))$ i.e. $b \in f(f^{-1}(B))$. Thus we have shown that $B \subset f(f^{-1}(B))$. This completes the proof of (D).

This problem, too, is very simple and involves only reasoning and hardly any computation. So it is well suited as a multiple choice question. However, as just observed, it fails to distinguish a student who marks the correct answer merely by eliminating the others from a student who can actually prove it.

The versions of this question reported by the candidates were incoherent. The version given above is more of a guess.

MAIN PAPER JEE 2005

In all there are 18 questions. The first eight questions carry 2 marks each, the next eight questions carry 4 marks each. The last two questions carry 6 marks each.

Problem 1: A player plays n matches ($n \geq 1$) and the total runs made by him in n matches is $\frac{1}{4}(n+1)(2^{n+1}-n-2)$. If he makes $k.2^{n-k+1}$ runs in the k-th match for $1 \leq k \leq n$, find the value of n.

Analysis and Solution: If we take away the garb of the matches and the runs made, the problem is clearly equivalent to solving the equation

$$\sum_{k=1}^{n} k2^{n-k+1} = \frac{(n+1)(2^{n+1}-n-2)}{4} \tag{1}$$

and get the value of n. The series on the L.H.S. is an example of what is called an arithmetico-geometric series (see Comment No. 6 of Chapter 5). The terms are products in which the first factors (viz., k) form an A.P. while the second factors (viz., 2^{n-k+1}) form a G.P. with common ratio $\frac{1}{2}$. Unlike the arithmetic or the geometric series, such series do not appear very frequently. So it is not worth to remember the formula for their sums. It is far better to derive it as and when needed, because, the trick involved in deriving it is the same as that in obtaining the sum of a geometric series, viz. to multiply each term by the common ratio.

In the present case, the common ratio is $\frac{1}{2}$. So, calling the sum on the L.H.S. of (1) as S, we get

$$\frac{1}{2}S = \sum_{k=1}^{n} k2^{n-k} \tag{2}$$

The trick now is to rewrite this with a change of index and subtract it from the original series. Let $r = k+1$. Then r varies from 2 to $n+1$ as k varies from 1 to n. Also, $k = r-1$. Hence (2) becomes

$$\frac{1}{2}S = \sum_{r=2}^{n+1} (r-1)2^{n-r+1} \tag{3}$$

We might now as well replace the dummy variable r by k again. Also there is no harm in adding an extra term corresponding to $r = 1$, since this additional term is 0 anyway. This gives,

$$\frac{1}{2}S = \sum_{k=1}^{n+1} (k-1)2^{n-k+1} \tag{4}$$

(After a little practice you can get (4) directly from (2), without introducing the intermediate variable r explicitly. You simply say, 'replacing k by $k+1$, we get (4) from (3)'.)

Recall that S is the sum $\sum\limits_{k=1}^{n} k2^{n-k+1}$. If we subtract (4) from this term by term, we get $\sum\limits_{k=1}^{n} 2^{n-k+1}$ and the last term in (4) (for which there is no matching term in S. Thus,

$$S - \frac{1}{2}S = \sum_{k=1}^{n} 2^{n-k+1} - n \tag{5}$$

Now we are on familiar grounds, because the series on the R.H.S. is a geometric series with the first term 2^n and common ratio $\frac{1}{2}$. So, using the well-known formula for geometric series, we get

$$\begin{aligned} \frac{1}{2}S &= 2^n \frac{(1/2)^n - 1}{(1/2) - 1} - n \\ &= 2(2^n - 1) - n \end{aligned} \tag{6}$$

whence $S = 4(2^n - 1) - 2n$.

Equation (1) now reduces to

$$16(2^n - 1) - 8n = (n+1)(2^{n+1} - n - 2) \tag{7}$$

Because of the presence of both the exponentials and also the powers of n, an equation like this does not fall under any standard type. It has to be solved by *ad-hoc* methods. If we collect all the exponential terms on one side, we get

$$2^{n+1}(n + 1 - 8) = (n+1)(n+2) - 8n - 16 \tag{8}$$

i.e.

$$2^{n+1}(n - 7) = n^2 - 5n - 14 \tag{9}$$

There is no golden method to solve this. But since the R.H.S. factorises as $(n-7)(n+2)$ we get $n = 7$ as a solution. The only other possibility is that $2^{n+1} = n + 2$. By inspection, $n = 0$ is a solution. But we are interested only in solutions that are positive integers. For $n = 1$, 2^{n+1} already exceeds $n + 2$. Moreover, the exponential function 2^{n+1} grows far more rapidly than the function $n + 2$ which is a polynomial (and of a very low degree, viz., 1). So, 2^{n+1} cannot equal $n + 2$ for any positive integer n. (A rigorous proof that $2^{n+1} > n + 2$ for every positive integer n can be given by induction in a multiplicative form, see Comment No. 14 of Chapter 6. A much easier way is to expand $2^{n+1} = (1+1)^{n+1}$ using the binomial theorem. All the terms are positive and the first two terms, viz. 1 and $n + 1$ already add up to $n + 2$. Considering that the question has only two marks and that its focal point is the summation of series rather than inequalities, such subsidiary proofs can generally be skipped.)

Hence finally, n, the number of matches played, is 7.

The problem is a combination of two essentially independent parts, the first the sum of an arithmetico-geometric progression and second, solving an equation involving exponentials. The time allowed (in proportion to the marks) is 4 minutes which is a bit too short.

Problem 2: Find the range of values of t, $t \in [-\pi/2, \pi/2]$ for which the equation

$$2\sin t = \frac{5x^2 - 2x + 1}{3x^2 - 2x - 1} \qquad (1)$$

has a solution in real x.

Analysis and Solution: This problem, too, is a combination of two essentially unrelated problems. The R.H.S. of (1) is a rational function (i.e. a ratio of two polynomials) and the first task is to find its range, i.e. to identify the set, say R, of all real y for which the equation

$$\frac{5x^2 - 2x + 1}{3x^2 - 2x - 1} = y \qquad (2)$$

has a solution in real x. Having found this set R, we then have to see which members of it can be expressed as $2\sin t$ for $t \in [-\pi/2, \pi/2]$.

Rewriting (2) as

$$(5 - 3y)x^2 - (2 - 2y)x + 1 + y = 0 \qquad (3)$$

we see that it will have a real solution (in x) if and only if its discriminant is non-negative, i.e. if and only if

$$(1 - y)^2 \geq (5 - 3y)(1 + y) \qquad (4)$$

which simplifies to

$$y^2 - y - 1 \geq 0 \qquad (5)$$

As the L.H.S. is a quadratic with leading coefficient positive and with roots $\dfrac{1 \pm \sqrt{5}}{2}$, (5) will hold when y lies outside the interval $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$. Thus the set R, i.e. the range of the R.H.S. of (1) is the set

$$R = \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \bigcup \left[\frac{1+\sqrt{5}}{2}, \infty\right) \qquad (6)$$

This finishes the first part of the work. Now we have to identify those t's in $[-\pi/2, \pi/2]$ for which $\sin t \in R$, or equivalently,

$$\sin t \in \left(-\infty, \frac{1-\sqrt{5}}{4}\right] \bigcup \left[\frac{1+\sqrt{5}}{4}, \infty\right) \qquad (7)$$

Since the values of $\sin t$ can only lie between -1 and 1, (7) can be replaced by

$$\sin t \in \left[-1, -\frac{1-\sqrt{5}}{4}\right] \bigcup \left[\frac{1+\sqrt{5}}{4}, 1\right] \qquad (8)$$

As we are further given that t lies in $[-\pi/2, \pi/2]$, which is the range of the inverse sine function we get the answer as

$$t \in \left[-\frac{\pi}{2},\ \sin^{-1}\left(\frac{1-\sqrt{5}}{4}\right)\right] \bigcup \left[\sin^{-1}\left(\frac{1+\sqrt{5}}{4}\right), \frac{\pi}{2}\right] \tag{9}$$

as the answer to the problem. This can be simplified further if we recall that both the arcsines are certain multiples of 18°. Specifically,

$$\sin 54^\circ = \sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4} \tag{10}$$

$$\text{and} \quad \sin 18^\circ = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} \tag{11}$$

whence (9) can be rewritten as

$$t \in [-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2] \tag{12}$$

It may be noted, however, that the formulas (10) and (11) are not as standard or commonly used as those for the trigonometric functions of multiples of 30°. So, even if the answer is left as (9), it ought to qualify for full credit since the mathematical part has been done correctly. This opens a needless controversy because those who do know the formulas like (10) and (11) are not sure whether it is worth their time to simplify (9) to (12). This controversy could have been avoided by asking the candidates to show that the range of t consists of two mutually disjoint intervals, with one of them twice as long as the other. That would force everybody to convert (9) to (12) and then note that the length of the first interval is $2\pi/5$ while that of the second is $\pi/5$.

Problem 3: Three circles of radii 3, 4, 5 touch each other externally. The tangents at the point of contact meet at P. Find the distance of P from the points of contact.

Analysis and Solution: This problem is strikingly similar to the Main Problem in Chapter 11. Replace the radii $3, 4, 5$ by r_1, r_2, r_3 respectively. Then P is simply the incentre and the desired distance is simply the inradius, say r, of the triangle whose vertices lie at the centres of the three circles. The solution to that problem already gives a formula for r in terms of r_1, r_2 and r_3, viz.

$$r^2 = \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} \tag{1}$$

In the present problem, $r_1 = 3, r_2 = 4$ and $r_3 = 5$. Substituting these values the desired distance comes out to be $\sqrt{\frac{3\times4\times5}{3+4+5}} = \sqrt{5}$ units.

Problem 4: Find the equation of the plane containing the lines $2x - y + z = 0$, $3x + y + z = 5$ and at a distance $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$.

Analysis and Solution: This is a straightforward problem in solid coordinate geometry. The difficulty, if any, is due more to the fact that this topic is newly introduced into the JEE syllabus rather than to the problem itself. Call the given line as L. Then a point (x, y, z) lies on L if and only if its coordinates satisfy the following two equations:

$$\begin{aligned} 2x - y + z - 3 &= 0 & (1) \\ \text{and} \quad 3x + y + z - 5 &= 0 & (2) \end{aligned}$$

Note that we have recast the second equation slightly so as to make its R.H.S. vanish. The reason for this will be clear later. Each one of these two equations represents a plane and the line L is precisely the intersection of these two planes.

We are after a certain plane containing this line L. The general equation of a plane is of the form

$$ax + by + cz + d = 0 \qquad (3)$$

where a, b, c, d are some constants (and at least one of a, b, c is non-zero). One way to tackle the problem would be to start with this general equation and to determine the values of the constants a, b, c, d from the data. (Even though superficially there are four constants, what matters is only their relative proportions and so we need a system of three equations to determine them.)

One equation is provided by the condition that the point $(2, 1, -1)$ is at a distance $\frac{1}{\sqrt{6}}$ from the plane given by (3). The formula for this distance is the analogue of the formula for the distance of a point from a line in the xy-plane. So we get,

$$\frac{2a + b - c + d}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{1}{\sqrt{6}} \qquad (4)$$

We need two more equations. These will come from the fact that every point which satisfies both (1) and (2) must also satisfy (3). So, first we have to identify the common solutions of (1) and (2) in a certain form. One way to do so is to take the parametric equations of the line L. For this, we first identify any one point, say P on L by inspection. For example, by setting $y = 0$ in (1) and (2) we get $x = 2, z = -1$. So we may take P as $(2, 0, -1)$. Now let L' be the line through the origin parallel to L. Then the equations of L' are simply the homogeneous system corresponding to (1) and (2), i.e.

$$\begin{aligned} 2x - y + z &= 0 & (5) \\ \text{and} \quad 3x + y + z &= 0 & (6) \end{aligned}$$

Using Theorem 7 in Comment No. 17 of Chapter 3, the general solution of this system can be written as

$$x = -2t, \; y = t, \; z = 5t \qquad (7)$$

where t is a real parameter. These are the parametric equations of the line L'. To get those of the line L from this, we merely add the coordinates of the point P which we have already identified as $(2, 0, -1)$. Thus we get

$$x = 2 - 2t, \; y = t, \; z = 5t - 1 \tag{8}$$

as the parametric equations of the given line L.

Substituting (8) into (3) gives

$$(-2a + b + 5c)t + 2a - c + d = 0 \tag{9}$$

If this equation is to hold for all (real) values of t, then the coefficient of t as well as the constant term must vanish identically. Thus

$$-2a + b + 5c = 0 \tag{10}$$

$$\text{and} \quad 2a - c + d = 0 \tag{11}$$

We now solve these two simultaneously with (4). Putting $b = 2a - 5c$ and $d = c - 2a$ into (4) and squaring gives

$$6(2a - 5c)^2 = a^2 + (2a - 5c)^2 + c^2 \tag{12}$$

This is a homogeneous equation in a and c. As we are interested only in the relative proportions of the constants a, b, c, d we are free to set the value of one of them arbitrarily. Putting $c = 1$ and simplifying we get

$$19a^2 - 100a + 124 = 0 \tag{13}$$

which is a quadratic in a with roots $a = 2$ and $a = \frac{62}{19}$. We have already taken c as 1. From (10), we get $b = -1$ or $\frac{29}{19}$ depending upon whether $a = 2$ or $a = \frac{62}{19}$. Similarly, (11) gives the corresponding values of d as -3 and $-\frac{105}{19}$.

Thus, there are two possible planes satisfying the data and their equations are :

$$2x - y + z - 3 = 0 \tag{14}$$

$$\text{and} \quad 62x + 29y + 19z - 105 = 0 \tag{15}$$

In this solution, we started with the general equation of a plane and determined the three (superficially four) constants in it. In effect, we considered the totality of all planes in the 3-dimensional space which is a 3-parameter family in the terminology of Comment No. 11 of Chapter 9. We then determined the values of these parameters from the data (viz. that the plane contains the given line L and is at a given distance from a given point).

We now present a more elegant way of doing the problem. The totality of all planes passing through a given line is a 1-parameter family because a particular member of it is determined by only one condition (such as that it passes through a given point). So, if we can somehow represent the equation of a typical plane containing L using only one parameter, then the work will be simplified considerably because we need only one equation to determine the value of this single parameter.

In order to represent the equation of a typical plane passing through the line L we follow a method which is analogous to that followed in Comment No. 13 of Chapter 9 to represent a typical line (in the xy-plane) which passes through the point of intersection of two given lines. In our case, the line L is the intersection of the two planes given by (1) and (2). Let us rewrite these equations as $E_1 = 0$ and $E_2 = 0$ where E_1 and E_2 are the expressions on the left hand sides of (1) and (2) respectively. Then E_1 and E_2 are linear (i.e. first degree) expressions in the variables x, y, z. Therefore any linear combination of E_1 and E_2, say $\alpha E_1 + \beta E_2$ will also be a linear expression in x, y, z. So $\alpha E_1 + \beta E_2 = 0$ will represent the equation of some plane. Further, this plane will contain the line L, because any point which satisfies $E_1 = 0$ and $E_2 = 0$ will satisfy $\alpha E_1 + \beta E_2 = 0$ no matter what α and β are. Different choices of α and β will give rise to different planes passing through L. But again, this plane will depend only on the relative proportions of α and β. So we might as well take one of them, say β, arbitrarily as 1. (*Caveat*: In doing so, there is a little danger which will be pointed out later.)

Summing up, we take the equation of the desired plane in the form $\alpha E_1 + E_2 = 0$, which upon simplification becomes

$$(2\alpha + 3)x + (1 - \alpha)y + (\alpha + 1)z - 3\alpha - 5 = 0 \tag{16}$$

where α is a parameter whose value is to be determined. For this we need an equation in α. This equation is obtained by equating the distance of the point $(2, 1, -1)$ from this plane with $\frac{1}{\sqrt{6}}$. Thus,

$$\frac{4\alpha + 6 + 1 - \alpha - \alpha - 1 - 3\alpha - 5}{\sqrt{(2\alpha + 3)^2 + (1 - \alpha)^2 + (\alpha + 1)^2}} = \pm \frac{1}{\sqrt{6}} \tag{17}$$

which, upon simplification, becomes $6(1 - \alpha)^2 = (2\alpha + 3)^2 + (1 - \alpha)^2 + (\alpha + 1)^2$, i.e.

$$24\alpha + 5 = 0 \tag{18}$$

So, $\alpha = -\frac{5}{24}$. If we put this value into (16), we get $62x + 29y + 19z - 105 = 0$ as the equation of the desired plane. But surely, we expect that there will be two planes satisfying the data. This is evident geometrically because the family of all planes passing through the line L can be obtained by rotating any member of it around L as the axis. In this rotation there will be two positions where the plane will be at a given distance from a given point not on L.

So, what happened to the other plane? The answer lies in our taking $\beta = 1$ arbitrarily in the expression $\alpha E_1 + \beta E_2$. We justified this by saying that it is only the relative proportion of α and β that matters. There is one exception to this. If $\beta = 0$, then the ratio of α to β is undefined. The plane $\alpha E_1 + \beta E_2 = 0$ is the same as the plane $E_1 = 0$ for every (non-zero) value of α. In other words, setting $\beta = 1$ and allowing α to vary will give us all planes in our 1-parameter family of planes, except the plane $E_1 = 0$, i.e. the plane given by (1). (By a similar reasoning, had we set $\alpha = 1$ arbitrarily, then various values of β would give all possible planes in the family with the exception of (2).) Normally, missing a single plane like this does not matter. But in the present problem, the missed plane is one of the planes we want, because it can be seen by a direct calculation that the distance of the point $(2, 1, -1)$ from the plane $2x - y + z - 3 = 0$ is indeed $\frac{1}{\sqrt{6}}$.

Hence, there are two planes which satisfy the data, viz. (14) and (15), which is the same answer as before. We leave it to the reader to do the problem by setting $\alpha = 1$ arbitrarily. In that case, the equation of a typical plane containing the line L will be of the form

$$(2 + 3\beta)x + (\beta - 1)y + (1 + \beta)z - 3 - 5\beta = 0 \tag{19}$$

and, instead of (18), we shall get

$$6(\beta - 1)^2 = 11\beta^2 + 12\beta + 6 \tag{20}$$

which is indeed a quadratic in β. Its two roots will give us the two planes. In other words the difficulty which we encountered earlier by taking $\beta = 1$ and letting α vary does not arise in this approach. It is obviously impossible to say that one approach is better than the other because that will depend on a particular problem. The thing to keep in mind is that one should not forget to consider the exceptional cases, for sometimes the answer, or at least a part of the answer, may lie in these exceptional cases.

If we do not want to run the risk of missing a part of the answer, we must not give any particular values to either α or β. Instead, let us keep them as they are. Then the equation of a plane containing the line L will be of the form

$$(2\alpha + 3\beta)x + (\beta - \alpha)y + (\alpha + \beta)z - 3\alpha - 5\beta = 0 \tag{21}$$

instead of either (16) or (19). The condition that the point $(2, 1, -1)$ is at a distance $\frac{1}{\sqrt{6}}$ from the plane will now give

$$6(\beta - \alpha)^2 = (2\alpha + 3\beta)^2 + (\beta - \alpha)^2 + (\alpha + \beta)^2 \tag{22}$$

which, upon, simplification, becomes

$$5\beta^2 + 24\alpha\beta = 0 \tag{23}$$

from which we get two possibilities : (i) $\beta = 0$ and (ii) $24\alpha + 5\beta = 0$. (i) leads to the plane we had missed earlier. (ii) leads to (15) as expected.

In our first solution, too, we obtained (13) from (12) by taking $c = 1$ arbitrarily. This did not create any problems because (13) is a quadratic in a. But trouble would have arisen if the coefficient of z would have vanished in the equation of one of the two planes. In that case, instead of (13) we would have gotten a linear equation in a and that ought to have alerted us to check the possibility $c = 0$ separately. Here too, if we want to play it safe, we do not give any particular values to either a or c in (12). Instead, we simplify it as it is and get

$$19a^2 - 100ac + 124c^2 = 0 \tag{24}$$

instead of (12). This can be factorised as $(a - 2c)(19a - 62c) = 0$ and leads to two possibilities : (i) $a = 2c$ and (ii) $a = \frac{62}{19}c$. Using (10) and (11), we express b and d in terms of c. Then we would get the same equations as (14) and (15) respectively, except that every term will be multiplied by c. The work involved is not substantially different because if we divide (24) by c^2, we get a quadratic in $\frac{a}{c}$, which is the same as the quadratic (13).

The problem is a good application of the method of a 1-parameter family of planes. It also illustrates the care that is necessary when we reduce the number of ostensible variables by setting one of them arbitrarily equal to 1.

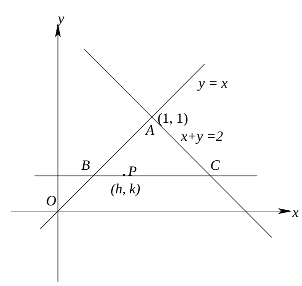
Problem 5: A line passes through $P(h, k)$ and is parallel to the x-axis. The area enclosed by the line and the lines $y = x$, $x + y = 2$ is $4h^2$. Find the locus of P.

Analysis and Solution: Normally, in a locus problem, we are *not* given the coordinates of the moving point. We usually take them as (h, k). We then translate the conditions on the moving point to get an equation in h and k, often with the help of one or more auxiliary variables (which we have to eliminate finally). In the present problem, we are given the condition satisfied by the moving point P directly in terms of the coordinates of P. This may confuse those students who are unduly linked to a particular format. But, in reality, it simplifies the problem, because some of the work which the candidate would have to do is already done by the paper-setter. In the conventional form, instead of specifying the area in terms of h, it could have, for example, been given as the area of a square centred at the origin and having the point P as a vertex. But that would have only complicated the problem.

Now, coming to the solution, let B and C be respectively, the points of intersection of the line $y = k$ with the lines $y = x$ and $x + y = 2$. Then clearly $B = (k, k)$ and $C = (2 - k, k)$. Let the lines $y = x$ and $x + y = 2$ meet at A. Then $A = (1, 1)$. The area of the triangle may be found in various ways. The simplest is to note that the length of the base BC is $|2 - k - k|$, i.e. $2|1 - k|$. The altitude is simply $|1 - k|$. Hence the condition on P gives

$$(k - 1)^2 = 4h^2 \tag{1}$$

So the locus of P is $4x^2 = (y - 1)^2$. This is a pair of straight lines with equations $y - 1 = \pm 2x$. (The problem is too simple for the Main Paper.)



Problem 6: If $f(x)$ satisfies $|f(x_1) - f(x_2)| \leq |x_1 - x_2|^2$ for all $x_1, x_2 \in I\!R$, find the equation of tangent to the curve $y = f(x)$ at the point $(1, 2)$.

Analysis and Solution: The hypothesis about the function f is identical to that in Exercise (16.29). So, from that exercise we get that f is identically constant and since the point $(1, 2)$ lies on the graph of the function this constant is 2. Therefore the equation of the tangent at $(1, 2)$ is simply $y = 2$.

Comment: Some students reported that in the statement of the problem, strict inequality was given to hold, i.e. it was given that the function f satisfies $|f(x_1) - f(x_2)| < |x_1 - x_2|^2$ for all $x_1, x_2 \in I\!R$. If so, this is a lapse on the part of the paper-setters, because the possibility $x_1 = x_2$ is not excluded and when this holds the condition reduces to $0 < 0$ which no function can satisfy! So the question becomes vacuous. In the present case this lapse is not very serious because in the solution, we need the given inequality only for $x_1 \neq x_2$. Still, a lapse like this can be confusing to a student, because in some problems (e.g. Problem No. 16 below) we are given some relationship to hold for all possible values of two variables and we get some useful information by specialising to the case where these two variables take equal values. Once again, the only practical advice is to make a record of the lapse and then ignore it.

Problem 7: Evaluate $\displaystyle\int_0^\pi e^{|\cos x|}\left\{3\cos\left(\frac{\cos x}{2}\right) + 2\sin\left(\frac{\cos x}{2}\right)\right\}\sin x\, dx$.

Analysis and Solution: Call the given integral as I, then

$$I = I_1 + I_2 \tag{1}$$

where

$$I_1 = \int_0^\pi e^{|\cos x|}\left\{3\cos\left(\frac{\cos x}{2}\right)\right\}\sin x\, dx \tag{2}$$

$$\text{and} \quad I_2 = \int_0^\pi e^{|\cos x|}\left\{2\sin\left(\frac{\cos x}{2}\right)\right\}\sin x\, dx \tag{3}$$

The substitution $\cos x = t$ changes these integrals to

$$I_1 = \int_{-1}^{1} e^{|t|} 3\cos(t/2)dt \tag{4}$$

$$\text{and} \quad I_2 = \int_{-1}^{1} e^{|t|} 2\sin(t/2)dt \tag{5}$$

The integrand in (4) is an even function of t while that in (5) is an odd function of t. As the interval of integration is symmetric about 0 in both the cases, we get

$$I_1 = 6\int_0^1 e^t \cos(t/2)dx \tag{6}$$

$$\text{and} \quad I_2 = 0 \tag{7}$$

To evaluate I_1 we use the formula for the antiderivative of a function of the form $e^{at}\cos bt$ given in Comment No. 24 of Chapter 17, viz. $\dfrac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$. We apply it with $a = 1$ and $b = 1/2$. So,

$$\begin{aligned} I_1 &= \frac{6}{1+1/4}\left[e^t(\cos(t/2)+\frac{1}{2}\sin(t/2))\right]_0^1 \\ &= \frac{24}{5}\left[e\cos(1/2)+\frac{e}{2}\sin(1/2)-1\right] \end{aligned} \tag{8}$$

which is also the value of I since $I_2 = 0$.

The problem is a straightforward combination of the method of substitution, a standard antiderivative and elementary properties of the definite integrals of even and odd functions over intervals that are symmetric w.r.t. 0. The purpose of the coefficients 3 and 2 in the integrand is not clear.

Problem 8: A person goes to office by a car, a scooter, a bus or a train, with probabilities $1/7, 3/7, 2/7$ and $1/7$ respectively. The probability of his reaching late if he takes a car, a scooter, a bus or a train are, respectively, $2/9, 1/9, 4/9$ and $1/9$. Given that he reaches the office in time, find the probability that he travelled by a car.

Analysis and Solution: This is a straightforward problem on conditional probability. Let E_1, E_2, E_3, E_4 be the events corresponding to taking a car, a scooter, a bus or a train respectively. These events are mutually exclusive and exhaustive and so their respective probabilities, viz., $1/7, 3/7, 2/7$ and $1/7$ do add up to 1 as they ought to. (In fact, one of these probabilities is redundant as it can be obtained knowing the remaining three.) Let L be the event that the person reaches the office late. Then L is the disjunction of the four mutually exclusive events $L \cap E_i$, $i = 1, 2, 3, 4$. So we have

$$P(L) = P(L\cap E_1) + P(L\cap E_2) + P(L\cap E_3) + P(L\cap E_4) \tag{1}$$

It is tempting to think that the given figures $2/9, 1/9, 4/9$ and $1/9$ are the respective probabilities of the events $P(L \cap E_i)$ for $i = 1, 2, 3, 4$, i.e. the terms on the R.H.S. of (1). But this is not so. The figure $2/9$ is the probability of reaching late *assuming that the person has taken a car.* In other words, $2/9$ is not the probability of the event $L \cap E_1$, but rather the conditional probability $P(L|E_1)$ which also equals $\frac{P(L\cap E_1)}{P(E_1)}$. In other words, we are given that

$$P(L\cap E_1) = P(E_1)\times\frac{2}{9} = \frac{1}{7}\times\frac{2}{9} = \frac{2}{63} \tag{2}$$

We can similarly calculate $P(L \cap E_i)$ for $i = 2, 3, 4$.

$$P(L\cap E_2) \quad = \quad P(E_2)\times\frac{1}{9} = \frac{3}{7}\times\frac{1}{9} = \frac{3}{63} \tag{3}$$

$$P(L \cap E_3) \quad = \quad P(E_3) \times \frac{4}{9} = \frac{2}{7} \times \frac{4}{9} = \frac{8}{63} \tag{4}$$

$$P(L \cap E_4) \quad = \quad P(E_4) \times \frac{1}{9} = \frac{1}{7} \times \frac{1}{9} = \frac{1}{63} \tag{5}$$

Putting these values in (1), we get

$$P(L) = \frac{2}{63} + \frac{3}{63} + \frac{8}{63} + \frac{1}{63} = \frac{14}{63} = \frac{2}{9} \tag{6}$$

Our interest is more in the complementary event L', i.e. reaching the office in time. From (6), we have

$$P(L') = 1 - \frac{2}{9} = \frac{7}{9} \tag{7}$$

We have to find the probability $P(E_1|L')$, i.e. the conditional probability of E_1, given L'. Again, this equals $\frac{P(L' \cap E_1)}{P(L')}$. We have already calculated $P(L')$. To find $P(L' \cap E_1)$, note that $L' \cap E_1$ and $L \cap E_1$ are two mutually exclusive events whose disjunction is E_1. So from (2),

$$P(L' \cap E_1) = P(E_1) - P(L \cap E_1) = \frac{1}{7} - \frac{2}{63} = \frac{9-2}{63} = \frac{1}{9} \tag{8}$$

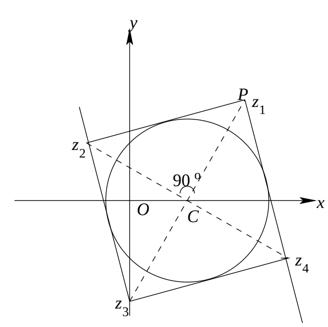
Hence, the desired probability is given by

$$P(E_1|L') = \frac{P(L' \cap E_1)}{P(L')} = \frac{1/9}{7/9} = \frac{1}{7} \tag{9}$$

More challenging problems on conditional probability have been asked in the past (see e.g. Exercise (22.19)). Coneptually, the present problem is too simple. But the calculations involved are repetitious and hence more prone to numerical errors if to be done in four minutes. It is difficult to see what is gained by introducing four vehicles. The conceptual part would have remained intact and the calculations much more reasonable, had there been only two vehicles and the problem would have been suitable for the screening paper.

Problem 9: If one of the vertices of a square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$, find the other vertices of the square.

Analysis and Solution: Let the centre of the circle be C. Then C is 1. It is clear that C is also the centre of the circumscribing square. Therefore, if one vertex, say P, is at z_1, then the remaining vertices are obtained simply by a counterclockwise rotation of the segment CP around C through multiples of 90 degrees. Counterclockwise rotation through 90 degrees corresponds to multiplication by the complex number i.



Hence,

$$\begin{aligned} z_2 - 1 &= i(z_1 - 1) & (1)\\ z_3 - 1 &= i^2(z_1 - 1) = -(z_1 - 1) & (2)\\ \text{and} \quad z_3 - 1 &= i^3(z_1 - 1) = -i(z_1 - 1) & (3) \end{aligned}$$

As $z_1 = 2 + \sqrt{3}i$, then $z_1 - 1 = 1 + \sqrt{3}i$. So, from (1), (2) and (3), the remaining vertices come out as $z_2 = 1 + i(1+\sqrt{3}i) = (1-\sqrt{3}) + i$, $z_3 = 1 - (1+\sqrt{3}i) = -\sqrt{3}i$ and $z_4 = 1 - i(1 + \sqrt{3}i) = (1 + \sqrt{3}) - i$.

The problem is simple conceptually and the calculations are short. It is hardly worth 4 points, since it involves less work than some of the two point questions above, e.g. Problem 2. It would have been better to interchange the credits of these two problems, possibly after making Problem 2 unambiguous in the manner indicated there.

Problem 10: Tangents are drawn from a point on the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of the mid-point of the chord of contact.

Analysis and Solution: Another routine problem on finding a locus, which can be done quite easily, almost mechanically, if you know the right formulas. As usual, let $P = (h, k)$ be the midpoint of the chord of contact. Then from the formula for the equation of a chord to a conic in terms of its mid-point, we see that its equation is

$$hx + ky = h^2 + k^2 \tag{1}$$

Now, we are also given that this line is the chord of contact (w.r.t. the circle) of some point on the given hyperbola. A typical point on the hyperbola is of the form $(3\sec\theta, 2\tan\theta)$. The equation of its chord of contact w.r.t. the circle $x^2 + y^2 = 9$ is

$$(3\sec\theta)x + (2\tan\theta)y = 9 \tag{2}$$

Since (1) and (2) represent the same line, comparing the coefficients, we get

$$\frac{h}{3\sec\theta} = \frac{k}{2\tan\theta} = \frac{h^2 + k^2}{9} \tag{3}$$

The required locus is obtained by eliminating θ from (3). This is easily done by observing that

$$\begin{aligned} \sec\theta &= \frac{3h}{h^2 + k^2} & (4)\\ \text{and} \quad \tan\theta &= \frac{9k}{2(h^2 + k^2)} & (5) \end{aligned}$$

and using the identity $\sec^2\theta - \tan^2\theta = 1$ to get

$$\frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1 \tag{6}$$

Simplifying and replacing h by x and k by y, we get the desired locus as

$$9x^2 - \frac{81}{4}y^2 = (x^2 + y^2)^2 \tag{7}$$

It may be noted that the use of the parametric equations of a hyperbola did not simplify the work substantially. Had we taken the point on the hyperbola as (x_1, y_1), then the equation of the chord of contact of the circle $x^2 + y^2 = 9$ would have been

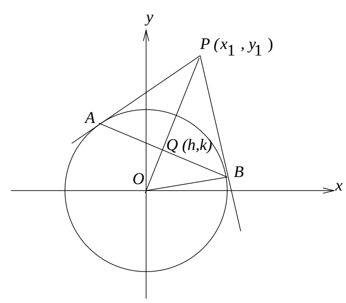
$$x_1 x + y_1 y = 9 \tag{8}$$

and comparing this with (1), we would have got

$$\frac{h}{x_1} = \frac{k}{y_1} = \frac{h^2 + k^2}{9} \tag{9}$$

instead of (3). The required locus can then be obtained by eliminating x_1 and y_1 from these two equations and the equation of the hyperbola, satisfied by (x_1, y_1), viz. $\frac{x_1^2}{9} - \frac{y_1^2}{4} = 1$. This demands the same work.

But there is no easy alternative to the use of formulas for the chord of contact of a point w.r.t. a conic and for the equation of a chord of a conic in terms of its mid-point, if the problem is to be done by coordinate geometry. Fortunately, the fact that the curve $x^2 + y^2 = 9$ is a circle makes it possible to give another solution which is based more on pure geometry than coordinates. Let $P = (x_1, y_1)$ be a point outside a circle of radius r centred at O. Let AB be the chord of contact of P w.r.t. this circle and Q be the midpoint of AB. Then by symmetry, O, Q, P are collinear. Also the similarity of the right-angled triangles OQB and OBP implies $OP.OQ = OB^2 = r^2$. (See Exercise (8.21) for more on the relationship between P and Q.)



Since $r = 3$ in the present problem, this gives us an alternate derivation of (9). This problem is a good test of a candidate's ability to pick up just the right formulas from the huge collection of formulas at his disposal. But conceptually, it is a mediocre problem and hardly deserves a place in the Main Paper, especially in presence of another locus problem.

Problem 11: Find the equation of the common tangent in the first quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the coordinate axes.

Analysis and Solution: The wording of the question is a little misleading. The phrase 'tangent in the first quadrant' does not mean that the entire tangent lies in the first quadrant. In fact, no straight line can lie completely in any single quadrant. The correct meaning of the phrase is that the points of contact of the tangent with the given curves lie in the first quadrant. With this understanding, a tangent in the first quadrant will necessarily have a negative slope.

Let L be the common tangent to the given two curves and let its points of contact with the circle and the ellipse be $(4\cos\alpha, 4\sin\alpha)$ and $(3\cos\beta, 2\sin\beta)$, where $\alpha, \beta \in [0, \frac{\pi}{2}]$ as the points are to lie in the first quadrant.

Then the equation of L is

$$4\cos\alpha\, x + 4\sin\alpha\, y = 16 \tag{1}$$

on one hand, and

$$\frac{5\cos\beta}{25}\, x + \frac{2\sin\beta}{4}\, y = 1 \tag{2}$$

on the other. As these two equations represent the same line, a comparison of coefficients gives

$$\frac{5\cos\alpha}{\cos\beta} = \frac{2\sin\alpha}{\sin\beta} = 4 \tag{3}$$

from which, $\cos\alpha = \frac{4}{5}\cos\beta$ and $\sin\alpha = 2\sin\beta$. Eliminating α, we get $\frac{16}{25}\cos^2\beta + 4\sin^2\beta = 1$ which simplifies to $\frac{84}{25}\sin^2\beta = \frac{9}{25}$, giving $\sin\beta = \frac{\sqrt{3}}{2\sqrt{7}}$ and $\cos\beta = \frac{5}{2\sqrt{7}}$ since β lies in $[0, \frac{\pi}{2}]$.

Putting these values in (2), we get the equation of the common tangent as

$$\frac{1}{2\sqrt{7}}x + \frac{\sqrt{3}}{4\sqrt{7}}y = 1 \tag{4}$$

from which it is clear that the intercepts along the x- and the y-axis are $2\sqrt{7}$ and $\frac{4\sqrt{7}}{\sqrt{3}}$ respectively. Hence the length of the intercept between the two axes is $\sqrt{28 + \frac{112}{3}} = \sqrt{\frac{196}{3}} = \frac{14}{\sqrt{3}}$ units.

An alternate way to do the problem is to start with the equation of a line in the form $y = mx + c$, where m is the slope and apply the condition for tangency to each of the two given curves. In the case of the circle $x^2 + y^2 = 16$ this gives

$$c = \pm 4\sqrt{1 = m^2} \tag{5}$$

while tangency to the ellipse gives

$$c = \pm\sqrt{25m^2 + 4} \tag{6}$$

Hence $16(1 + m^2) = 25m^2 + 4$, which is a quadratic in m with roots $\pm\frac{2}{\sqrt{3}}$. As the tangent to a point in the first quadrant has negative slope, we take the negative sign. This gives the equation of the common tangent as $y = -\frac{2}{\sqrt{3}}x + 4\frac{\sqrt{7}}{\sqrt{3}}$. The rest of the work is the same.

This is yet another mediocre problem which deserves no place in the Main Paper. It is difficult to guess the motive behind asking the last part of the question, viz. finding the length of the intercept between the axes. This is sheer computation and a student who cannot do it does not deserve to clear the screening paper. No purpose is served by asking it in the Main Paper.

Problem 12: Let $p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$ and $p(1) = -6$. Suppose, further, that $p(x)$ has a maximum at $x = -1$ and $p'(x)$ has a minimum at $x = 1$. Find the distance between the local maximum and the local minimum of the curve $y = p(x)$.

Analysis and Solution: This problem is very analogous conceptually to the Main Problem in Chapter 15 and also to Exercise (17.23). In all three problems, we have to determine a function from some data regarding its values and the values of its derivative. (Intertestingly, in all these three problems, the function involved is a polynomial of degree 3. In the present problem, this is given explicitly. In Exercise (17.23), this can be inferred from the fact that its derivative is given to be a polynomial of degree 2.)

The general form of a cubic polynomial $p(x)$ is

$$p(x) = ax^3 + bx^2 + cx + d \tag{1}$$

where a, b, c, d are some constants. To determine these, we need four equations in a, b, c, d. The conditions $p(-1) = 10$ and $p(1) = -6$, imply, respectively, that

$$\begin{aligned} -a + b - c + d &= 10 && (2) \\ \text{and} \quad a + b + c + d &= -6 && (3) \end{aligned}$$

The fact that $p(x)$ has a local maximum at $x = -1$ implies that its derivative $p'(x)$ vanishes at $x = -1$. Since $p'(x) = 3ax^2 + 2bx + c$, we get

$$3a - 2b + c = 0 \tag{4}$$

Finally, the fact that $p'(x)$ has a minimum at $x = 1$ gives that $p''(1) = 0$. Since $p''(x) = 6ax + 2b$, we get

$$6a + 2b = 0 \tag{5}$$

We now solve (2) - (5) simultaneously. Although the system is linear, it instead of applying the general standard methods (e.g. the one based on Cramer's rule) for solutions of such systems, it is usually better to apply *ad-hoc* methods, taking advantage of the particular simplifying features. For example, (2) and (3) can be replaced by the easier ones

$$\begin{aligned} b + d &= 2 && (6) \\ \text{and} \quad a + c &= -8 && (7) \end{aligned}$$

which are obtained by adding and subtracting (2) and (3). (This simplification is possible because the points at which the values of $p(x)$ are given happened to be negatives of each other.)

Similarly adding (5) and (6), we eliminate b and get $9a + c = 0$. Coupled with (7), this gives $8a = 8$, i.e. $a = 1$. So $c = -9$ and $b = -3$. From (7), d comes out as 5. Thus, the polynomial $p(x)$ is $x^3 - 3x^2 - 9x + 5 = 0$.

Instead of starting with the general form of a cubic polynomial, viz., (1), which involves as many as four variables, we could have made use of some of the data to reduce the number of variables. For example, since $p(x)$ is of degree 3, we already know that $p''(x)$ is of degree 1 and the knowledge that $p''(1) = 0$ allows us to take $p''(x)$ as $\lambda(x-1)$. Integrating, $p'(x) = \frac{\lambda}{2}(x-1)^2 + \mu$ for some constant μ and the knowledge that $p'(-1) = 0$ gives us $2\lambda + \mu = 0$ and hence $p'(x) = \frac{\lambda}{2}(x-1)^2 - 2\lambda$. Another integration, coupled with $p(-1) = 10$ and $p(1) = 6$ will determine $p(x)$. Although such short cuts are normally desirable, in the present problem the straightforward determination of $p(x)$ starting from (1) was equally easy.

To complete the solution, we need to locate the points of maximum and the minimum of $p(x)$. We have $p'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$ which shows that $x = -1$ and $x = 3$ are the critical points of p. Further, $p'(x) > 0$ for $x < -1$ and for $x > 3$ while $p'(x) < 0$ for $-1 < x < 3$. This shows that $p(x)$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. (This can also be seen by calculating $p''(x) = 6x - 6$ and applying the second derivative test.)

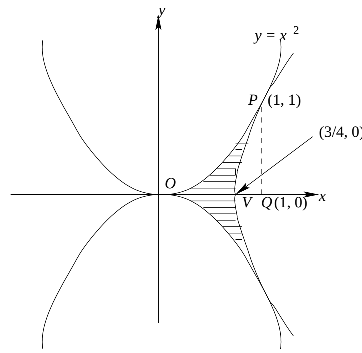
We are already given that $p(-1) = 10$. By a direct computation, $p(3) == 27 - 27 - 27 + 5 = -22$. Hence on the graph of $y = p(x)$, the points $(-1, 10)$ and $(3, -22)$ are the points of local maximum and local minimum respectively. The distance between them is $\sqrt{(3+1)^2 + (-22-10)^2} = \sqrt{1040} = 4\sqrt{65}$ units.

As in the last problem, the second part of the problem is pure drudgery. Even the first part really does not deserve to be asked in the Main Problem. (It may be noted that the analogous problem in Exercise (17.23) was asked in 1998 JEE. But at that time there was no screening paper. Moreover, that problem also required some simplification of determinants.)

Problem 13: Find the area bounded by the curves $y = x^2, y = -x^2$ and $y^2 = 4x - 3$.

Analysis and Solution: All the three curves are parabolas. The first two are reflections of each other in the x-axis and touch each other at their common vertex, the origin. The third parabola has a horizontal axis and vertex at $(3/4, 0)$ as we see easily by rewriting its equation in the form $y^2 = 4(x - \frac{3}{4})$. Call the vertex as V.

To identify the the region bounded by these three parabola, we need to determine the points of intersection of the third parabola with the first two. If we solve $y = x^2$ and $y^2 = 4x - 3$ simultaneously, we get $x^4 - 4x + 3 = 0$. Being a fourth degree equation, we have to solve it by inspection. Clearly, $x = 1$ is a solution and hence $x-1$ is a factor. Factoring, $(x-1)(x^3+x^2+x-3)=0$. The cubic factor has $x=1$ as a root and hence $x-1$ is again a factor. So, $(x-1)^2(x^2+2x+3)=0$. The quadratic factor x^2+2x+3 equals $(x+1)^2+1$ and hence never vanishes. It follows that $x=1$ is a double root and that there are no other roots. Geometrically, this means that the parabolas $y=x^2$ and $y^2=4x-3$ touch each other when $x=1$, i.e. at the point $(1,1)$. By a similar reasoning (or by symmetry), the parabolas $y=-x^2$ and $y^2=4x-3$ touch each other at the point $(-1,1)$. Call this point P. Therefore the region, say R, bounded by these three parabolas is as shown by shading in the figure. Clearly, it is symmetric about the x-axis.



To find the area of R we double the area of its upper half. The latter can be obtained by finding the area OPQ (where Q is the projection of P onto the x-axis) and subtracting the area VPQ. Both these areas can be obtained by recognising them as the areas bounded by the curves $y=x^2$ and $y=\sqrt{4x-3}$ on the top respectively and by the x-axis below. Therefore,

$$\begin{aligned}
\text{area of } R &= 2\times(\text{area } OPQ) - \text{area } VPQ) \\
&= 2\times\left(\int_0^1 x^2dx - \int_{3/4}^1 \sqrt{4x-3}dx\right) \\
&= 2\times\left(\frac{x^3}{3}\Big|_0^1 - \frac{1}{6}(4x-3)^{3/2}\Big|_{3/4}^1\right) \\
&= 2\times(\frac{1}{3}-\frac{1}{6}) \\
&= \frac{1}{3}\text{ sq. units} \qquad (1)
\end{aligned}$$

The problem is a good combination of various abilities such as curve sketching, solving a system of simultaneous equations and evaluating integrals, the calculations needed in each step are reasonable. It is just that since problems of this type are asked every year, there is usually a set procedure for solving them and therefore such problems do not test any originality of thinking. It is especially difficult to see what purpose it serves in the Main paper when the screening paper already has a similar problem (Q. 21 above). It is true that the screening paper and the main paper are set by different teams, working independently of each other. Still, a duplication like this can be avoided by restricting the problems in the Main Paper only to those which do not form a set pattern.

Problem 14: Find the family of curves with the property that the intercept of the tangent to it at any point between the point of contact and the x-axis is of unit length.

Analysis and Solution: Let $y = f(x)$ be the equation of a typical member of this family and $P = (x_0, y_0)$ be a typical point on it. The equation of the tangent at P then is

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{1}$$

This cuts the x-axis at the point $(x_0 - \frac{f(x_0)}{f'(x_0)}, 0)$. The distance between this point and the point $(x_0, f(x_0))$ is given to be 1 unit. This translates as

$$(\frac{f(x_0)}{f'(x_0)})^2 + (f(x_0))^2 = 1 \tag{2}$$

i.e. as

$$(f'(x_0))^2 = \frac{(f(x_0))^2}{1 - (f(x_0))^2} \tag{3}$$

This means that the curve satisfies the differential equation

$$\frac{dy}{dx} = \pm \frac{y}{\sqrt{1 - y^2}} \tag{4}$$

which can be recast in the separate variables form as

$$\frac{\sqrt{1 - y^2}}{y} dy = \pm dx \tag{5}$$

To solve this, we need to find the indefinite integral $\int \frac{\sqrt{1-y^2}}{y} dy$. The substitution $y = \sin\theta$ suggests itself because it will get rid of the radical. With this substitution we get

$$\begin{aligned}
\int \frac{\sqrt{1 - y^2}}{y} dy &= \int \frac{\cos\theta}{\sin\theta} \cos\theta d\theta \\
&= \int \frac{\cos^2\theta}{\sin\theta} d\theta \\
&= \int \frac{1 - \sin^2\theta}{\sin\theta} d\theta \\
&= \int (\text{cosec}\ \theta - \sin\theta)\ d\theta \\
&= -\ln(\text{cosec}\ \theta + \cot\theta) + \cos\theta \\
&= -\ln(\frac{1 + \sqrt{1 - y^2}}{y}) + \sqrt{1 - y^2} \qquad (6)
\end{aligned}$$

Hence, the general solution of the differential equation (4) is

$$\sqrt{1 - y^2} - \ln(\frac{1 + \sqrt{1 - y^2}}{y}) = \pm x + c \tag{7}$$

where c is an arbitrary constant. Therefore this is the equation of a typical member of the given family of curves. Note that this is of the form $x = g(y)$ and cannot be written explicitly in the form $y = f(x)$. If one wants, the term $-\ln(\frac{1+\sqrt{1-y^2}}{y})$ can be written as $\ln(\frac{y}{1+\sqrt{1-y^2}})$ and further, after rationalisation, as $\ln(\frac{1-\sqrt{1-y^2}}{y})$. But this serves only a cosmetic purpose and is entirely optional. (Some people feel a little more comfortable when a minus sign is replaced by a plus sign and a fraction is rewritten so that the denominator looks simple and succinct!)

There are two parts to this problem : (i) construction of a differential equation from some geometric data and (ii) solving this differential equation. In the second part, the equation itself was the easiest to solve, being of the separate variable type as in (5). But in integrating one of the two sides, the antiderivative took a little work. As for the first part, analogous problems have been asked many times in the JEE (e.g. in 1994, 1995, 1998, 1999). This again makes the problem unsuitable to test the originality of thinking on the part of a candidate. What it really tests is a drill. To some extent this is inevitable because although the differential equations have a rich variety of applications, those that can be asked at the JEE level are severely limited, usually falling into one of the two types, the geometric ones such as the present problem or those involving some physical process (e.g. the draining of a liquid from a reservoir).

Problem 15: If

$$\begin{aligned} f(x-y) &= f(x)g(y) - g(x)f(y) \qquad (1) \\ \text{and} \quad g(x-y) &= g(x)g(y) + f(x)f(y) \qquad (2) \end{aligned}$$

for all $x, y \in R$, and the right handed derivative of $f(x)$ at 0 exists, find $g'(x)$ at $x = 0$.

Analysis and Solution: Here we are given a system of two functional equations involving two functions f and g. Therefore the problem is of the same spirit as Exercise (20.14). But no initial conditions are given. Also the problem does not ask us to solve the functional equations, i.e. to determine the functions f and g. Instead, we merely have to find $g'(0)$, it being given that $f'(0^+)$ exists.

Although the problem does not ask us to solve the system, it is easy to see by inspection that $f(x) = \sin x$ and $g(x) = \cos x$ is a solution. Moreover, these functions are differentiable everywhere. So certainly, the condition about the existence of the right handed derivative of f at 0 is satisfied. In this particular case, $g'(x) = -\sin x$ and so $g'(0) = 0$. No other hypothesis is given in the problem. So, if at all the data of the problem determines $g'(0)$ uniquely, then this unique value must be 0.

This is, of course, not a valid solution. But it gives us some insights into a possible line of attack. For example, in the special case we considered, the functions $f(x)$ and $g(x)$ are odd and even respectively. Further $[f(x)]^2 + [g(x)]^2 = 1$ for all $x \in I\!R$. So, let us begin by checking which of these properties hold for the given functions $f(x)$ and $g(x)$, i.e. which of these properties can be derived from (1) and (2) (because that is all that is given to us about the functions f and g).

Putting $x = 0 = y$ in (1) gives $f(0) = 0$. If we do the same to (2), we get

$$g(0) = [g(0)]^2 + [f(0)]^2 = [g(0)]^2 \tag{3}$$

(since we already know $f(0) = 0$). From this we get that $g(0)$ equals either 0 or 1. We have to consider these cases separately.

If $g(0) = 0$, then putting $y = 0$ in (2) we get $g(x) = g(x - 0) = g(x)g(0) + f(x)f(0) = 0$ for all x. This makes g an identically constant function. So, in this case, $g'(x) \equiv 0$, and in particular, $g'(0) = 0$.

Now suppose $g(0) = 1$ (which would be the case if $g(x)$ were equal to $\cos x$). Let us put $x = 0$ in (1). Then we get

$$\begin{aligned} f(-y) = f(0 - y) &= f(0)g(y) - g(0)f(y) \\ &= 0 - f(y) \\ &= -f(y) \end{aligned} \tag{4}$$

for all $y \in I\!\!R$. This means that f is an odd function. Therefore, if $f'(0^+)$ exists then so does $f'(0^-)$ and the two are equal. This is essentially a consequence of Exercise (18.6)(d). But a direct proof can be given as

$$\begin{aligned} f'(0^-) &= \lim_{h \to 0^+} \frac{f(0 - h) - f(0)}{-h} \\ &= \lim_{h \to 0^+} \frac{f(-h)}{-h} \\ &= \lim_{h \to 0^+} \frac{-f(h)}{-h} \\ &= \lim_{h \to 0^+} \frac{f(h)}{h} \\ &= f'(0^+) \end{aligned} \tag{5}$$

Our goal is to get $g'(0)$. For this we need to consider ratios of the form $\dfrac{g(h) - g(0)}{h}$. Unfortunately, neither (1) nor (2) is of much direct use here, because they only relate the values of the functions f and g at some points with their values at some other points. They do not relate the values of these functions with the values of the arguments as such. But a little experimentation may help. Both (1) and (2) hold for arbitrary values of x and y. We are free to choose them anyway we like. Let us choose y to be x. Then (1) gives nothing new. But (2) combined with $g(0) = 1$ gives

$$1 = g(0) = g(x - x) = [g(x)]^2 + [f(x)]^2 \tag{6}$$

for all $x \in I\!\!R$. It is tempting to differentiate both the sides of (6) at $x = 0$ and get

$$\begin{aligned} 0 &= 2g(0)g'(0) + 2f(0)f'(0) \\ &= 2g'(0) + 0 \end{aligned} \tag{7}$$

from which we get that $g'(0) = 0$. Note that the actual value of $f'(0)$ cannot be found from the data. Nor is it needed because it is getting multiplied by $f(0)$ which is 0 anyway. But it is vital to know that $f'(0)$ exists. Otherwise, we cannot differentiate the R.H.S. of (6).

Strictly speaking, in the argument above, we also need to know beforehand that $g'(0)$ exists. The argument above does *not* show that g is differentiable at 0. It only *finds* $g'(0)$ in case it exists. It is tempting to overcome this difficulty by rewriting (6) as

$$g(x) = \pm\sqrt{1 - [f(x)]^2} \tag{8}$$

and then showing that regardless of the choice of the sign, we have

$$g'(0) = \pm\frac{-f(0)f'(0)}{\sqrt{1 - [f(0)]^2}} = 0 \tag{9}$$

But even this argument is faulty because it presupposes that the *same* sign (i.e. either $+$ or $-$) holds in (8) *for all* $x \in I\!R$ (or, at least, for all x in some neighborhood of 0). This need not be so. All that we can conclude from (6) is that for every $x \in I\!R$, either $g(x) = \sqrt{1 - [f(x)]^2}$ or $g(x) = -\sqrt{1 - [f(x)]^2}$. To conclude from this that either the first possibility holds for all x or that the second possibility holds for all x is as absurd as saying that since every human being is either a man or a woman, therefore either every human being is a man or every human being is a woman!

Another possible attempt to overcome this difficulty is to rewrite (6) as $[g(x)]^2 - 1 = -[f(x)]^2$. Factorising, we get

$$\begin{aligned}\frac{g(x)-1}{x} &= -\frac{x}{g(x)+1}\left(\frac{f(x)}{x}\right)^2 \\ &= -\frac{x}{g(x)+1}\left(\frac{f(x)-f(0)}{x}\right)^2 \qquad (10)\end{aligned}$$

for all $x \neq 0$.

We now take the limits of both the sides as x tends to 0. The second factor on the R.H.S. tends to $(f'(0))^2$. The numerator of the first factor (viz., x) tends to 0. Unfortunately, there is not much we can say about the limit of the denominator. If g were given to be continuous at 0, then $g(x)+1 \to 2$ as $x \to 0$ and, in that case, the ratio $\frac{x}{g(x)+1}$ would indeed tend to 0 as $x \to 0$, which would imply that $g'(0)$ exists and equals 0. Actually, it would suffice if we know that $g(x) + 1$ stays away from 0 when x is sufficiently close to 0. (Alternately, one can argue that under this hypothesis, the positive sign must hold in the R.H.S. of (8) for all x in a neighbourhood of 0 and so the derivation based on (8) would be valid.)

The wording of the question is to *find* $g'(0)$. One possible interpretation is to *assume* that it exists and that the problem merely asks its value. Another interpretation is that the candidates are also expected to *prove* that $g'(0)$ exists.

An analytically minded student is likely to take the second interpretation. He will then spend a lot of time in searching for a valid proof, while a sloppy candidate who does not even perceive the need for it will go scotfree.

There is another reason reason why a perceptive and thoughtful student is more likely to take the second (i.e. the harder) interpretation. If the problem was merely to find out $g'(0)$, there is a much easier way. Just as we proved in (4) that the function f is odd, we can prove, using the given functional equation (2) alone, that g is an even function. Indeed, since the R.H.S. of (2) is unaffected if we interchange x and y, we immediately get that $g(x-y) = g(y-x)$ for all real numbers x and y. Since every real number z can be expressed as $x - y$ for some x and y (e. g., by taking $x = z$ and $y = 0$), it follows that $g(z) = g(-z)$ for all $z \in I\!R$. Once we know that g is an even function, it follows that if at all it is differentiable at 0, then $g'(0)$ must be 0. Again, this is essentially a consequence of Exercise (18.6)(d). But a direct proof can be given by taking steps similar to those in the derivation of (5) to show that $g'(0^-) = -g'(0^+)$.

Thus, if the purpose of the question was merely to find $g'(0)$ *assuming* its existence, then the existence of the right handed derivative of f at 0 is not at all needed. In a serious examination like the JEE, unless the very purpose of a question is to test a candidate's ability to weed out the irrelevant parts of the hypothesis, there is a general presumption that every bit of the hypothesis is needed in finding an honest solution. So when two interpretations are possible, one which requires the full use of the data and the other not requiring some part of it, the former is the more reasonable interpretation. Moreover, when a question specifically states that the right handed derivative of f at 0 exists it is difficult for a scrupulous person to think that the same question would allow you to assume the existence of $g'(0)$ without saying so explicitly. That would clearly amount to double standards.

To get a rigorous proof of the existence of $g'(0)$ (having known that $f'(0)$ exists), we have to again go back to the analogy between the functions $f(x)$, $g(x)$ and the sine and the cosine functions and imitate the proof of differentiability of the cosine function at 0, knowing that of the sine function. The standard way to to this is to write the difference quotient $\dfrac{\cos x - \cos 0}{x - 0}$ as $-\dfrac{2\sin^2(x/2)}{x}$ and hence as $-\sin(x/2)\dfrac{\sin(x/2)}{x/2}$ and then take the limit as $x \to 0$. The crucial step here is to express $\cos x$ in terms of $\sin(x/2)$.

Let us see if we can do the same to the function $g(x)$. Now that we have proved that the functions f and g are, respectively, even and odd, if we replace y by $-y$ in the given functional equations (1) and (2), we get

$$f(x+y) = f(x)g(y) + g(x)f(y) \tag{11}$$

$$\text{and} \quad g(x+y) = g(x)g(y) - f(x)f(y) \tag{12}$$

for all $x, y \in I\!R$ which are a little more convenient to work with than (1) and (2) respectively.

Putting $y = x$ in (12) gives $g(2x) = (g(x))^2 - (f(x))^2$ or in other words,

$$g(x) = (g(x/2))^2 - (f(x/2))^2 \tag{13}$$

for all $x \in I\!R$. Combining this with (6), we get

$$g(x) = 1 - 2(f(x/2))^2 \tag{14}$$

It is now a simple matter to prove that $g'(0)$ exists and also to find its value. Indeed,

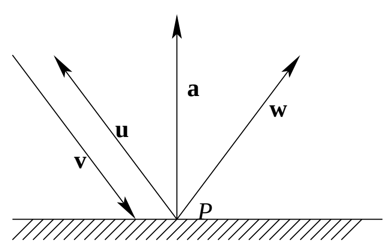
$$\begin{aligned} g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} &= \lim_{x \to 0} \frac{g(x) - 1}{x} \\ &= \lim_{x \to 0} \frac{-(f(x/2))^2}{x/2} \\ &= -\lim_{x \to 0} \frac{f(x/2)}{x/2} \times \lim_{x \to 0} f(x/2) \\ &= -f'(0) \times f(0) = 0 \end{aligned} \tag{15}$$

where in the last step we have used the differentiability and the continuity of f at 0 respectively to evaluate the two limits in the previous step.

It is only now that we have a complete solution to the problem. This problem is a good example of a problem whose degree of difficulty varies drastically depending upon the interpretation. If we are allowed to assume that $g'(0)$ exists and are to find only its value, the problem is absolutely trivial. If we are given that g is continuous at 0 then the problem is moderately challenging. But as it stands, the problem is fairly non-trivial.

Problem 16: A light ray falls on a plane surface and is reflected by it. If $\mathbf{v}$ is a unit vector along the incident ray, $\mathbf{w}$ is a unit vector along the reflected ray and $\mathbf{a}$ is a unit vector along the outward normal to the plane at the point of incidence, express $\mathbf{w}$ in terms of $\mathbf{v}$ and $\mathbf{a}$.

Analysis and Solution: The law of reflection says that the reflected ray lies in the plane spanned by the incident ray and the outward normal to the surface at the point of incidence and that the outward normal is equally inclined to the incident ray as the reflected ray.



In the present problem the unit vector $\mathbf{v}$ is given to be along the incident ray. The natural interpretation is that it is directed towards the point of incidence, say P. The vector unit vector $\mathbf{w}$, on the other hand, is directed away from P. Therefore, to apply the law of reflection, we replace $\mathbf{v}$ by its negative $\mathbf{u}$, which is a unit vector along the incident ray but directed away from P. Then, by the law of reflection, $\mathbf{a}$ is equally inclined to $\mathbf{u}$ and $\mathbf{w}$. To translate this in a useful manner, let us resolve $\mathbf{u}$ and $\mathbf{w}$ along and perpendicular to $\mathbf{a}$. Then the law of reflection can be paraphrased to say that the components of $\mathbf{u}$ and $\mathbf{w}$ along $\mathbf{a}$ are equal while their components perpendicular to $\mathbf{a}$ are negatives of each other.

Resolving $\mathbf{u}, \mathbf{v}$ along and perpendicular to $\mathbf{a}$ gives

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{a})\mathbf{a} + (\mathbf{u} - (\mathbf{u}\cdot\mathbf{a})\mathbf{a}) \tag{1}$$

$$\text{and} \quad \mathbf{w} = (\mathbf{w}\cdot\mathbf{a})\mathbf{a} + (\mathbf{w} - (\mathbf{w}\cdot\mathbf{a})\mathbf{a}) \tag{2}$$

By the law of reflection, the first terms on the R.H.S. of these two equations are equal to each other, while the last two terms are negatives of each other. Hence

$$\begin{aligned} \mathbf{w} &= (\mathbf{u}\cdot\mathbf{a})\mathbf{a} - (\mathbf{u} - (\mathbf{u}\cdot\mathbf{a})\mathbf{a}) \\ &= 2(\mathbf{u}\cdot\mathbf{a})\mathbf{a} - \mathbf{u} \end{aligned} \tag{3}$$

Recalling that $\mathbf{u} = -\mathbf{v}$, we finally get

$$\mathbf{w} = \mathbf{v} - 2(\mathbf{v}\cdot\mathbf{a})\mathbf{a} \tag{4}$$

as the expression of $\mathbf{w}$ in terms of $\mathbf{a}$ and $\mathbf{v}$.

The key idea in the problem is the resolution of a vector along and perpendicular to a given direction. Once the law of reflection is paraphrased in terms of this resolution, there is not much left in the problem. This problem, therefore, is more suited for two points than for 4 points.

Problem 17: Suppose $f(x)$ is a differentiable function, $g(x)$ is a doubly differentiable function such that $f'(x) = g(x)$ and $|f(x)| \leq 1$ for all $x \in [-3, 3]$. If further $[f(0)]^2 + [g(0)]^2 = 9$, prove that there exists some $c \in (-3, 3)$ such that $g(c)g''(c) < 0$.

Analysis and Solution: The formulation of the problem is a little puzzling. If $g(x) = f'(x)$, there is really no need to introduce the function g. The whole problem could have been stated in terms of f and its derivatives (upto order 3). The introduction of the function g suggests that the focus of the problem is on the behaviour of the function g rather than that of its primitive f. This means that the given condition $f'(x) = g(x)$ is probably more likely to be useful in a paraphrased form, viz.

$$\int_0^x g(t)dx = f(x) - f(0) \tag{1}$$

for all $x \in [-3, 3]$.

In view of this reformulation, the condition $|f(x)| \leq 1$ becomes

$$\left|\int_0^x g(t)dt\right| \;\leq\; |f(x)| + |f(0)| \;\leq\; 2 \tag{2}$$

for all $x \in [-3, 3]$.

We are further given that $[f(0)]^2 + [g(0)]^2 = 9$. Combined with $|f(0)| \leq 1$, this gives $2\sqrt{2} \leq |g(0)| \leq 3$ and hence

$$\begin{array}{rrcccl} \text{either} & 2\sqrt{2} & \leq & g(0) & \leq & 3 \qquad (3) \\ \text{or} & -3 & \leq & g(0) & \leq & -2\sqrt{2} \qquad (4) \end{array}$$

Without loss of generality, we suppose that (3) holds. For otherwise we replace f by $-f$. Then g, g'' get replaced by $-g$ and $-g''$ respectively, but the expression gg'' remains unaffected.

Let us see how the graph of $g(x)$ may look like for $-3 \leq x \leq 3$. At $x = 0$, the graph has a height at least $2\sqrt{2}$ by (3). As g is continuous, it follows that in a neighbourhood of 0 too, $g(x)$ will be at least α where α is a real number slightly less than $2\sqrt{2}$. We may, for example take α to be 2. But as x moves from 0 to 3, the height of the graph of $g(x)$ cannot remain bigger than 2 all the time, for if it did, then taking $x = 3$ in (2), we get

$$\begin{aligned} \left| \int_0^3 g(t)dt \right| &= \int_0^3 g(t)\ dt \text{ (since } g(t) \geq 0) \\ &\geq \int_0^3 2\ dt = 6 \qquad (5) \end{aligned}$$

which is a contradiction to (2).

Thus we get that there exists some $x \in (0, 3]$ such that $g(x) < 2$. Since $g(0) >\geq 2\sqrt{2} > 2$, by the Intermediate Value property, the graph of g cuts the line $y = 2$ at least once in $(0, 3]$. We let a be the smallest value of x (in $(0, 3]$) for which this happens. (The existence of such a smallest value is an intuitively clear but non-trivial property of continuous functions whose proof requires completeness.)

Summing up, we have shown that there exists a such that

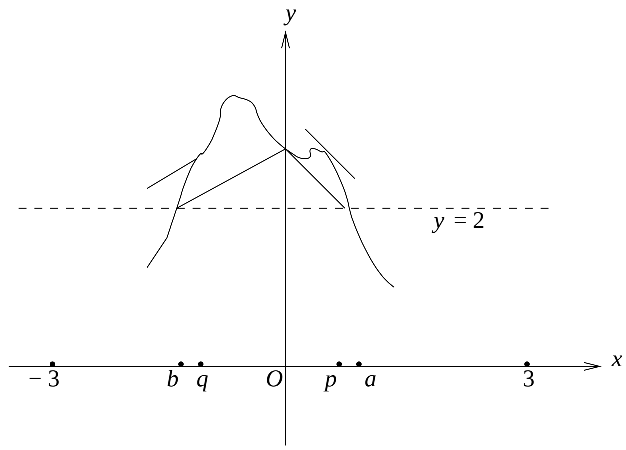
$$a \in (0, 3], \ \ g(a) = 2 \ \text{ and } \ g(x) > 2 \ \text{ for } 0 \leq x < a \qquad (6)$$

(Here the number 2 is essentially arbitrary. Any number between 2/3 and $2\sqrt{2}$ would do.)

By an entirely analogous argument, we get that there exists some b such that

$$b \in [-3, 0), \ \ g(b) = 2 \ \text{ and } \ g(x) > 2 \ \text{ for } b < x \leq 0 \qquad (7)$$

The situation is shown graphically in the figure below.



We now apply Lagrange's MVT to get points $p \in (0, a)$ and $q \in (b, 0)$ respectively such that

$$g'(p) = \frac{g(a) - g(0)}{a} \tag{8}$$

$$\text{and} \quad g'(q) = \frac{g(0) - g(b)}{-b} \tag{9}$$

Clearly, $g'(p) < 0$ while $g'(q) > 0$. Applying Lagrange's MVT to g' on $[q, p]$, we get some $c \in (q, p)$ such that

$$g''(c) = \frac{g'(p) - g'(q)}{p - q} \tag{10}$$

Since $g'(q) > 0 > g'(p)$, we have $g''(c) < 0$. But since $c \in (q, p) \subset (b, a)$, we also have $g(c) \geq 2$ and hence $g(c) > 0$. Therefore we have $g(c)g''(c) < 0$ as desired.

The problem is a simple application of Lagrange's MVT once the key idea strikes, viz., that the value of $g(x)$ at $x = 0$ is fairly high but that it cannot remain high as x moves on either side of 0. But the condition given viz., $[f(0)]^2 + [g(0)]^2 = 9$ is somewhat misleading because the number 9 has no particular role. It could have been replaced by any number bigger than $1 + (2/3)^2$. A student who notices that 9 is the square of 3 which is the length of either half of the interval $[-3, 3]$ is likely to be misled and spend some time to see if this relationship has some significance in the problem. It doesn't.

Problem 18: Suppose a, b, c are three distinct real numbers and $f(x)$ is a quadratic function which satisfies

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix} \tag{1}$$

The maximum of $f(x)$ occurs at a point V. A is the point of intersection of the graph $y = f(x)$ with the x-axis abd B is a point such that the chord AB subtends a right angle at V. Find the area enclosed by $y = f(x)$ and the chord AB.

Analysis and Solution: The problem is a clumsy combination of many unrelated ideas. (In this respect it resembles the last problem solved in Comment No. 17 of Chapter 3.) We first have to determine a quadratic function $f(x)$. Then we have to find the highest point, V on its graph. Then we have to determine the chord AB (A being given) which subtends a right angle at V. And, finally, we have to evaluate a certain area bounded by the chord and the curve.

The first part, viz., that of determining a quadratic $f(x)$ is purely algebraic. A quadratic is uniquely determined by its values at three points. In the present problem, these three points are $-1, 1$ and 2. But we are not given the values $f(-1), f(1)$ and $f(2)$ directly. Instead, we are given a system of three linear equations, viz. (1), in a matrix form. If we write these equations out separately, they become

$$\begin{aligned} 4a^2 f(-1) + 4af(1) + f(2) &= 3a^2 + 3a && (2) \\ 4b^2 f(-1) + 4bf(1) + f(2) &= 3b^2 + 3b && (3) \\ \text{and} \quad 4c^2 f(-1) + 4cf(1) + f(2) &= 3c^2 + 3c && (4) \end{aligned}$$

We can solve this system in a straightforward manner. Subtracting (3) from (2) and dividing by $a - b$ (which is non-zero since a, b, c are all distinct) we get

$$4(a + b)f(-1) + 4f(1) = 3(a + b) + 3 \tag{5}$$

Similarly subtracting (4) from (3) gives

$$4(b + c)f(-1) + 4f(1) = 3(b + c) + 3 \tag{6}$$

Subtracting (6) from (5) gives

$$4(a - c)f(-1) = 3(a - c) \tag{7}$$

which implies $f(-1) = \frac{3}{4}$ because $a \neq c$. Putting this into (5) (or (6)) we get $f(1) = \frac{3}{4}$. Putting these into any one of (2) to (4) gives $f(2) = 0$.

There is a more elegant way of determining $f(-1), f(1)$ and $f(2)$ which takes advantage of the peculiar nature of the coefficients in (2) to (4). If we rewrite (2) as

$$(4f(-1) - 3)a^2 + (4f(a) - 3)f(1) + f(2) = 0 \tag{8}$$

we see that a is a root of the quadratic

$$(4(-1) - 3)x^2 + (4f(1) - 3) + f(2) = 0 \tag{9}$$

Similarly, from (3) and (4), b and c are also roots of the same quadratic. But a, b, c are given to be distinct. No quadratic can have more than two distinct roots unless it is an identity, i.e. it is of the form $0 = 0$. Hence the coefficients of x^2, x and the constant term in (9) must all vanish. This gives $f(-1) = 3/4, f(1) = 3/4$ and $f(2) = 0$.

The next task is to determine $f(x)$, from the values of $f(-1), f(1)$ and $f(2)$. The standard method once again is to take $f(x)$ as Ax^2+Bx+C and determine A, B, C from the equations

$$A - B + C = \frac{3}{4} \tag{10}$$

$$A + B + C = \frac{3}{4} \tag{11}$$

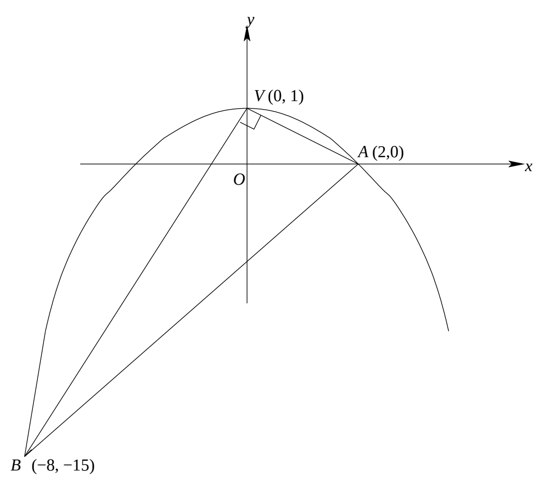
$$\text{and} \quad 4A + 2B + C = 0 \tag{12}$$

The first two equations imply $B = 0$. Hence the last equation gives $C = -4A$. Along with $A + C = \frac{3}{4}$, we get $A = -1/4$ and $C = 1$. So,

$$f(x) = -\frac{1}{4}x^2 + 1 \tag{13}$$

A slightly less pedestrian method would be to note that since $f(2) = 0$, $f(x)$ can be taken as $(x-2)(Dx+E)$ and determine D and E from the values of $f(-1)$ and $f(1)$.

The algebraic part of the problem is over. Let us now work with the geometric part. Being a quadratic with a negative leading coefficient, the graph of $y = f(x)$ is a downward parabola. Its topmost point, V, is $(0, 1)$. The point A is specified as the point where the graph cuts the x-axis. Actually, there are two such points, viz., $(2, 0)$ and $(-2, 0)$. The statement of the problem does not specify which of these is to be taken as A. But clearly, these points are symmetrically located w.r.t. the axis of the parabola and so even though the chord AB will change with A, the area between the chord and the graph will be the same whether we take A as $(2, 0)$ or as $(-2, 0)$. We take it to be $(2, 0)$.



We have to determine a point B on the parabola $y = -\frac{1}{4}x^2 + 1$ such that the chord AB subtends a right angle at V. Take B as $(t, -\frac{1}{4}t^2 + 1)$ where t is to be determined. The slope of VA is $-\frac{1}{2}$. In order that AB subtends a right angle at V, we must have the slope of BV equal to 2. This gives

$$\frac{1 + \frac{1}{4}t^2 - 1}{0 - t} = 2 \tag{14}$$

i.e. $-t = 8$. Thus we get $t = -8$ and B comes out to be $(-8, -15)$. (Had we taken A as $(-2, 0)$, B would have come out to be $(8, -15)$.)

Finally, we have to find the area bounded by the chord AB and the parabola. This is quite straightforward since the parabola is always above the chord. The slope of the chord is $\frac{15}{10} = \frac{3}{2}$ and its equation is $y = \frac{3}{2}(x-2)$, i.e. $y = \frac{3}{2}x - 3$. Hence

$$\begin{aligned}
\text{area} &= \int_{-8}^{2} 1 - \frac{x^2}{4} - \frac{3x}{2} + 3dx \\
&= (4x - \frac{x^3}{12} - \frac{3x^2}{4})\Big|_{-8}^{2} \\
&= 40 - \frac{8}{12} - \frac{512}{12} - 3 + 48 \\
&= \frac{255}{3} - \frac{130}{3} = \frac{125}{3} \text{ sq. units}
\end{aligned} \tag{15}$$

The problem demands long, laborious work and the credit 6 points (which means 12 minutes in terms of time) is not adequate. Moreover, except possibly for the elegant method by which (1) can be solved, there is nothing conceptually challenging in the problem especially since the solution of (1) can be obtained in the straightforward manner anyway. Finding the point B so that AB subtends a right angle at V is hardly something that deserves to be asked in the Main Paper. The part involving the determination of the quadratic $f(x)$ from the values $f(-1), f(1)$ and $f(2)$ is conceptually very similar to the determination of the cubic polynomial $p(x)$ in Problem 12. It is difficult to see what is gained by this duplication. And, as if this duplication was not enough, there is the determination of the area between two curves, which has already been tested in Problem 13.

The question is a good one considering the various skills it demands. It would have been better to increase the credit of this question to, say 8 points and drop Problems 12 and 13 altogether.

CONCLUDING REMARKS

Overall, the questions in the screening paper are reasonable. There are no mistakes or unsettling obscurities except possibly for the use of the phrase 'primitive integral equation' in Q. 25 and the unmotivated nature of the data in Q. 26. The probability question is somewhat unusual in that it asks for infinitistic probability. However, those who have done similar problems before will have little difficulty with it.

There is some avoidable duplication of the type of work needed in the two questions on differential equations. Some areas such as logarithms, classification of conics and number theory have not been represented at all. Most conspicuously absent is, of course, the greatest integer function. Apparently, like everybody else, the papersetters are also tired of it! A more serious omission is that a candidate's ability to correctly visualise verbally given data has not been tested anywhere. This could have been done by simply not giving the diagrams in Q. 2 and Q. 5.

There is a good balance between the number of thought oriented problems and the number of computational problems. Unfortunately, the computations needed are often too lengthy for the time allowed, which comes to be just a little over two minutes per question (because in all there are 84 questions to be answered in three hours). Some of the computations can be shortened with the knowledge of advanced techniques such as the Cayley-Hamilton theorem or Gram-Schmidt orthogonalisation. But this is no consolation for a student who is just clearing HSC. In fact, in a competitive examination where a candidate does not have to reveal the methods he uses, such problems give an unfair advantage to some students, especially when time is so short.

The long time demanded by many computational questions such as Q. 12, 13, 16, 17, 21, 24, 25, 26 becomes an all the more serious matter when we compare them with some chemistry questions in the same screening papers. Here are a couple of sample chemistry questions in JEE Screening 2005.

Q. 1 Which of the following ores contains both Fe and Cu ?
(A) Chalcopyrite (B) Malachite (C) Cuprite (D) Azurite

Q. 2 The ratio of the diffusion rates of He and CH_4 (under identical conditions) is
(A) 1/2 (B) 3 (C) 1/3 (D) 2

Q. 1 is purely memory oriented. You simply have to know the answer before you enter the examination hall. If you do, the question can be answered in less than ten seconds. Otherwise, no amount of logical thinking will help you get the answer.

Q. 2 is not quite such a giveaway. It is based on Graham's law of rates of diffusion of gases, viz., that under identical conditions the rate of diffusion is inversely proportional to the square root of its density and hence in turn, to the square root of the molecular weight of the gas. The derivation of this law is not listed as a part of the JEE syllabus. Even if it is, the present question does not ask for the derivation. The question can be answered if you merely know the law and not its derivation. That makes the question almost as memory oriented as Q. 1. The only difference is that instead of having to remember the names of some ores, you have to remember some law. True, even after recollecting the correct law, you have to do some work. For example, you have to find the molecular weights of helium and methane. That again is a matter of memory (except the tiny bit of arithmetic needed to find the molecular weight of methane from the atomic weights of carbon and hydrogen!). You also have to take square roots of 16 and 4 and then take their ratio. All this computational work is nowhere close to that needed in finding the area in Q. 21 or solving the differential equation in Q. 24.

Summing up, neither of the two chemistry questions given here takes more than 15 seconds to answer. Not all chemistry questions can be answered so quickly. Still, most of them are single idea questions, unlike Q. 21 in mathematics where you have to first identify the three given curves, then the region bounded by them, then solve some equations to get the points of intersection so as to set up some integral and finally evaluate that integral.

As a result, candidates who are strong in chemistry may get an unfair advantage over those who are strong in mathematics. They can easily score twice as much in half the time. If the three screening papers were separate, this would not cause much harm. But as all the three subjects are combined and the selection will take place only on the total, it is possible that those who are strong in mathematics but weak in chemistry might not even get to the Main Paper. Even with a combined paper things would not be so bad if the number of questions was not so large, or if the questions were given credit proportional to the expected time to solve them.

In fact, because of the recent advances in storage and retrieval of information, a time has come to seriously address the role of questions similar to the chemistry questions given above. Memory is not such a cherished virtue as it was till a decade ago. Just as pocket calculators have drastically reduced the importance of the ability to multiply or divide fast, the powerful search engines easily available today have reduced the role of memory to store information.

As an experiment, the search word 'chalcopyrite' was given on google.com. Pat came the reply that its chemical name is copper iron sulfide. This is all the information needed to answer Q. 1 in chemistry. The situation with Q. 2 was not very different. When the search string 'law of diffusion of gases' was given, the very first link gave not only Graham's law, but a portrait of Graham too!

Something like this is not, as yet, possible for Q. 21 in mathematics where you have to find the area bounded by three given curves. Surely, there are many websites which will give you formulas for integration. There are also websites which give tutorials that can train you to solve similar problems. But that means you still have to put in a lot of efforts. The sites may tell you the way. But you have to do your own walking. And not everybody can do that.

Even with all the advances in artificial intelligence, it is unlikely that you can get the answer to Q. 21 with a click of a mouse the way you get the answers to the sample chemistry questions above.

A computer with an ever-ready INTERNET connection will be a part of the minimum basic facilities with which the IIT graduates will be working from now onwards, no matter which walk of life they enter. If they ever need to know what chalcopyrite is an ore of, they can get it instantaneously. Is there any point in testing this knowledge at the JEE?

The Main Paper is a disappointment. The only problems where some originality of thinking is needed are Problems 15, 16 and 17. Together their credit is 14 points which is less than one-fourth of the paper. Even this figure is questionable because of the ambiguity in Problem 15. If the existence of $g'(0)$ is to be proved then the problem is indeed thought provoking. But if it may be assumed without proof, then the problem loses steam. In the process of evaluation, chances are that the candidates are given the benefit of doubt, which results in favouring the sloppy over the scrupulous.

All other problems are accessible to those who have been thoroughly drilled in solving similar problems. Areas such as inequalities, determinants and matrices, number theory, combinatorics, binomial identities and geometric constructions are totally absent while probability is represented very insignificantly. These are among the areas where some challenging problems can be asked for which drill type coaching will not work. Instead, what we see here is a plethora of routine problems in coordinate geometry, evaluation of areas, determination of polynomials (with considerable repetition as pointed out in the comments on Problem 18). Because of their large number, even a student who *can* think originally will have little time to tackle a problem like Problem 17. (It would be intersting to know how many candidates among the successful ones solved Problem 17 correctly. Unfortunately, the IIT's never make this information public. It is, in fact, doubtful if they even bother to collect this information. In many important examinations such as the Putnam competition, announced along with the results are complete profiles which give for each question asked, the number of candidates who attempted that question and also the score distribution for it among those who attempted it. Needless to say, this data is very valuable to see how effective a particular problem was in the selection.)

So chances are that the Main Paper too, will favour the mediocre candidates with good training. There is no convincing argument why problems like 2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 14 and 18 are asked in the Main Paper, which is supposed to test things like mental alertness and analytical thinking rather than memory, drill and speed. On the other hand, some questions in the screening paper (e.g. the two matrix questions) would have been more suitable for the Main Paper.

Perhaps the answer lies in the requirement that all the questions in the screening paper be of the same credit. At present, each question in the screening paper is allowed only two minutes on the average. This permits only simple, single idea based questions. Designing such questions may not be a problem in a subject like Chemistry (as illustrated above). But in mathematics, many questions, although conceptually not difficult, involve work that takes well over 5 minutes. As a result, such questions are barred from the screening paper and have to be asked in the Main Paper. But that vitiates the main paper.

Things would be better if the papersetters of the screening paper are allowed the freedom to vary the credit for each question depending on the time it takes. The Main paper should consist only of a few well chosen, challenging questions with ample time. By clubbing together challenging questions with routine ones (as has been happening almost every year), the selection is invariably dominated by the latter. (Again, to substantiate such statements, we need the scorecard for each question as elaborated above. The human work needed to prepare such data is minimal. All that is necessary is to prepare a scorecard for each candidate, listing how many points he scored on each question. The rest of the work, which consists of retrieving this data and listing it questionwise, so as to give, for each question, the numbers of candidates getting a particular score, is purely mechanical and efficient softwares are already available for it.)