EDUCATIVE COMMENTARY ON JEE 2006 MATHEMATICS PAPER

Contents

It is a pity that the IIT's have still not adopted the practice of making the JEE question papers public immediately after the examination. So, these comments were based on memorised versions of the questions. After about two months, an official text of the mathematics question paper was obtained using the Right to Information Act. As a result, some of the answers and comments had to be revised. All these changes have been listed at the end. The opinions expressed are the author's personal ones and all references are to the book *Educative JEE Mathematics* by the author, unless stated otherwise.

The question paper is divided into five sections. Section I (Q. 1 to 12) consists of multiple choice questions with only one correct answer. Each question carries $(3, -1)$ points, meaning that 3 points are given for a correct answer while -1 point (i.e. one negative point) is given for each wrong answer. Section II (Q. 13 to 20) consists of multiple choice questions where one or more answers are correct, each question carrying $(5, -1)$ points. Section III (Q. 21 to 32) has four passages, each passage carrying three multiple choice questions with only one correct answer, each question having $(5, -2)$ points.

Section IV (Q. 33 to 36) has four questions, where numerical answers have to be filled in and each question has $(6, 0)$ points. The last section, Section V (Q. 37 to 40) has four questions, each asking for matching the pairs and carrying $(6, 0)$ points.

The author will be happy to receive comments by e-mail (kdj@math.iitb.ac.in). Corrections/additions will be made from time to time and displayed here.

SECTION I

Only one of the given answers is correct.

Q. 1 If $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}, t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, where $\theta \in (0, \pi/4)$, then

(A) $t_4 < t_2 < t_1 < t_3$ (B) $t_4 < t_1 < t_3 < t_2$
(C) $t_4 < t_3 < t_2 < t_1$ (D) $t_2 < t_1 < t_3 < t_4$

Answer and Comments: (D). Superficially, this is a problem in trigonometry. But in reality it is a problem about inequalities, specifically, about relating inequalities of powers to those of the bases and of the exponents. Let $\alpha = \tan\theta$. Then $\cot\theta = \frac{1}{\alpha}$. As θ is given to lie between 0 and $\frac{\pi}{4}$ we have $0 < \alpha < 1$. Conversely, every positive real number $\alpha \in (0, 1)$ can be expressed as $\tan\theta$ for some (unique) $\theta \in (0, \frac{\pi}{4})$. So translated in terms of α, the problem amounts to asking you to arrange the four real numbers $\alpha^\alpha, \alpha^{1/\alpha}, (\frac{1}{\alpha})^\alpha$ and $(\frac{1}{\alpha})^{1/\alpha}$ in the ascending order, given that $0 < \alpha < 1$. The first step is to recognise that since α is positive and less than 1, so are all powers of α in which the exponents are positive. Similarly, $\frac{1}{\alpha} > 1$ and therefore all its powers with positive exponents are also greater than 1. Thus we immediately have that t_1, t_2 are both less than 1 while t_3, t_4 are both greater than 1. (Both these statements are obvious when the exponents are positive integers. For the general case, it is better to take logarithms. Thus, if $a > 1$, then $\ln a > 0$ which makes $b \ln a > 0$ whenever $b > 0$. But $b \ln a = \ln(a^b)$ and hence $a^b > 1$. On the other hand, for $0 < a < 1$, we have $\ln a < 0$ and therefore $\ln(a^b) = b \ln a < 0$ for $b > 0$, which gives $a^b < 1$.)

It now only remains to compare t_1 with t_2 and t_3 with t_4. The latter comparison is easier. When the bases are the same and greater than 1, the powers are in the same order as the exponents. In symbols, if $a > 1$ and $b_1 < b_2$ then $a^{b_1} < a^{b_2}$. Once again, this appears obvious. But a rigorous proof can be given using logarithms, because $\ln(a^{b_1}) = b_1 \ln a < b_2 \ln a = \ln(a^{b_2})$.) Now, since $\frac{1}{\alpha} > 1$, we also have $\alpha < 1/\alpha$.

Therefore, by what we just said, $(\frac{1}{\alpha})^{\alpha} < (\frac{1}{\alpha})^{1/\alpha}$. Or, in the notation of the given problem, $(\cot\theta)^{\tan\theta} < (\cot\theta)^{\cot\theta}$, i.e. $t_3 < t_4$.

But the comparison between t_1 and t_2, i.e. between α^{α} and $\alpha^{1/\alpha}$ is a little tricky. Here the base α is less than one. As a result, the powers are in the opposite orders as the exponents. In symbols, if $0 < a < 1$ and $b_1 < b_2$ then $a^{b_1} > a^{b_2}$. Once again the best way to see this is to take logarithms, keeping in mind that $\ln a < 0$. (Note that when we talk of powers, we tacitly assume that the bases are positive. Also, the proofs above are independent of whether we take logarithms w.r.t. the base e or w.r.t. some other base as long as that base is greater than 1. If the base of the logarithms is taken to be positive but less than 1, then $x < y$ is equivalent to $\log x > \log y$ and not to $\log x < \log y$. As a result, the inequalities in the proofs above will be reversed. But the ultimate results will not change. That is, powers with equal bases are related the same way as their exponents are when the base is greater than 1 and oppositely when it is less than 1.)

So, from $0 < \alpha < 1$ and $\alpha < 1/\alpha$, we get $\alpha^{\alpha} > \alpha^{1/\alpha}$. In terms of the given notations, $t_1 > t_2$. Putting this together with $t_3 < t_4$ and the fact $t_1 < t_3$ (since $t_1 < 1$ and $t_3 > 1$), we get $t_2 < t_1 < t_3 < t_4$, i.e. (D) as the right answer.

The essential facts needed from trigonometry are only that $\tan\theta$ and $\cot\theta$ are reciprocals of each other and that the former lies between 0 and 1 for the given values of θ. The crux of the problem lies in the comparisons of powers with common bases. When the common base is greater than 1, they behave in the expected way. But one has to be wary when the common base is positive but less than 1. A person who is careful enough to realise can be safely credited to know the reasons for doing so, without having to spell them out. That makes this problem ideal to be asked as a multiple choice question. It is reported that the problem appears in the book *103 Trigonometry Problems* by Titu Andreescu and Zuming Feng, a Birkhauser Publication.

Q. 2 For $x > 0$, $\lim\limits_{x\to 0}\left((\sin x)^{\frac{1}{x}} + (\dfrac{1}{x})^{\sin x}\right)$ equals

(A)	0	(B)	-1
(C)	1	(D)	2

Answer and Comments: (C). A most natural approach is to write the given function, say $f(x)$ as the sum of two functions, say $f_1(x) = (\sin x)^{1/x}$ and $f_2(x) = (\frac{1}{x})^{\sin x}$ and consider the separate limits of each of these two functions as x tends to 0 from the right. If both these limits exist then their sum will give us the desired limit.

So, let us first consider $\lim_{x\to 0^+} (\sin x)^{1/x}$. Here the expression is a power and as x approaches 0 (from the right), the base $\sin x$ tends to 0 while the exponent $1/x$ tends to ∞. So it is clear that the expression $(\sin x)^{1/x}$ tends to 0 as $x \to 0^+$. (For a formal proof, we can take its log and consider the limit of the log. We have $\ln((\sin x)^{1/x}) = \dfrac{\ln \sin x}{x}$. Rewrite this as $\dfrac{\ln \sin x}{\sin x} \times \dfrac{\sin x}{x}$. The second factor tends to 1 as x tends to 0. So we need to consider only the limit of the first factor. Writing y for $\sin x$, this is the same as $\lim_{y\to 0^+} \dfrac{\ln y}{y}$. As the numerator tends to $-\infty$ while the denominator tends to 0^+, it is clear that the ratio tends to $-\infty$. So $\lim_{x\to 0^+} \ln((\sin x)^{1/x}) = \dfrac{\ln \sin x}{x} = -\infty$ and therefore the original limit, viz. $\lim_{x\to 0^+} (\sin x)^{1/x}$ equals 0.)

Let us now tackle the second limit, viz. $\lim_{x\to 0^+} (\dfrac{1}{x})^{\sin x}$. This is an indeterminate form of the ∞^0 type. As the expression is a power, we once again convert the problem by taking logs. We have $\ln((\frac{1}{x})^{\sin x}) = \sin x \ln(\frac{1}{x}) = -\sin x \ \ln x$. Once again we divide and multiply by x to rewrite this as $-\dfrac{\sin x}{x}(x \ln x)$, which enables us to focus our attention on $\lim_{x\to 0^+} x \ln x$. Here the first factor tends to 0 while the second tends to $-\infty$. So, this is again an indeterminate form of the $0 \times \infty$ type. To find the limit we put $y = 1/x$ and write the limit as $\lim_{y\to\infty} -\dfrac{\ln y}{y}$. Here the numerator and the denominator both tend to ∞. But it is well-known that the logarithm of y tends to ∞ much slower than y (in fact, slower than any power y^α, as long as the exponent α is positive). This is a consequence of the fact that any polynomial growth is slower than an exponential growth, see Exercise (6.51). Or, one can apply L'Hôpital's rule to convert the limit to $\lim_{y\to\infty} \dfrac{1/y}{1} = \lim_{y\to\infty} \dfrac{1}{y}$.

So, we have proved that $\lim\limits_{x \to 0^+} \ln((\frac{1}{x})^{\sin x} = 0$ and therefore $\lim\limits_{x \to 0^+} (\frac{1}{x})^{\sin x}$ equals e^0 i.e. 1. This gives us the limit of the second term in the given problem. As the limit of the first term is 0, the final answer is 1.

Although we have given the reasoning rather elaborately, most of the thinking is intuitive. That the limit of the first term is 0 is clear from the fact that the base gets smaller and tends to 0 while the exponent gets larger and tends to ∞ as x tends to 0 from the right. The calculation of the limit of the second term is a little subtle. But the technique of taking logarithms is a very common one while dealing with limits of powers. In a compact form, it says that the limit of a log is the log of the limit. The theoretical justification is based on the continuity of the exponential function, as pointed out at the beginning of Comment No. 8 of Chapter 15. The trick of replacing $\sin x$ by x is also used frequently. (See Comment No. 6 of Chapter 15.) In fact, after gaining some practice, these things can be done mentally. The really non-trivial fact needed in the evaluation of the second limit is that $x \ln x \to 0$ as $x \to 0^+$. This is quite well known and is often expressed by saying that logarithmic infinity is weaker than an algebraic infinity.

In essence, the problem involves the evaluation of two separate limits and adding them. Not all problems about the limit of a sum are so straightforward. It may happen that neither $\lim\limits_{x \to c} f_1(x)$ nor $\lim\limits_{x \to c} f_2(x)$ exists and still $\lim\limits_{x \to c}(f_1(x) + f_2(x))$ exists. For example, take $f_1(x) = \frac{1}{\sin x}$ and $f_2(x) = -\frac{1}{x}$ and $c = 0$. In such cases, the simple-minded method we applied for the present problem does not work and more delicate methods have to be used. (For the particular example just given, see Comment No. 7 of Chapter 15.)

Finally, it may happen that one of the two limits exists and the other does not. In that case then the limit of the sum will not exist. (The proof is basically the same as that of Exercise (15.20).) The problem could have been a little more revealing by giving the non-existence of the limit as one possible option. Such an option is likely to tempt those who are not very scrupulous.

Q. 3 One angle of an isosceles triangle is $120°$ and the radius of its incircle is $\sqrt{3}$ units. Then the area of the triangle in sq. units is

(A) $7+12\sqrt{3}$	(B) $12-7\sqrt{3}$
(C) $12+7\sqrt{3}$	(D) 4π

Answer and Comments: (C). There are several formulas which express the area Δ of a triangle ABC in terms of its inradius r and something else. The most well-known of these is

$$\Delta = rs \tag{1}$$

where s is the semi-perimeter of the triangle. In the present problem, this is not of much direct use, because although we are given r, we are not given the sides. So, to find the semi-perimeter, we shall have to first find the sides from the angles and the inradius. This can indeed be done and yields the following formula which gives the area Δ of a triangle ABC most directly in terms of the inradius r and the angles.

$$\Delta = r^2(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}) \tag{2}$$

(see Equation (1) in the solution to the Main Problem of Chapter 11). In the present problem, if we take $\angle A = 120°$, then $\angle B = \angle C = 30°$. As $r = \sqrt{3}$, we have $\Delta = 3(\cot 60° + 2\cot 15°)$. The value of $\cot 60°$ is standard and equals $\tan 30° = \frac{1}{\sqrt{3}}$. The value of $\cot 15°$ is not so standard but can be calculated easily from $\tan 30°$. Call $\tan 15°$ as α. Then we have

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{2\alpha}{1-\alpha^2} \tag{3}$$

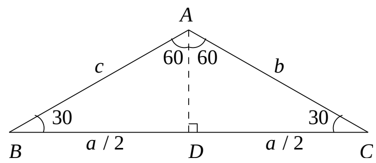
This can be cast as a quadratic in α, viz. $\alpha^2 + 2\sqrt{3}\alpha - 1 = 0$, which gives $\alpha = -\sqrt{3} \pm 2$ and hence $\alpha = 2 - \sqrt{3}$ since $\tan 15° > 0$. Thus $\cot 15° = \dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3}$ after rationalisation. Putting these values and $r = \sqrt{3}$ in (2) gives $\Delta = 3(\frac{1}{\sqrt{3}} + 4 + 2\sqrt{3}) = 12 + 7\sqrt{3}$ sq. units.

Although we have calculated the value of $\cot 15°$ by solving a quadratic, some standard textbooks list the values of $\sin 15°$ and $\cos 15°$ among the standard ones. A student who remembers them correctly will save some time. In that case the method given here is the quickest solution to the problem.

What if you cannot recall formula (2), which is, after all, not so standard as (1)? Even then, everything is not lost. A solution based directly on (1) (i.e. bypassing (2)) is also possible. Note that since the angles of the triangle ABC are known, we already know the relative proportions of its sides. Therefore each side can be expressed as a multiple of any one of the sides, say a. In that case, the semi-perimeter s can also be expressed in terms of a. We are already given the value of r. As a result, using (1) we can now express Δ in terms of a.

To work out the details, from the figure we see that $a = 2b\cos 30° = 2c\cos 30°$, i.e.

$$b = c = \frac{a}{\sqrt{3}} \tag{4}$$



(This can also be obtained by applying the sine rule to ΔABC.)
From (4) we get,

$$s = \frac{1}{2}(a+b+c) = (\frac{1}{2} + \frac{1}{\sqrt{3}})a \tag{5}$$

Therefore from (1) we have

$$\Delta = \sqrt{3}\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)a = \frac{\sqrt{3}+2}{2}a \tag{6}$$

(An alert reader will notice something amiss here. The L.H.S. is an area and hence the square of a length while the R.H.S. is a length. This happens because we have replaced r, which is a length, by the scalar $\sqrt{3}$.)

We are still not at the answer. To get to it from (6) we need to know the value of a. We are given the value of r. If we can write r in terms of a, then we shall get the value of a and hence that of Δ.

There are indeed formulas for expressing the inradius r in terms of the sides of a triangle. But the most standard formula of this type is (1) itself recast slightly, viz. $r = \frac{\Delta}{s}$. Obviously, this will take us to a vicious cycle where we have to know Δ to get a and to get Δ we have to know a. Here again, a less well-known formula can do the trick. One such formula is

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} \tag{7}$$

Obviously, in the present problem, it is easier to apply this with A instead of B or C. From (5) we get

$$r = (\frac{1}{\sqrt{3}} - \frac{1}{2})a\tan 60^\circ = (\frac{2-\sqrt{3}}{2})a \tag{8}$$

But we are already given that $r = \sqrt{3}$. So, from (8) we get $a = \frac{2\sqrt{3}}{2-\sqrt{3}}$. Hence from (5) we get $s = (\frac{1}{2} + \frac{1}{\sqrt{3}})\frac{2\sqrt{3}}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})}$. As we already know $r = \sqrt{3}$, (1) now gives $\Delta = \frac{2+\sqrt{3}}{2-\sqrt{3}}\sqrt{3}$ which gives the same answer as before upon rationalisation.

There is one more way out which does not use the relatively obscure formula (7). Equation (6) is an equation in the two unknowns Δ and a. If we can get some other equation relating these two unknowns, then solving it simultaneously with (6) we can get the value of Δ (in which we are interested) and also that of a (in which we are not interested *per se*).

So, we look for another equation involving Δ and a. This can be done using either of the two other standard formulas for Δ, viz.

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{9}$$

$$\text{or,}\quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} \tag{10}$$

We can now put (4) and (5) into (10). Or we can put (4) into (9) and use the fact that we know the angles A, B, C. Opting for the second method, we get

$$\Delta = \frac{a^2}{6}\sin 120^\circ = \frac{a^2}{4\sqrt{3}} \tag{11}$$

As we are not interested in a, we eliminate a between (6) and (11) to get

$$\left(\frac{2\Delta}{\sqrt{3}+2}\right)^2 = 4\sqrt{3}\Delta \tag{12}$$

As $\Delta \neq 0$, we cancel it from both the sides and get

$$\Delta = \sqrt{3}(\sqrt{3}+2)^2 = \sqrt{3}(7+4\sqrt{3}) = 12+7\sqrt{3} \tag{13}$$

which is the same answer as before.

Although the problem is a simple problem about solutions of triangles, it amply illustrates the good as well as the bad effects of the rich variety of formulas about triangles. For example, we listed four formulas, viz. (1), (2), (9) and (10), for the area of a triangle. If you use the right formula (which is (2) for the present problem) you get the answer almost instantaneously. Otherwise you may wind up working in circles. But if you analyse the failure, then the situation can be salvaged.

Q. 4 Suppose a, b, c are the sides of a triangle and no two of them are equal. Let $\lambda \in \mathbb{R}$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$

(C) $\lambda \in (\frac{1}{3}, \frac{5}{3})$ (D) $\lambda \in (\frac{4}{3}, \frac{5}{3})$

Answer and Comments: (A). As with many other problems in JEE, here a problem from one area is artificially combined with some tidbit from another. The garb of quadratic equations here is quite superficial and easy to lift if we see that the given condition is merely a twisted way of giving the inequality $(a+b+c)^2 \geq 3\lambda(ab+bc+ca)$ or equivalently,

$$\lambda \leq \frac{(a+b+c)^2}{3(ab+bc+ca)} \tag{1}$$

(The equivalence follows because a, b, c and hence $ab+bc+ca$ are all positive.)

Thus the problem is reduced to a problem of trigonometric optimisation, viz. finding the range of the possible values of the expression $\dfrac{(a+b+c)^2}{3(ab+bc+ca)}$ where a, b, c are the sides of a scalene triangle (i.e. a triangle in which no two sides are equal). In essence, this is the same as the 1979 JEE problem, given in Exercise (14.4). It says that for any triangle ABC, we must have

$$3 \leq \frac{(a+b+c)^2}{ab+bc+ca} \leq 4 \tag{2}$$

Further, equality holds on the left only when the triangle is equilateral. Although not given in the exercise, equality holds on the right only for degenerate triangles where one of the angles is 0. (In such a case, in the middle expression, the other two sides have to be equal by the triangle inequality. Or, we can replace it by the ratio $\dfrac{(\sin A + \sin B + \sin C)^2}{\sin A \sin B + \sin B \sin C + \sin C \sin A}$ which makes sense and equals 4 even when one of A, B, C is 0 while the other two are positive and add up to π.)

As we assume that the given triangle is not degenerate, it follows immediately from (1) and (2) that the value of the real number λ is less than $\frac{4}{3}$. Hence (A) is a correct alternative. It may be argued that when $\lambda < \frac{4}{3}$, we also have $\lambda < \frac{5}{3}$ and therefore (C) is also a correct answer. But this claim is fallacious because even if $\lambda \leq \frac{1}{3}$, the quadratic equation given in the statement of the problem will have real roots. So the statement of the problem does not logically force λ to lie in the open interval $(\frac{1}{3}, \frac{4}{3})$.

There is, however, some impropriety in the problem. The coefficient 3 of λ in the statement of the problem has absolutely no role, except to unnecessarily complicate the problem a little. More seriously, the hypothesis that no two sides of the triangle are equal is nowhere needed in the solution. The problem basically deals with the second, and not the first, inequality in (2) where, to ensure strictness of the inequality all we need is that the triangle be non-degenerate. It makes no difference whether any of the sides are equal or not. Even if the inequality on the left in (2) was relevant in the problem, for its strictness all you need is that a, b, c are not all equal and not that no two of them are equal.

It appears that the problem has been designed by taking the 1979 JEE problem stated above and putting the garb of quadratic equations on it, The impropriety just mentioned suggests that this conversion was done without giving much thought. (See the remarks at the end of Comment No. 4 of Chapter 12 about how hastily converted problems sometimes result in comic situations.)

Q. 5 If $0 < \theta < 2\pi$, then the interval(s) of values for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(A) $(0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, 2\pi)$ (B) $(\frac{\pi}{8}, \frac{5\pi}{6})$
(C) $(0, \frac{\pi}{8}) \cup (\frac{\pi}{6}, \frac{5\pi}{6})$ (D) $(\frac{41\pi}{48}, \pi)$

Answer and Comments: (A). Another straightforward problem, where trigonometric equations are combined with a little bit about the sign of a quadratic expression. Call $\sin\theta$ as t. The roots of the quadratic equation $2t^2 - 5t + 2 = 0$ are $t = \dfrac{5 \pm \sqrt{9}}{4}$, i.e. $t = 2$ and $t = \frac{1}{2}$. Also the leading coefficient 2 is positive. So the expression $2t^2 - 5t + 2$ is positive for all values of t outside the two roots (see Comment No. 11 of Chapter 2), i.e. for $t \notin [\frac{1}{2}, 2]$. So the given inequality is satisfied if and only if $\sin\theta < \frac{1}{2}$ or $\sin\theta > 2$. The second possibility is vacuous because $\sin\theta$ can only take values between -1 and 1. So the problem now is to decide for which values of $\theta \in (0, 2\pi)$, we have $\sin\theta < \frac{1}{2}$. Keeping in mind that for $\theta \in (0, 2\pi)$, $\sin\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{6}$ and for $\theta = \frac{5\pi}{6}$ and the increasing/decreasing behaviour of the graph of the sine function, this set comes out to be the union of the intervals $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, 2\pi)$.

Q. 6 If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$ satisfies the condition that $\left(\frac{w - \overline{w}z}{1-z}\right)$ is purely real, then the set of values of z is

(A) $\{z : |z| = 1\}$ (B) $\{z : z = \overline{z}\}$
(C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

Answer and Comments: (D). A sneaky way to answer this problem is by eliminating the incorrect alternatives. (A) is nipped in the bud because the set $\{z : |z| = 1\}$ contains the point 1 which is specifically excluded by the statement of the problem. To rule out (B) and (C) it suffices to show that the respective sets contain some points z which do not satisfy the condition that $\left(\frac{w - \overline{w}z}{1-z}\right)$ is purely real. By inspection, $z = 0$ is one such point because for $z = 0$, this expression becomes w $(= \alpha + i\beta)$ which is not real since β is given to be non-zero. This single value eliminates both (B) and (C) simultaneously. As one of the given answers has to be correct, it must be (D).

Although good enough (and indeed recommended in order to save time) for a multiple choice examination, this solution can hardly be defended from an educative point of view. So let us see if there is some 'honest' way to arrive at the answer. As usual, we denote a typical complex number z as $x + iy$ where x, y are real. Now $w = \alpha + i\beta$ is a fixed complex number. Let us denote the complex number $\dfrac{w - \overline{w}z}{1 - z}$ by $Z = X + iY$. The problem asks us to identify those $z \neq 1$ for which the corresponding Z is real. (The word 'purely' is added only for emphasis. 'Purely real' is the same as 'real'. Although the expression 'purely imaginary' is used commonly, 'purely real' is not so commonly used. A candidate who sees it for the first time in an examination may get a little confused.)

A brute force way of doing this would be to work in terms of the real numbers x, y, X and Y instead of the complex numbers z and Z. Thus we express X and Y in terms of x and y and then determine the conditions on x and y under which Y is zero. A straightforward calculation gives

$$\begin{aligned} X + iY = Z &= \frac{w - \overline{w}z}{1 - z} \\ &= \frac{(\alpha + i\beta) - (\alpha - i\beta)(x + iy)}{(1 - x) - iy} \\ &= \frac{(\alpha - \alpha x - \beta y) + i(\beta - \alpha y + \beta x)}{(1 - x) - iy} \\ &= \frac{((\alpha - \alpha x - \beta y) + i(\beta - \alpha y + \beta x))((1 - x) + iy)}{(1 - x)^2 + y^2} \end{aligned} \tag{1}$$

As we are interested only in the imaginary part, viz. Y of Z, we need not expand the numerator of the R.H.S. fully. We take only its imaginary part and set it equal to 0. Thus we get that Z is real if and only if

$$(\beta - \alpha y + \beta x)(1 - x) + (\alpha - \alpha x - \beta y)y = 0 \tag{2}$$

which represents a circle because the coefficients of x^2 and y^2 are equal (viz. $-\beta$ each) and non-zero and there is no xy-term. So if we want to abandon the honest approach half way, we have already narrowed the choice to (A) and (D). But since (A) contains the forbidden point $z = 1$, we choose (D) as the correct answer.

But let us follow the honest approach all the way through and identify the circle represented by (2). Upon simplification, (2) can be rewritten as

$$\beta(1 - x^2 - y^2) = 0 \tag{3}$$

Canceling β which is given to be non-zero, this is simply $x^2 + y^2 = 1$, i.e. the equation of the unit circle, which in the complex form is $|z| = 1$. But since we must exclude the point $(1, 0)$, the correct answer is (D) and not (A).

Although the computations in the solution above are not prohibitive, while tackling a problem about complex numbers, it is always a good idea to first see if the complex numbers can be handled using the constructions peculiar to complex numbers, rather than routinely translating everything in terms of their real and imaginary parts. Just as problems of pure geometry can sometimes be solved more elegantly using methods of pure geometry than by brute force conversion to coordinates, the same thing holds for problems about complex numbers.

The present problem is a good illustration of this. The requirement that Z be real was translated above to mean that $Y = 0$. But we can also express this requirement in terms of complex conjugates. Thus Z is real if and only if it coincides with its complex conjugate, i.e. if and only if $\overline{Z} = Z$. The advantage of this approach is that the complex conjugation preserves sums, differences, products and quotients. Therefore if we have an expression for Z which involves only these basic operations, then an expression for $\overline{Z}$ can be written down instantaneously. The present problem is of this type. We are given that

$$Z = \frac{w - \overline{w}z}{1 - z} \tag{4}$$

Therefore, by properties of complex conjugates, we have

$$\overline{Z} = \frac{\overline{w} - w\overline{z}}{1 - \overline{z}} \tag{5}$$

The requirement that Z be real now translates as a requirement about z, viz.,

$$\frac{w - \overline{w}z}{1 - z} = \frac{\overline{w} - w\overline{z}}{1 - \overline{z}} \tag{6}$$

Cross-multiplication gives

$$w + \overline{w}z\overline{z} = \overline{w} + wz\overline{z} \tag{7}$$

Rearranging,

$$w - \overline{w} = (w - \overline{w})z\overline{z} \tag{8}$$

Since $w = \alpha + i\beta$, we have $w - \overline{w} = 2\beta$. So, (8) becomes

$$2\beta = 2\beta z\overline{z} \tag{9}$$

Now for the first time we use the fact that β is non-zero. This gives $z\overline{z} = 1$. Thus the point z lies on the unit circle. Note, however, that $z \neq 1$, because for $z = 1$, Z is not even defined. So the correct answer is (D) and not (A).

Note that the equations (3) and (9) are essentially the same. But the manner in which they are arrived at is different. (3) was obtained by a brute force conversion of a complex number to an ordered pair of real numbers, while (9) came out elegantly because of the handy properties of complex conjugation. Obviously, the latter approach is far better. Unfortunately, the wording of the problem is a little likely to mislead a candidate. There was absolutely no need to introduce the real and imaginary parts, viz. α and β respectively, of the complex number w. Instead of saying that $\beta \neq 0$, it would have been fine to simply say that w is not real. That is equivalent to saying that $w - \overline{w} \neq 0$ and so the desired conclusion, viz. $z\overline{z} = 1$ would have followed directly from (8). (The unscrupulous will anyway cancel $(w - \overline{w})$ from both the sides of (8) without bothering to check if it is non-zero.) By unnecessarily introducing β (and α too), a candidate is tempted to convert the whole problem in terms of the real and imaginary parts.

In this problem, Z was a complex valued function of the complex variable, given by $Z = \dfrac{w - \overline{w}z}{1 - z}$. This function is an example of what is called a **linear fractional transformation** or a **Mobius transformation**. More generally, the term is applied to any transformation, say T of the form $T(z) = \dfrac{az + b}{cz + d}$ where a, b, c, d are complex numbers. We generally assume that $ad - bc \neq 0$ as otherwise the transformation degenerates into a constant. (In the present problem, $a = -\overline{w}, b = w, c = -1$ and $d = 1$. The requirement $ad \neq bc$ translates into $\overline{w} \neq w$.)

Such a Mobius transformation is defined at all points of the complex plane except where its denominator vanishes, i.e. except the point $-d/c$. It is easy to show that this transformation is one-to-one and almost onto in the sense that its range consists of all points of the Z-plane except the point a/c. Further it is easy to show that the inverse transformation of a Mobius transformation is also a Moius transformation. In fact, if $Z = \frac{az+b}{cz+d}$ is a Mobius transformation, then the inverse Mobius transformation is given explicitly by $z = \frac{-dZ+b}{cZ-a}$.

In the present problem we are given a Mobius transformation $Z = T(z) = \dfrac{w - \overline{w}z}{1 - z}$ and are asked to find the inverse image, under T, of the real-axis in the Z-plane. The real axis is a straight line. But its inverse image in the z-plane came out to be a circle (except one point). Note further that the real axis in the Z-plane divides it into two halves, the 'upper' half plane where the imaginary part of Z is positive and the 'lower' half plane. It is not hard to show that the inverse image of one of these is the disc inside the unit circle in the z-plane while the inverse image of the other is the exterior of this circle. (Which one goes to which will depend upon the sign of β.)

This is typical of all Mobius transformations. They take straight lines to either circles or to other straight lines and the same holds for circles. And accordingly, they take discs to discs or to half planes and half planes to discs or half planes. Exactly which possibility holds depends on the values of the constants a, b, c, d and also on the particular region. If $c = 0$, then nothing very strange happens. Straight lines go to straight lines and circles to circles. But if $c \neq 0$ (as is the case in the present problem) then such strange things do happen and they can be used advantageously. Just as in the evaluation of definite integrals we sometimes change the interval of integration to a more convenient one by a suitable substitution (accompanied, of course, by a corresponding change of the integrand too), sometimes in dealing with certain problems in the complex plane, we want to convert a disc domain to a half plane. A suitable Mobius transformation is employed to do the trick.

The lacunae created by the vanishing of the denominator can be patched up by adding an extra point, denoted by ∞ (and called, simply, the point at infinity). This point is common to all straight lines. If we include that as a part of the real axis in the Z-plane in the present problem, then its inverse image will be the entire unit circle, including the point $z = 1$ because the transformation $Z = \frac{w - \overline{w}z}{1 - z}$ maps 1 to ∞. Note also that with the inclusion of this point in the domain, every Mobius transformation becomes onto, because if $T(z) = \frac{az + b}{cz + d}$, then we can set $T(\infty) = \frac{a}{c}$.

These things are, of course, beyond the JEE syllabus. But once in a while the JEE papers contain problems whose origin lies in something from the outside the syllabus. For example, the 1998 JEE problem given in Exercise (21.27) is based on a property of quaternions given in Exercise (21.26). Similarly, the JEE 2005 Mathematics papers included a problem based on lattices in the plane and also a problem based on the Cayley Hamilton equation of a matrix. (See the author's educative commentary on the JEE 2005 Mathematics Papers.) Although such problems can always be done, and indeed are expected to be done, without a knowledge of the more sophisticated concepts or theorems, they can be truly appreciated only with some idea of where they originate.

Q. 7 If r, s, t are prime numbers and p, q are positive integers such that the L.C.M. of p, q is $r^2 s^4 t^2$, then the number of ordered pairs (p, q) is

(A) 224 (B) 225
(C) 252 (D) 256

Answer and Comments: (B). This is a simple problem in number theory, once you understand what it really is. If the given setting is too abstract for you, it is a good idea to work out some special cases as illustrations (for example, by taking r, s, t to be the smallest three primes, viz. 2, 3, 5 respectively, in which case $r^2 s^4 t^2 = 8100$). Once you solve the problem for this special case you will realise that the method is quite independent of which three particular primes are represented by r, s, t. With a little more maturity, you can see this instinctively and proceed with the general problem as we now do. But before tackling it honestly, we observe that there is an unwarranted short cut to eliminate the incorrect alternatives. The desired ordered pairs (p, q) are of two types: those in which $p = q$ and those in which $p \neq q$. Clearly, there is only one pair of the first type, because the only way the l.c.m. of two equal numbers can be a given number is when both of them are equal to that number. For every ordered pair of the second type, viz. (p, q) where $p \neq q$, the pair (q, p) is also to be counted ans is distinct from (q, p). So, without actually counting, the number of pairs of the second type is even. Therefore the total number of desired pairs is odd and since this happens only for (B), if at all one of the answers is correct, it has to be (B).

Now, for an honest solution, we are told that p and q are two positive integers whose l.c.m. is $r^2s^4t^2$. This first of all means that neither p nor q can have any prime factor besides r, s and t. So each of them is a product of powers of some of these three primes. We can therefore write p, q in the form

$$p = r^a s^b t^c \quad \text{and} \quad q = r^u s^v t^w \tag{1}$$

where a, b, c, u, v, w are non-negative integers. Then the l.c.m., say e, of p and q is given by

$$e = r^i s^j t^k \tag{2}$$

where

$$i = \max\{a, u\}, \quad j = \max\{b, v\} \quad \text{and} \quad k = \max\{c, w\} \tag{3}$$

This is the key idea of the problem. The problem is now reduced to finding the number of triplets of ordered pairs of the form $\{(a, u), (b, v), (c, w)\}$ where a, b, c, u, v, w are non-negative integers that satisfy

$$\max\{a, u\} = 2, \quad \max\{b, v\} = 4 \quad \text{and} \quad \max\{c, w\} = 2 \tag{4}$$

Let us see in how many ways the first entry of this triplet, viz., (a, u) can be formed. We want at least one of a and u to equal 2. If we let $a = 2$, then the possible values of u are $0, 1$ and 2. These are three possibilities. Similarly, with $u = 2$ there will be three possibilities, viz. $a = 0, 1$ or 2. So, in all the first ordered pair (a, u) can be formed in 6 ways. But the possibility $(2, 2)$ has been counted twice. So, the number of ordered pairs of the type (a, u) that satisfy the first requirement in (4) is 5 and not 6.

By an entirely analogous reasoning, the number of ordered pairs of the form (b, v) which satisfy the second requirement in (4) is $2 \times 5 - 1$, i.e. 9 while that of ordered pairs of the type (c, w) satisfying the third requirement in (4) is 5. But the ways these three ordered pairs are formed are completely independent of each other. So the total number of triplets of ordered pairs of the form $\{(a, u), (b, v), (c, w)\}$ where a, b, c, u, v, w are non-negative integers that satisfy (4) is $5 \times 9 \times 5 = 225$. Hence (B) is the correct answer.

The number theory involved in the problem is very elementary. Thereafter it is essentially a counting problem. Although the reasoning takes a long time to write down, once you hit the essential idea it does not take much time to work out the details mentally. That makes this problem ideal as a multiple choice question. It is, in fact, a very good problem.

One of the common pitfalls in reasoning in this problem is to count the ordered pairs $(2, 2), (4, 4)$ and $(2, 2)$ twice each. In that case the answer would come out to be $6 \times 10 \times 6 = 360$. As this is not given as a possible answer, a good student is alerted that he is making some mistake. If the paper-setters have done this intentionally, it is commendable on their part because it shows that they are trying to help a good student rather than pounce on his single weakness when he has done most of the work correctly. This is important because unlike in a conventional examination, where you can get some partial credit, in a multiple choice question, a silly slip is even more fatal than total inability to solve a problem. It costs you heavily both in terms of the time spent and the negative credit you earn in spite of doing most of the work correctly. However, if the paper-setters are really so concerned about a sincere student, they ought to have included at least one fake answer with an odd number to preclude the sneaky short cut given at the start.

Q. 8 $\int \dfrac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$ equals

(A) $\dfrac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + c$ (B) $\dfrac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + c$

(C) $\dfrac{\sqrt{2x^4 - 2x^2 + 1}}{x} + c$ (D) $\dfrac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + c$

Answer and Comments: (A). This is evidently a problem about finding an antiderivative of a given function. Had it been asked in the conventional form, then one would really have to *find* it. But the multiple choice format obviates the need to do so. If one wants, one can simply differentiate each of the given alternatives and see which derivative equals the given integrand. Moreover, as only one answer is correct, the search stops as soon as you have found one match. As fate would have it, the way the answers are ordered in the present problem, the derivative of the very first one tallies with the integrand. So, (A) is the right answer. (Sometimes, to reduce the chances of copying, the alternatives are shuffled among themselves in the various versions of the same question. In that case, if in a problem like this, the correct alternative is listed as (A) in some question papers and as (D) in some other, that means between two crooks one is luckier than the other because unless the crook is extra smart, he would try the answers one by one from (A) to (D) till he gets the right one!)

Instead of differentiating each alternative separately, a smart crook can concentrate on their similarity. Forgetting the constant of integration c, each of the four alternatives is of the form $\dfrac{\sqrt{2x^4 - 2x^2 + 1}}{u(x)}$ where $u(x)$ is some function of x. If we differentiate this, the derivative, after a little simplification, comes out to be

$$\frac{(4x^3 - 2x)u(x) - (2x^4 - 2x^2 + 1)u'(x)}{u(x)^2\sqrt{2x^4 - 2x^2 + 1}} \tag{1}$$

This will match with the integrand if and only if

$$(x^2 - 1)(u(x))^2 = (4x^3 - 2x)x^3u(x) - u'(x)x^3(2x^4 - 2x^2 + 1) \tag{2}$$

The choices given for the answer correspond to $u(x) = 2x^2, x^3, x$ and x^2 respectively. Of these four, the possibilities $u(x) = x^3$ and $u(x) = x$ are ruled out by considerations of the degrees of the two sides as polynomials in x. The case $u(x) = x^2$ is dismissed by comparing the leading coefficients of both the sides.

Let us now leave aside these crooks and get the answer honestly, i.e. by *finding* an antiderivative of $\dfrac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}$. Note that everywhere we have only even powers of x except in the factor x^3 in the denominator. We can write $\dfrac{1}{x^3}$ as $\dfrac{x}{x^4}$ and combine the x in the numerator nicely with dx so that it becomes $\frac{1}{2}d(x^2)$. This suggests that the substitution $u = x^2$ may work. Trying it, the integral, say I, becomes

$$I = \frac{1}{2}\int \frac{u-1}{u^2\sqrt{2u^2-2u+1}}du \tag{3}$$

It is now tempting to get rid of the radical by calling the radical itself as a new variable, say v, i.e. by substituting $v = \sqrt{2u^2-2u+1}$. If we do so, then the expression for dv becomes $\dfrac{(2u-1)du}{\sqrt{v}}$. So, this would have been just the substitution we need had the numerator in the integrand in (3) been $2u-1$ instead of $u-1$ and had there been no u^2 in the denominator. But unfortunately, we are stuck with these.

Let us now see if we can transform the radical $\sqrt{2u^2-2u+1}$ to a more manageable radical, say, a radical of the form $\sqrt{f(u)}$ where $f(u)$ is some function of u. The integrand in (3) would then look like $\dfrac{g(u)}{\sqrt{f(u)}}$ where $g(u)$ is also some function of u. In fact, $g(u)$ depends on $f(u)$. More specifically,

$$g(u) = \frac{(u-1)\sqrt{f(u)}}{u^2\sqrt{2u^2-2u+1}} \tag{4}$$

By a 'more manageable' radical $\sqrt{f(u)}$ we mean that if we put $z = \sqrt{f(u)}$, then the expression for dz, viz. $\dfrac{f'(u)du}{2z}$ should be equal to $g(u)$ except possibly for some constant factor. If this happens then the substitution $z = f(u)$ will work. In simpler terms, we would like $g(u)$ to be a constant multiple of $f'(u)$.

Let us now look for this magic radical $\sqrt{f(u)}$. As the things already stand, in the integrand in (3), we have $f(u) = 2u^2 - 2u + 1$ and $g(u) = \frac{u-1}{u^2}$. Here $f'(u) = 4u - 2$ and $g(u)$ is not a constant multiple of $f'(u)$. So this choice of $f(u)$ is no good as we already know anyway. But let us take out a factor u^2 from this $f(u)$ and pass it to $g(u)$. So our new $f(u)$ now equals $(2 - \frac{2}{u} + \frac{1}{u^2})$ and our new $g(u)$ is $\frac{u-1}{u^3}$. This time $f'(u) = \frac{2}{u^2} - \frac{2}{u^3} = \frac{2(u-1)}{u^3}$ which is indeed a constant multiple of $g(u)$.

So, we have hit the right choice. Calculating the antiderivative is now a clerical matter. Continuing from (3) we have

$$\begin{aligned} I &= \frac{1}{2}\int \frac{u-1}{u^2\sqrt{2u^2-2u+1}}du \\ &= \frac{1}{2}\int \frac{u-1}{u^3\sqrt{2-\frac{2}{u}+\frac{1}{u^2}}}du \end{aligned} \tag{5}$$

We now put $z = \sqrt{2 - \frac{2}{u} + \frac{1}{u^2}}$. Then we have

$$dz = \frac{\frac{2}{u^2} - \frac{2}{u^3}}{2\sqrt{2-\frac{2}{u}+\frac{1}{u^2}}}du = \frac{(u-1)}{u^3\sqrt{2-\frac{2}{u}+\frac{1}{u^2}}}du \tag{6}$$

Putting (5) and (6) together, we get

$$\begin{aligned} I = \frac{1}{2}\int 1\,dz &= \frac{1}{2}z + c \\ &= \frac{1}{2}\sqrt{2-\frac{2}{u}+\frac{1}{u^2}} + c \\ &= \frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + c \\ &= \frac{\sqrt{x^4-2x^2+1}}{2x^2} + c \end{aligned} \tag{7}$$

So, at long last, we have *arrived at* (A) as the correct choice. We had to use two substitutions to get to it. The first one was rather common. But the second one was quite tricky. The crucial idea was to take out a factor u from the radical $\sqrt{2u^2 - 2u + 1}$ in the denominator. There is a slightly easier way to think of this than the one outlined above. We rewrite the expression under the radical sign as $u^2 + (u-1)^2$. Then taking out the factor u^2 from it, we can rewrite the radical as $u\sqrt{1 + (1 - \frac{1}{u})^2}$ and hence the integral as

$$I = \frac{1}{2}\int \frac{u-1}{u^3\sqrt{1+(1-\frac{1}{u})^2}}du \tag{8}$$

Now we observe that $\dfrac{u-1}{u^3}$ can be rewritten as $(1 - \dfrac{1}{u}) \times \dfrac{1}{u^2}$. But the second factor $\dfrac{1}{u^2}$ is simply the derivative of $1 - \dfrac{1}{u}$ w.r.t. u. It follows therefore that if we put $t = 1 - \dfrac{1}{u}$, then $\dfrac{(u-1)du}{u^3}$ is simply tdt. So, with this substitution, I becomes $\dfrac{1}{2}\displaystyle\int \frac{t}{\sqrt{1+t^2}}dt$. This is easy enough to evaluate directly as $\dfrac{1}{2}\sqrt{1+t^2}$. (If one wants, one can try one more substitution, say $y = t^2 + 1$. But after the tricky substitution we have used, this one is too straightforward and common.) Converting from t to u and then from u to x we get the same answer as before.

There does not seem to be any easier way of finding the antiderivative. Even for the methods given above it is difficult to say whether one could have thought of them without looking at the format of the answers. Had the answers not been given this problem would have been quite challenging and a credit of only three points is grossly inadequate for it. In terms of proportionate time, this means that the question was meant to be answered in less than two minutes! It is possible that the intention of the paper-setters was that it should be answered only by trying the given alternatives one-by-one. Otherwise, instead of asking for the indefinite integral $\displaystyle\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$, they could have asked some definite integral, say $\displaystyle\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$. In that case the answer would have been numerical and it would have been next to impossible to arrive at it by any short cut.

Q. 9 Suppose $f''(x) = -f(x)$ and $g(x) = f'(x)$. Let $F(x) = \left(f(\frac{x}{2})\right)^2 + \left(g(\frac{x}{2})\right)^2$. Given that $F(5) = 5$, $F(10)$ equals

(A) 0	(B) 5
(C) 10	(D) 15

Answer and Comments: (B). Interlinked equations seem to be getting popular with the JEE paper-setters. Last year (2005), there was a question about two interlinked functional equations. This time we have a pair of interlinked differential equations. In view of the second equation $g(x) = f'(x)$, the first equation could have as well been given as $f(x) = -g'(x)$. From this one can, of course, derive that $f''(x) = -f(x)$. But the paper-setters have spared the candidates by giving it directly

$$f''(x) = -f(x) \tag{1}$$

Apparently, the idea here is to help the candidates. Even though second order differential equations are not meant to be solved at the JEE level, this particular equation is very familiar to students as it occurs in the study of simple harmonic motions in physics. It can be solved as indicated in Exercise (19.13) and the general solution is

$$f(x) = A\sin x + B\cos x \tag{2}$$

where A and B are arbitrary constants. Since $g(x) = f'(x)$, we have

$$g(x) = A\cos x - B\sin x \tag{3}$$

We are not given sufficient data to determine the values of the constants A and B. But that hardly matters, because no matter what they are, from (2) and (3) we always have

$$(f(x))^2 + (g(x))^2 = A^2 + B^2 \tag{4}$$

for all x. Introducing the notation in the statement of the problem, this means

$$F(2x) = A^2 + B^2 \tag{5}$$

for all x. But this means that the function F is identically constant. So if its value at some point is 5, then this is also the value at all points. In particular, $F(10) = 5$.

This is a perfectly legitimate, albeit somewhat sneaky way to solve the problem which has become possible because of the knowledge of the general solution of (1). A solution not using (2) is also possible. Let us multiply both the sides of (1) by $2f'(x)$. Then we get

$$2f''(x)f'(x) + 2f(x)f'(x) = 0 \tag{6}$$

We recognise the two terms on the L.H.S. as the derivatives of $(f'(x))^2$ and $(f(x))^2$ respectively. Keeping in mind that $f'(x) = g(x)$, (6) can be recast as

$$\frac{d}{dx}\left((g(x))^2 + (f(x))^2\right) = 0 \tag{7}$$

which implies that $((g(x))^2 + (f(x))^2)$ is identically constant. (As hammered in the remarks about Theorem 3 in Comment No. 9 of Chapter 13, a rigorous proof of this fact is non-trivial and requires the Lagrange Mean Value Theorem. But in a multiple choice test, what matters is the result and not whether you can justify it!) So, we have proved (4) without using (2). The rest of the work remains the same. (We can also get $F'(x) \equiv 0$ by differentiating $F(x)$ as given and then substituting from the data.)

Problems where the value of a function at some point is to be evaluated by showing that the function is identically constant have appeared in the JEE before. (See for example, the Main Problem of Chapter 16 or the 1996 JEE problem at the end of Comment No. 7 of Chapter 24.) And, almost invariably, this is done by showing that the derivative of the function vanishes identically. (For an exception, see the 1997 JEE problem in Comment No. 4 of Chapter 16.) A gambler who draws on familiarity with such problems may instinctively think that the present problem is also of this type and he is right (and rewarded too by the time saved)!

Q. 10 The axis of a parabola is along the line $y = x$. The distance of its vertex from the origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If the vertex and the focus both lie in the first quadrant, the equation of the parabola is

(A)	$(x+y)^2 = (x-y-2)$	(B)	$(x-y)^2 = (x+y-2)$
(C)	$(x-y)^2 = 4(x+y-2)$	(D)	$(x-y)^2 = 8(x+y-2)$

Answer and Comments: (None). The focus and the vertex of any parabola always lie on its axis. Call the focus as F and the vertex as V. In the present problem, the axis is given as the line $y = x$ and therefore $v = (a, a)$ and $F = (b, b)$ for some real numbers a, b which are positive as both V and F lie in the first quadrant. We are also given that the distance of V from the origin is $\sqrt{2}$. This determines a as 1 and hence $V = (1, 1)$. We are further given that the distance between V and F is $2\sqrt{2}$. This means $(b-1)^2 + (b-1)^2 = 8$, which implies $b = 1 \pm 2 = 3$ since the $-$ sign is excluded by the positivity of b.

So, we have determined the focus F as $(3, 3)$. If we can now determine the directrix, say L, of the parabola then we shall get the equation of the parabola. The directrix is always perpendicular to the axis, and hence in the present problem its slope is -1. To determine it, we need to know any one point, say G on it. The best choice is to take G as the point of intersection of the directrix and the axis. This means G is of the form (c, c) for some $c \in I\!R$. Further, the vertex always lies midway between the focus F and this point G. As we already know the focus as $(3, 3)$ and the vertex as $(1, 1)$, c is determined by the equation $\frac{c+3}{2} = 1$, i.e. $c = -1$. Hence the point G is $(-1, -1)$. As we already know the slope of the directrix L to be -1, its equation comes out to be

$$x + y = -2 \tag{1}$$

Having known the directrix L and the focus F, we get the equation of the parabola straight from its definition, viz. the locus of a point which is equidistant from L and F. The distance of a point $P = (x, y)$ from the line (1) is $\dfrac{|x+y+2|}{\sqrt{2}}$ while its distance from F is $\sqrt{(x-3)^2 + (y-3)^2}$. Hence the equation of the parabola is

$$(x+y+2)^2 = 2((x-3)^2 + (y-3)^2) \tag{2}$$

which, upon expansion and simplification, becomes

$$(x-y)^2 = 16(x+y-2) \tag{3}$$

which does not match with any of the given alternatives. Had the focus been at $(2, 2)$ (instead of at $(3, 3)$), the point G would have been the origin (instead of $(-1, -1)$) and the equation of the directrix L would have been $x + y = 0$. The equation of the parabola would then have been $(x+y)^2 = 2((x-2)^2+(y-2)^2)$ and after simplification this would have matched the answer (D).

Apparently, the confusion arises because of the phrase 'and that from its focus' in the statement of the problem. As the problem reads, it means the distance of V from F rather than the distance of the origin from F. If the latter meaning was intended then instead of saying that 'the distance of its vertex from the origin is $\sqrt{2}$ ', the correct wording should have been 'the distance of the origin from its vertex is $\sqrt{2}$ '. Mathematically, the distance is symmetric and so it does not matter whether you say 'the distance of A from B' or 'the distance of B from A'. But the way the word 'that' is used in English, the interpretation of the second part (involving the focus) changes drastically. It is interesting to note that if instead of saying 'that *from* its focus', the problem had said 'that *of* its focus' then it would have meant that the distance of the focus from the origin is $2\sqrt{2}$. This would have fixed the focus at $(2, 2)$ and, as shown above, (D) would have been the answer. How the change of a single word can change the problem! This needless ambiguity could have been avoided by simply giving the vertex and the focus directly. It would be a pity if a student loses 4 marks (not to mention some precious time) just because of poor grammar whether on his own part or on the part of the paper-setters! (It may, of course, very well be that the mistake is not in the original question paper, but in its memorised version.)

In fact, even from a strictly mathematical point of view, it is difficult to see what has been achieved by giving the vertex and the focus of the parabola in such a twisted manner. The problem is about parabolas and its central idea is that the vertex and the focus lie on the axis and further that the vertex is equidistant from the directrix and the focus. Even if the vertex and the focus had been given explicitly as $(1, 1)$ and $(3, 3)$ (or $(2, 2)$ depending on the intention), the major work needed would have been the same. In fact, then the quantum of the work would have been fair for a three point question. The work needed to identify the vertex and the focus first has little to do with the subsequent work and therefore represents only an additional burden which serves little purpose as far as testing the main idea is concerned.

Q. 11 If $f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$ and $g(x) = \int\limits_0^x f(t)dt$, then $g(x)$ has

(A) a local maximum at $x = 1 + \ln 2$ and a local minimum at $x = e$

(B) a local maximum at $x = 1$ and a local minimum at $x = 2$

(C) no local maxima

(D) no local minima

Answer and Comments: (A). To determine the maxima/minima of $g(x)$ we first identify its critical points, i.e. points where $g'(x)$ either vanishes or fails to exist. As the function $g(x)$ is defined by an integral, its derivative $g'(x)$ is given by the second form of the Fundamental Theorem of Calculus (Theorem 2 in Comment No. 11 of Chapter 17). It is simply $f(x)$. Thus we have

$$g'(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases} \tag{1}$$

As the exponential function is always positive, the first line on the R.H.S. of (1) shows that $g'(x)$ has no zero in $[0, 1]$. Since $2 < e < 3$, the third line shows that $g'(x)$ has one zero, viz. e in $(2, 3]$. The zeros in $(1, 2]$ are precisely the solutions of the equation $e^{x-1} = 2$, or equivalently $x - 1 = \ln 2$. There is only one solution, viz. $x = 1 + \ln 2$. It is important to note that this point indeed lies in the interval $(1, 2]$ because $0 < \ln 2 < 1$ (since $1 < 2 < e$).

Summing up, $g'(x)$ vanishes at $1 + \ln 2$ and e. So these are among the candidates for the local maxima/minima of $g(x)$. Further, differentiating (1), we get

$$g''(x) = \begin{cases} e^x, & 0 \leq x < 1 \\ -e^{x-1}, & 1 < x < 2 \\ 1, & 2 < x \leq 3 \end{cases} \tag{2}$$

In particular, we have $g''(1 + \ln 2) = -e^{\ln 2} = -2 < 0$, which means that $g(x)$ has a local maximum at $x = 1+\ln 2$. Similarly, $g'(e) = 1 > 0$ whence there is a local minimum at $x = e$. So the statement (A) is true.

The solution is over because only one option is supposed to be correct. But in case the question had been designed so that one or more options are correct, then we would have to check if (B) holds. (We already know that (C) and (D) are false.) Since the checking of (B) involves some interesting ideas, we present it here, even though it is not a part of the solution as the problem now stands.

In addition to the points where $g'(x)$ vanishes, we also have to look for those points where $g'(x)$ fails to exist. From (1), it may appear that $g'(x)$ exists at all points in $[0, 3]$. But it is not quite so. To explain this apparent anomaly, we need to look at the second fundamental theorem of calculus more carefully. In its usual form it says that if $f(x)$ is continuous on $[a, b]$ and $g(x)$ is defined by $g(x) = \int\limits_a^x f(t)dt$ for $a \leq x \leq b$, then $g'(x)$ exists and equals $f(x)$ for all $x \in [a, b]$. Here, if x equals either of the two end-points a or b, then by continuity of f we mean only the appropriate left or right continuity. The same holds for differentiability of $g(x)$ at the end-points.

What happens if $f(x)$ has a removable discontinuity at a? This means the right handed limit $\lim\limits_{x \to a^+} f(x)$ exists but does not equal $f(a)$. As long as $f(x)$ is bounded on $[a, b]$, a finite number of such discontinuities does not affect the integral. So we can still define $g(x)$ as $\int\limits_a^x f(t)dt$ for $x \in [a, b]$. But now $g'_+(a)$, i.e. the right handed derivative of $g(x)$ at a will equal $\lim\limits_{x \to a^+} f(x)$ and not $f(a)$. Similarly, if there is a removable discontinuity at b, then the left handed derivative $g'_-(b)$ equals $\lim\limits_{x \to b^-} f(x)$ and not $g(b)$.

The same considerations apply if $f(x)$ has a discontinuity at some intermediate point, say $c \in (a, b)$. As long as the two limits $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist, there is no difficulty in defining the integral of f over $[a, b]$ and hence the function $g : [a, b] \longrightarrow I\!R$ by $g(x) = \int\limits_a^x f(t)dt$. But now we can no longer say that $g(x)$ is differentiable at c. All we can say is that $g(x)$ is right differentiable at c with $g'_+(c) = \lim_{x \to c^+} f(x)$ and similarly, it is left differentiable at c with $g'_-(c) = \lim_{x \to c^-} f(x)$.

Although these fine considerations are usually omitted at the JEE level, it is precisely this hair splitting that is needed in the present problem. Here the integrand $f(x)$ given in the statement of the problem has two possible discontinuities in its domain $[0, 3]$, one at $x = 1$ and the other at $x = 2$. At $x = 1$, the left and right handed limits of $f(x)$ are e and 1 respectively. So at $x = 1$, Equation (1) is not correct as stated but instead has to be interpreted to say that

$$\begin{aligned} g'_-(1) &= e \qquad (3) \\ \text{and} \quad g'_+(1) &= 1 \qquad (4) \end{aligned}$$

Similarly, at $x = 2$, the left and right handed limits of $f(x)$ are $2 - e$ and $2 - e$. This makes f continuous at this point and so $g'(x) = 2 - e$. In other words, $x = 2$ is not a critical point of $g(x)$. So there is neither a local maximum nor a local minimum at $x = 2$. This makes (B) false even without checking what happens at $x = 1$. Nevertheless we go into it from an academic point of view.

Since 1 is a point where even the first derivative of $g(x)$ does not exist, the question of the second derivative does not arise. So, the popular and handy 'second derivative test' is useless in determining whether there is a local maximum of $g(x)$ at $x = 1$. Instead we have to go by something more basic, and far more elementary. If c is an interior point of the domain of a (not necessarily differentiable) function $g(x)$, then g will have a local maximum at c if g changes its behaviour from increasing to decreasing as x passes over the point c from the left to right. In other words, if there is some small neighbourhood of c, say $(c - \delta, c + \delta)$ such that $g(x)$ is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then $g(x)$ has a local maximum at c. In particular this is the case if $g'(x) > 0$ for all $x \in (c - \delta, c)$ and $g'(x) < 0$ for all $x \in (c, c + \delta)$. Note that $g'(c)$ is not involved here at all, much less $g''(c)$. An entirely analogous criterion holds for $g(x)$ to have a local minimum at c.

In the present problem, from (1) we see that $g(x)$ is increasing on the interval $(0, 1)$ and also increasing on $(1, 1 + \delta)$ if δ is positive and sufficiently small (specifically, if $0 < \delta < \ln 2$). So, $g(x)$ is increasing on both the sides of 1. Hence there is neither a local maximum nor a local minimum at $x = 1$.

In fact, this simple criterion, based on a change of the increasing/decreasing behaviour could have been applied even without using the fundamental theorem of calculus to find $g'(x)$. The basic idea is simply that if the integrand is positive throughout an interval, then the integral increases as the interval gets larger, while if it is negative then the integral decreases. More specifically, suppose $g(x) = \int\limits_a^x f(t)dt$ and $[c, d]$ is an interval contained in the domain of $f(x)$. If $f(x) > 0$ for all x in the open interval (c, d), then the function $g(x)$ is monotonically increasing (in fact, strictly monotonically increasing) on the interval $[c, d]$. This follows by writing $g(x+h) - g(x)$ as $\int_x^{x+h} f(t)dt$ and noting that for $h > 0$, the integral is positive as the integrand is positive on $(x, x + h)$. Here we are using only very elementary properties of definite integrals which can be derived directly from their definitions as limits of Riemann sums. A deep result like the fundamental theorem of calculus is nowhere needed.

This simple-minded observation works wonders in the present problem. We simply keep track of where the integrand $f(x)$ is positive and where it is negative. We use the same reasoning as we applied above to determine the sign of $g'(x)$. But now we are applying it directly to $f(x)$ as given in the statement of the problem. The answer is, of course the same, viz. that $f(x)$ is positive on $[0, 1 + \ln 2)$, negative on $(1 + \ln 2, e)$ and then positive again on $(e, 3]$. So, by our simple criterion, $g(x)$ has a local maximum at $x = 1 + \ln 2$, a local minimum at e and no other local maxima or minima. So, once again (A) is the only true answer.

When done by the conventional method (based on derivatives), the problem requires certain subtleties of the fundamental theorem of calculus to check whether (B) is also a correct answer. The problem would have been far more interesting if, on the interval $(1, 2]$, $f(x)$ were given as $e^{x-1} - 2$ instead of $2 - e^{x-1}$. In that case the function $f(x)$ would have changed its sign at all the four points $1, 1 + \ln 2, 2$ and e and both (A) and (B) would have been correct. In fact such a problem would have been commendable as an eye-opener to those who have acquired the dirty habit of indiscriminately applying derivatives to tackle any problems of local maxima and minima. This habit becomes so dominating that many people think that to say that a function is strictly increasing over an interval *means* that its derivative is positive at every point of that interval, totally forgetting that the concept of monotonicity is in terms of comparison of functional values and that derivatives are only a convenient tool which works often but not always. It is probably a confusion like this which resulted in an incorrect problem in the Screening Paper of JEE 2004 Mathematics. (See Q. 2 in the author's commentary on the same.)

Q. 12 Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\dfrac{1}{\sqrt{3}}$, is

(A) $4\hat{i} - \hat{j} + 4\hat{k}$ (B) $3\hat{i} + \hat{j} - 3\hat{k}$
(C) $2\hat{i} + \hat{j} + 2\hat{k}$ (D) $4\hat{i} + \hat{j} - 4\hat{k}$

Answer and Solution: (C). Although the methods needed to solve this problem are straightforward, what makes it unusual is that there is no unique answer to it. The given requirements on the desired vector, say $\vec{v}$, do not determine it uniquely. Let P be the plane spanned by the vectors $\vec{a}$ and $\vec{b}$. Resolve $\vec{c}$ along and perpendicular to P, i.e. write $\vec{c}$ as $\vec{u} + \vec{w}$ where $\vec{u}$ is in P and $\vec{w}$ is perpendicular to P. As $\vec{v}$ is given to lie in P, its projection on $\vec{c}$ is the same as its projection on $\vec{u}$. Resolve $\vec{v}$ as $\vec{x} + \vec{y}$, where $\vec{x}$ is parallel to $\vec{u}$ and $\vec{y}$ is perpendicular to $\vec{u}$. Then the projection of $\vec{v}$ on $\vec{u}$ depends only on $\vec{x}$ and not on $\vec{y}$. By changing the magnitude of $\vec{y}$, we get infinitely many vectors which satisfy the same conditions as $\vec{v}$.

That is why the problem only specifies 'a' and not 'the' vector which satisfies the given conditions. The right way to solve the problem is to identify the set of all vectors that satisfy its conditions and then see which of the given alternatives belongs to it.

Once this point is understood, the problem itself is simple. Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be a vector which satisfies the given conditions. The first condition implies that $\vec{v}$ is perpendicular to the cross product $\vec{a} \times \vec{b}$. By a direct computation,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{k} \tag{1}$$

As $\vec{v}$ is perpendicular to $\vec{a} \times \vec{b}$, we get

$$v_1 = v_3 \tag{2}$$

The second requirement on $\vec{v}$ is that its projection along $\vec{c}$ is $\dfrac{1}{\sqrt{3}}$. Since $\vec{c} = \hat{i} + \hat{j} - \hat{k}$, a unit vector along $\vec{c}$ is $\dfrac{1}{\sqrt{3}}\vec{c} = \dfrac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$. Therefore this requirement gives $\dfrac{\vec{v} \cdot \vec{c}}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, i.e. $v_1 + v_2 - v_3 = 1$. In view of (2), this means

$$v_2 = 1 \tag{3}$$

(2) and (3) give the general form of a vector which satisfies the given conditions. Out of the given choices, (C) is the only one where they are satisfied.

A slightly different approach is to start by taking $\vec{v}$ as a linear combination of the vectors $\vec{a}$ and $\vec{b}$, say,

$$\begin{aligned} \vec{v} &= \alpha\vec{a} + \beta\vec{b} \\ &= \alpha(\hat{i} + 2\hat{j} + \hat{k}) + \beta(\hat{i} - \hat{j} + \hat{k}) \\ &= (\alpha + \beta)\hat{i} + (2\alpha - \beta)\hat{j} + (\alpha + \beta)\hat{k} \end{aligned} \tag{4}$$

where α, β are some scalars. As before, the second requirement becomes

$$[(\alpha + \beta)\hat{i} + (2\alpha - \beta)\hat{j} + (\alpha + \beta)\hat{k}] \cdot (\hat{i} + \hat{j} - \hat{k}) = 1 \tag{5}$$

i.e.

$$2\alpha - \beta = 1 \tag{6}$$

Putting (6) into (4) we get

$$\vec{v} = (3\alpha - 1)\hat{i} + \hat{j} + (3\alpha - 1)\hat{k} \tag{7}$$

Here α can have any real value. This gives us the same set of vectors as before and out of the given options (C) is the only one where the given vector belongs to this set.

SECTION II

One or more of the given answers is/are correct.

Q. 13 The equation(s) of the common tangent(s) to the parabolas $y = x^2$ and $y = -(x-2)^2$ is/are

(A) $y = 4(x-1)$ (B) $y = 0$
(C) $y = -4(x-1)$ (D) $y = -30x - 50$

Answer and Comments: (A), (B). This is a straightforward problem. It can be done by various methods which differ more in presentations than in substance. The best method is to take a typical line $y = mx+c$ and determine for which value(s) of m and c it touches both the parabolas. (In doing this there is a danger that we may miss common tangents whose equations cannot be written down in this form. These are the vertical lines, i.e. lines of the form $x = a$ for some constant a. But in the present problem, the first parabola, viz. $y = x^2$ is such a standard figure that even without drawing it, it is clear that it has no vertical tangents. So, nothing is missed. But this point represents an inherent limitation of an objective type test. A careful student will spend some time to rule out this possibility, while to an unscrupulous student, it may simply not occur! And there is no way to distinguish because no reasoning is to be given. In short, in a question like this, ignorance is bliss.)

So suppose $y = mx+c$ is a line in the plane. Its points of intersection with $y = x^2$ correspond to the roots of the quadratic equation

$$x^2 = mx + c \tag{1}$$

the line will be a tangent to the parabola if and only if these roots coincide, i.e. the discriminant vanishes. This gives

$$m^2 + 4c = 0 \tag{2}$$

as the condition for tangency. (For a parabola in the standard form, viz. $y^2 = 4ax$, the condition for tangency of a line $y = mx + c$ is a standard formula. If you can modify it correctly, you can apply it here and save some time. But this is prone to error. At any rate, it is always a good idea to know the underlying reasoning, which requires no modification whether the parabola is in the standard form or not.)

By an analogous reasoning, $y = mx + c$ touches the parabola $y = -(x-2)^2$ if and only if the quadratic $(x-2)^2 + mx + c = 0$ has a vanishing discriminant, i.e.

$$(m-4)^2 - 4(c+4) = 0 \tag{3}$$

Adding (2) and (3), we get $2m^2 - 8m = 0$, which means $m = 0$ or $m = 4$. We can now find the corresponding values of c from either (2) or (3). But that is not necessary. In the given options (A) and (B) are the only ones where the lines given have slopes 0 or 4. So we mark them without further ado. (Such cheap short cuts are academically insignificant. But in an examination they can save you precious time.)

Q. 14 If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with the major and minor axes of the ellipse, and the product of the eccentricities is 1, then

(A) the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(B) the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(C) focus of the hyperbola is $(5, 0)$
(D) focus of the hyperbola is $(5\sqrt{3}, 0)$

Answer and Comments: (A), (C). Looks more like a problem designed to test the knowledge of the vocabulary about conics! Less pretentiously, the second condition simply means that the equation of the hyperbola can be taken to be in the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

So the hyperbola will be determined as soon as we know the values of a and b (which can be taken to be positive). For this we need two equations in these two unknowns. And these are provided by the other parts of the data. The condition about passing through the focus of the ellipse is a little faultily expressed. Every ellipse has two foci and so the language 'the focus' is incorrect. Fortunately, in the present case both the foci of the ellipse are symmetric about the y-axis. As the hyperbola is also symmetric about the y-axis, once it passes through either focus, it also passes through the other.

Now, coming to the solution, the eccentricity of the given ellipse is given by $5\sqrt{1-e^2} = 4$ which determines e as $\frac{3}{5}$. Hence the foci of the ellipse are at $(\pm 3, 0)$. As both these points satisfy (1), we have

$$\frac{9}{a^2} = 1 \tag{2}$$

which gives $a = 3$.

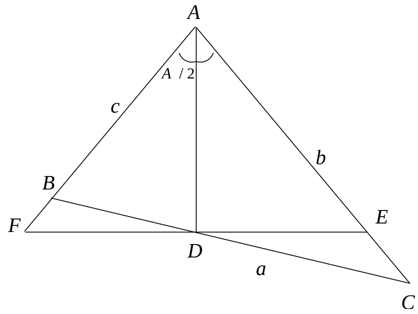
The third condition in the data determines the eccentricity, say e' of the hyperbola as $e' = \frac{5}{3}$. Since the eccentricity of the hyperbola (1) is $\frac{\sqrt{a^2+b^2}}{a}$ and $a = 3$, we get $\frac{5}{3} = \frac{\sqrt{b^2+9}}{3}$ which determines b as $\sqrt{25-9} = 4$. Finally, since the foci of (1) lie at $(\pm ae', 0)$, from $e' = \frac{5}{3}$ and $a = 3$ we get that the foci of the hyperbola lie at $(\pm 5, 0)$.

Q. 15 Internal bisector of $\angle A$ of a triangle ABC meets the side BC at D. A line drawn through D and perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent the sides of ΔABC, then

(A) AE is H.M. of b and c (B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (D) the triangle AEF is isosceles

Answer and Comments: (All). Since AD is the altitude as well as the internal angle bisector through A of the triangle AEF (D) is obvious. The remaining three statements are based on the formula for the length of an angle bisector of a triangle. (B) is this very formula. Although not as standard as others, it is a fairly well-known formula, see e.g. p. 246 of *Trigonometry* by S. L. Loney. Once we know AD we get AE immediately as $AD \sec \frac{A}{2} = \dfrac{2bc}{b+c}$. This makes (A) true. Finally, we also have $ED = AD \tan \frac{A}{2} = \dfrac{2bc}{b+c} \sin \frac{A}{2}$. Since we already know that the triangle AEF is isosceles, AD is also a median. Therefore $EF = 2ED$. Hence (C) is also true.



All except one of the alternatives are easy consequences of a relatively less known formula in trigonometry. Those who have memorised this formula get a rather substantial advantage in terms of time saved. In this respect, the problem resembles Q. 3 in Section A.

Q. 16 If $f(x) = \min\{1, x^2, x^3\}$, then

(A) $f(x)$ is continuous at every $x \in I\!R$
(B) $f'(x) > 0$ for every $x > 1$
(C) $f(x)$ is continuous but not differentiable for every $x \in I\!R$.
(D) $f(x)$ is not differentiable for two values of x

Answer and Comments: (A), (C). A common problem about testing continuity and differentiability of a given function $f(x)$. The only difference is that $f(x)$ is given in a somewhat unusual manner, viz. as the smallest of the three numbers $1, x^2$ and x^3. So the first task is to decide which of these three expressions equal $f(x)$ in which of the intervals of the real line.

There are three functions here, the constant function 1, the function x^2 and the function x^3. They are all continuous. Now, if $f_1(x), f_2(x)$ are continuous functions, then to decide which is greater for which x, we identify all points where the two are equal. Equivalently, we find the zeros of the difference function $f_1(x) - f_2(x)$. In between any two consecutive zeros, we shall have that either $f_1(x) > f_2(x)$ for all x or else $f_1(x) < f_2(x)$ for all x. (This conclusion is intuitively obvious and we often use it as a preliminary step in finding things like the area between the graphs of the two functions. But a rigorous proof requires the Intermediate Value Property of continuous functions applied to the function $f_1 - f_2$.)

Now, coming to the present problem, we are fortunate that all the three functions $1, x^2$ and x^3 agree at $x = 1$. The first two also agree at $x = -1$ while the last two also agree at $x = 0$. Therefore $-1, 0$ and 1 are the points we have to be wary about. The graphs of these functions are very well-known. But even without them, it is easy to see that among these three functions, x^3 is the smallest for $x < 0$, because it is negative while the other two are positive. Also, for $x > 1$, both x^2 and x^3 exceed 1 and so the minimum of the three functions is 1. For $x \in (0, 1)$, we have $x^3 < x^2 < 1$ and so the minimum of the three functions is x^3.

We are now in a position to cast the given function $f(x)$ in a more conventional form, viz.

$$f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \tag{1}$$

Evidently, $f(x)$ is differentiable (and hence continuous) at every $x \neq 1$. Also, the right and the left handed limits of $f(x)$ equal 1 each. So, it is also continuous at 1. For differentiability at 1, we need to check the left and right handed derivatives of $f(x)$ at $x = 1$. These are respectively, 3 and 0. (The latter is obvious. The simplest justification for the former is that the function x^2 is continuously differentiable on $(-\infty, 1]$. So its (left handed) derivative is the limit of its derivative $3x^2$ as $x \to 1^{-1}$.)

Thus we see that $f(x)$ is continuous everywhere and differentiable everywhere except at one point, viz. $x = 1$. This renders (A) and (C) true and (D) false. As for (B), $f(x)$ is a constant for $x > 1$ and so its derivative vanishes identically.

Once the function $f(x)$ is identified in the form (1), the problem is very simple. Coming up with (1) requires an elementary knowledge of inequalities involving powers. The problem is a good combination of two elementary ideas and requires virtually no computation.

Were the problem only about continuity of $f(x)$, we could have bypassed (1). The minimum (and also the maximum) of two continuous functions is continuous. This follows by writing $\min\{f_1(x), f_2(x)\}$ and $\max\{f_1(x), f_2(x)\}$ as $\frac{1}{2}(f_1(x)+f_2(x) \mp |f_1(x)-f_2(x)|)$ respectively and using the continuity of the absolute value function. By induction, the result holds for the max/min of any finite number of continuous functions. Unfortunately, for differentiability there is no such short cut.

Q. 17 $f(x)$ is a cubic polynomial which has a local maximum at $x = -1$. If $f(2) = 18, f(1) = -1$ and $f'(x)$ has local maximum at $x = 0$, then

(A) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minimum is $2\sqrt{5}$

(B) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(C) $f(x)$ has local minimum at $x = 1$

(D) $f(0) = 5$

Answer and Comments: (B), (C). This problem is strikingly similar to Problem 12 of the Main paper of 2005 JEE Mathematics (see the author's commentary on the same), which, in turn, is of the same spirit as the Main Problem of Chapter 15 or Exercise (17.23). In all these problems, we are given some data about a cubic polynomial and the starting point is by converting the data to a system of equations whose solution will determine the cubic uniquely.

So, in the present problem, we let $f(x) = ax^3 + bx^2 + cx + d$. (This choice of notation is a little dangerous because the same symbol a is used in the statement of the problem to denote something else, viz. the point where $f(x)$ has a local minimum. But no confusion need arise because we are first going to determine a, b, c, d from the given conditions and there is no harm if later on the same symbols are used for something else. Still, those who want to play it safe, may take $f(x)$ as px^3+qx^2+rx+s or, more clumsily, as $a_0x^3+a_1x^2+a_2x+a_3$.) We need four equations to determined these four unknowns a, b, c, d. The four conditions in the data give these equations. But instead of writing them mechanically one after another, it is better to see if some short cuts are possible by using the pieces of the data in a more clever order. The last piece of data implies that $f''(x)$ vanishes at $x = 0$. Since $f(x)$ has degree 3, $f''(x)$ is of degree 1. So an equation involving it is much easier to deal with. Specifically, we have $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. So $f''(0) = 0$ gives us $b = 0$. This makes $f(x) = ax^3 + cx + d$ and simplifies the other three equations we are going to get from the other three pieces of the data. Now that $f'(x) = 3ax^2 + c$, the first piece implies that $f'(-1) = 0$ i.e. $c = -3a$. So, now we take $f(x)$ as $ax^3 - 3ax + d$. We now have only two unknowns, viz. a and d. To get their values, we use the remaining two pieces of data, viz. $f(2) = 18$ and $f(1) = -1$, which translate, respectively, into

$$\begin{aligned} 2a + d &= 18 \\ \text{and} \quad -2a + d &= -1 \end{aligned} \tag{1}$$

which can be solved by inspection to get $a = \frac{19}{4}$ and $d = \frac{34}{4}$. Hence we also get $c = -3a = -\frac{57}{4}$. Thus $f(x) = \frac{1}{4}(19x^3 - 57x + 34)$.

Now that we have got hold of $f(x)$, we can answer any questions about it one-by-one. But once again, it is better to begin with the more direct ones first. (D) requires only the constant term in $f(x)$, which is $\frac{34}{4}$ and not 5. So, (D) is false. Both (B) and (C) require the derivative $f'(x) = \frac{57}{4}(x^2 - 1)$ which vanishes at $x = \pm 1$. As we are already given that there is a local maximum at $x = -1$, it follows from the general properties (given in Comment No. 13 of Chapter 13) of a cubic that there is a local minimum at the other critical point, viz. $x = 1$. Of course, we can also verify this directly by computing $f''(1)$ as $\frac{57}{2}$ which is positive. So, (C) is true. Again, $f'(x) > 0$ for all $x > 1$ and so f is strictly increasing on $[1, \infty)$ which includes the interval $[1, 2\sqrt{5}]$. Hence (B) is also true. Finally, for (A), we already know that $a = 1$. (We remark again that this a is different from the one that appeared earlier and in particular in (1) above. We now know the value of the former a, viz. $a = \frac{19}{4}$. The present a is the point where $f(x)$ has a local minimum.) Hence $(a, f(a)) = (1, f(1)) = (1, -1)$. The distance of this point from $(-1, 2)$ (which is not a point on the graph of $f(x)$) is $\sqrt{4+9} = \sqrt{13}$ and not $2\sqrt{5}$. So, (A) is false.

The computations involved are simple but prone to errors. Conceptually, there is nothing exciting in the problem. The problem would have been more interesting had the data been insufficient to identify

$f(x)$ uniquely, but nevertheless sufficient to answer the given questions. As it stands, it is straightforward almost to the point of being a drudgery. The significance of the particular number $2\sqrt{5}$ which appears twice in the problem is far from clear. Maybe it is a random figure inserted only to test if a candidate can save himself from getting confused. Part (A) merely involves finding the distance between two points in the plane. It is not clear what purpose it serves, except to make life miserable for a student by increasing his labour and chances of numerical errors. The language in Part (B) is a little faulty too. A function is said to be increasing (or decreasing) *over* an interval, rather that *at a point* of the interval.

Q. 18 A is a 3×3 matrix and $\vec{u}$ is a column vector. If $A\vec{u}$ and $\vec{u}$ are orthogonal for all real $\vec{u}$, then the matrix A is

(A) singular	(B) non-singular
(C) symmetric	(D) skew-symmetric

Answer and Comments: (A), (D). Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

Then by a direct computation,

$$A\vec{u} = \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} \tag{2}$$

Orthogonality of $A\vec{u}$ and $\vec{u}$ means that their dot product vanishes, i.e.

$$x(a_1x + b_1y + c_1z) + y(a_2x + b_2y + c_2z) + z(a_3x + b_3y + c_3z) = 0$$

or, after simplification,

$$a_1x^2 + b_2y^2 + c_3z^2 + (a_2 + b_1)xy + (c_2 + b_3)yz + (c_1 + a_3)zx = 0 \tag{3}$$

If this is to hold for all x, y, z, the coefficient of every term must vanish. So $a_1 = b_2 = c_3 = 0$ and $a_2 = -b_1, c_2 = -b_3, a_3 = -c_1$. Therefore the matrix A looks like

$$A = \begin{bmatrix} 0 & b_1 & c_1 \\ -b_1 & 0 & c_2 \\ -c_1 & -c_2 & 0 \end{bmatrix} \tag{4}$$

It is immediate that A is skew-symmetric. Also, its determinant comes out to be 0. So it is singular too. (More generally, one can show that for odd n, every skew-symmetric matrix of order n is singular. For, if A is such a matrix, then by definition, A^t, i.e. the transpose of A, equals $-A$. Since every matrix has the same determinant as its transpose, we get that A and $-A$ have the same determinant. But $|-A| = (-1)^n|A| = -|A|$ as n is odd. So, $|A| = -|A|$, which means $|A| = 0$.)

If one wants, the first part (that of showing the skew-symmetry of A) can also be done more elegantly. The orthogonality of $A\vec{u}$ with $\vec{u}$ can be paraphrased to say that $\vec{u}\,^tA\vec{u} = 0$. Taking transposes of both the sides, $\vec{u}\,^tA^t\vec{u} = 0$ and hence $\vec{u}\,^tP\vec{u} = 0$ for all $\vec{u}$, where $P = A + A^t$. Symmetry of P then forces it to vanish. But even without these elegant solutions, the problem is a good one as the computations do not dominate the concepts.

Note that the L.H.S. of (3) is simply $\vec{u}\,^tA\vec{u}$ where A and $\vec{u}$ are as in (1). It is a homogeneous polynomial of degree 2 in the three variables x, y, z. Such expressions are called **quadratic forms**. They have many applications, including classification of conics. And matrices, especially the symmetric ones, play an invaluable role in their analysis.

Q. 19 A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and the y-axis at A and B respectively so that $BP : AP = 3 : 1$. Given $f(1) = 1$,

(A) the equation of the curve is $x\dfrac{dy}{dx} - 3y = 0$

(B) normal at $(1, 1)$ is $x + 3y = 4$

(C) the curve passes through $(2, 1/8)$

(D) the equation of the curve is $x\dfrac{dy}{dx} + 3y = 0$

Answer and Solution: (C), (D). Another problem which is strikingly similar to a JEE problem in the past, specifically the 1998 JEE problem, solved in Comment No. 16 of Chapter 19. The only difference in fact, is that in the earlier problem it was given that $BP = AP$ while in the present problem we have $BP : AP = 3 : 1$.

So, once again we take $P = (x_0, y_0)$ as a typical point on the given curve and let m be the slope of the tangent to the curve at this point. Then the equation of the tangent to the curve at P is

$$y = y_0 + m(x - x_0) \tag{1}$$

The points A and B come out to be respectively

$$A = (x_0 - \frac{y_0}{m}, 0) \quad \text{and} \quad B = (0, y_0 - mx_0) \tag{2}$$

We are given that the point P divides the segment AB in the ratio $1 : 3$. Using the section formula and (2) this gives

$$x_0 = \frac{3(x_0 - \frac{y_0}{m})}{4} \quad \text{and} \quad y_0 = \frac{y_0 - mx_0}{4} \tag{3}$$

(These two equations are exactly the same. This is a consequence of the collinearity of A, P and B. In fact, with a little foresight, we could have stopped after writing either one of them down as we already know that the other one can convey no new information.)

Using either of these two equations we get

$$mx_0 + 3y_0 = 0 \tag{4}$$

We now replace x_0, y_0, m (which were introduced to avoid confusion in an equation like (1)), by x, y and $\frac{dy}{dx}$ respectively and get

$$x\frac{dy}{dx} + 3y = 0 \tag{5}$$

as the equation of the curve. So (D) is true and (A) is false. (Actually, this is a differential equation and represents a one-parameter family of curves, of which the given curve is one member. So the language 'equation of the curve' used in the problem is slightly misleading.)

The next task is to solve (5). This is very easy. Rewriting it in the differential form as $xdy = -3ydx$ and then in the separate variables form as $\frac{dy}{y} = -3\frac{dx}{x}$, the general solution is $\ln y = -\ln(x^3) + c$ or equivalently, $yx^3 = k$ where c and k are some constants. As the curve passes through $(1,1)$ we get $k = 1$. Therefore the equation of the curve is

$$y = \frac{1}{x^3} \tag{6}$$

Now that we know the curve completely, we can answer any questions about it. It is obvious that it passes through the point $(2, \frac{1}{8})$. So, (C) holds. Finally, to find the normal at $(1,1)$, we already know from (5) that the slope of the tangent at $(1,1)$ is -3. So the slope of the normal is $\frac{1}{3}$. But the line given in (B) has slope $-\frac{1}{3}$. So, even without finding the normal at $(1,1)$, we know that (B) is false.

The essential part of the problem is over as soon as (6) is obtained. Determining whether the statements (B) and (C) are true is an additional drudgery which has absolutely nothing to do with the crux of the problem. It also makes it possible that a candidate who gets the heart of the problem correctly, later makes a silly slip in working over these appendages and thereby gets -1 mark instead of 5 which he fully deserves. In a keenly competitive examination like JEE, losing 6 points may translate into the difference between getting in and not getting in.

In a conventional examination, the problem would most likely have asked the candidates only to find the equation of the curve. But in a multiple choice test, it is dangerous to do so, because if (6) is given as one of the answers, one can simply *verify* it rather than obtain it honestly by forming and solving a differential equation. An excellent example of how a particular format of an examination gives undue importance to some peripheral things.

For the analogous 1998 JEE problem mentioned above, the work was essentially the same (and, in fact, a little simpler) and the equation of the curve came out to be $y = 1/x$, because even the initial condition (viz. $f(1) = 1$) is the same for both the problems! It is indeed shocking that essentially the same question is repeated. The explanation perhaps is that although differential equations form an extremely vast area of mathematics, only a very tiny fragment of it is included at the JEE level and so there is not much scope to come up with qualitatively new problems every year.

There is also the time factor. In the 1998 JEE, the question carried 8 points in a 200 point test to be done in 3 hours. Proportionately, this meant a little over 7 minutes for the problem. The present problem has 5 points in a 184 marks test for 2 hours. So, proportionately, you get less than half the time as in 1998. And you have to do more work. Working with the midpoint is not as time consuming as working with a point which divides a segment in some other proportion. Moreover, even after getting the equation of the curve you have to tackle the useless appendages as pointed out earlier. It is no consolation that in 1998 you had to show the work while now you need not. This argument would hold some water in the case of problems like Q. 1 or Q. 7 above. But in a highly computational problem like the present, one still has to do the work in the rough, whether one displays it later or not. Maybe the intention behind asking a familiar type problem was to make up for this severe reduction in the time allowed.

Q. 20 Let $\vec{A}$ be a vector parallel to the line of intersection of the planes P_1 and P_2 through the origin. P_1 is parallel to the vectors $2\hat{\jmath} + 3\hat{k}$ and $4\hat{\jmath} - 3\hat{k}$ and P_2 is parallel to $\hat{\jmath} - \hat{k}$ and $3\hat{\imath} + 3\hat{\jmath}$. Then the angle between the vectors $\vec{A}$ and $2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{6}$ (D) $\dfrac{3\pi}{4}$

Answer and Comments: (B), (D). This is a straightforward problem about the cross and the dot products of vectors. We first need to determine the direction of the vector $\vec{A}$. Let $\vec{N_1}$ and $\vec{N_2}$ be vectors normal to the planes P_1 and P_2. As $\vec{A}$ is parallel to line of intersection of P_1 and P_2, it is perpendicular to both N_1 and N_2. Therefore, it is parallel to $\vec{N_1} \times \vec{N_2}$. This does not determine $\vec{A}$ uniquely. But that is not needed either, because all we want to find is the angle between $\vec{A}$ and the vector $2\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

So, we must first determine $\vec{N_1}$ and $\vec{N_2}$. This is easy because we are given two (linearly independent) vectors in each plane and forming the cross products of these pairs, we get $\vec{N_1}, \vec{N_2}$. Thus,

$$\begin{aligned}
\vec{N_1} &= (2\hat{\jmath} + 3\hat{k}) \times (4\hat{\jmath} - 3\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 3 \\ 0 & 4 & -3 \end{vmatrix} = -18\hat{\imath} \qquad (1)\\
\text{and } \vec{N_2} &= (\hat{\jmath} - \hat{k}) \times (3\hat{\imath} + 3\hat{\jmath}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & -1 \\ 3 & 3 & 0 \end{vmatrix} = 3\hat{\imath} - 3\hat{\jmath} - 3\hat{k} \qquad (2)
\end{aligned}$$

(A slight short-cut to (1) is possible. Note that we are interested only in the direction of $\vec{N_1}$ which is a normal to the plane P_1. Since P_1 contains the vectors $2\hat{\jmath} + 3\hat{k}$ and $4\hat{\jmath} - 3\hat{k}$, neither of which has any component parallel to $\hat{\imath}$, it is obvious that this plane P_1 is simply the yz-plane. Therefore we may as well take $\vec{N_1} = \hat{\imath}$ without having to do any computations. Such a clever thinking is not possible for $\vec{N_2}$. And even when it is possible, the time to recognise its applicability is probably the same as the time to get the result by computation. But it does serve some purpose in confirming an answer.)

As noted above, we may take $\vec{A} = \vec{N_1} \times \vec{N_2}$. This gives,

$$\vec{A} = (-18\hat{\imath}) \times (3\hat{\imath} - 3\hat{\jmath} - 3\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -18 & 0 & 0 \\ 3 & -3 & -3 \end{vmatrix} = -54\hat{\jmath} + 54\hat{k} \qquad (3)$$

Let θ be either of the two angles between $\vec{A}$ and the vector $2\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Then,

$$\begin{aligned}
\cos\theta &= \pm\frac{\vec{A} \cdot (2\hat{\imath} + \hat{\jmath} - 2\hat{k})}{|\vec{A}||(2\hat{\imath} + \hat{\jmath} - 2\hat{k})|} \\
&= \pm\frac{(-54\hat{\jmath} + 54\hat{k}) \cdot (2\hat{\imath} + \hat{\jmath} - 2\hat{k})}{|(-54\hat{\jmath} + 54\hat{k})||(2\hat{\imath} + \hat{\jmath} - 2\hat{k})|} \\
&= \pm\frac{-162}{54\sqrt{2} \times \sqrt{9}} \\
&= \pm\frac{-1}{\sqrt{2}} \qquad (4)
\end{aligned}$$

which gives $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. So, (B) and (D) are correct.

Note that the vector $\vec{A}$ is the cross product $\vec{N}_1 \times \vec{N}_2$, where each of the vectors $\vec{N}_1$ and $\vec{N}_2$ is itself a cross product of two vectors. So $\vec{A}$ is a vector of the form

$$\vec{A} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) \tag{5}$$

where $\vec{a} = 2\hat{j} + 3\hat{k}, \vec{b} = 4\hat{j} - 3\hat{k}, \vec{c} = \hat{j} - \hat{k}$ and $\vec{d} = 3\hat{i} + 3\hat{j}$. Now, there is an identity for the vector appearing in the R.H.S. of (5), which expresses it in terms of certain scalar triple products (or 'box' products) of vectors, viz.

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}\, \vec{c}\, \vec{d})\vec{b} - (\vec{b}\, \vec{c}\, \vec{d})\vec{a} \tag{6}$$

(For a proof as well as an application, see Comment No. 12 and also Comment No. 18 of Chapter 21.) So we could as well get (3) directly using (6), bypassing (1) and (2). Of course, computations of the box products require determinants anyway and so there is not much saving of time. Moreover, the identity (6) itself is not as standard as some other identities about vectors.

The computations in this problem are simple but highly repetitious and prone to errors. The conceptual part is very elementary and straightforward. All it involves is the computations of the dot and the cross products and the norm of a vector. All these three were also needed in the solution to Q. 12 in Section A. It is not clear what purpose is served by this duplication.

The answer, perhaps, lies once again in the helplessness of the paper-setters imposed by the format of the examination. In a conventional examination, candidates can be asked to prove identities like (6). This is precluded in a multiple choice test. Numerical problems, on the other hand, have clear cut, crisp answers and are more suitable to be asked in the multiple choice format. But that puts some constraint on the topics that can be covered. So problems involving the same formulas about the dot and the cross products of vectors appear twice.

But with a little imaginativeness, the situation could have been salvaged. For example, instead of asking a proof of (6), the paper-setters could have given an equation of the form

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \alpha\vec{a} + \beta\vec{b} \tag{7}$$

where α and β are some scalars. The candidates could then have been asked to identify them from a set of four choices. Such a question could have replaced one of the two questions based on the same formulas.

SECTION III

There are four comprehensions, each with three questions with single correct answers.

Comprehension I

There are n urns each containing $n+1$ balls such that the i-th urn contains i white balls and $n+1-i$ red balls. Let U_i be the event of selecting the i-th urn $i = 1, 2, \ldots, n$, and W denote the event of getting a white ball when a ball is drawn at random from the selected urn.

Q. 21 If $P(U_i) \propto i$ for $i = 1, 2, \ldots, n$, then $\lim_{n\to\infty} P(W)$ equals

(A) 1 (B) $\frac{2}{3}$

(C) $\frac{3}{4}$ (D) $\frac{1}{4}$

Answer and Comments: (B). We are given that $P(U_i) = \lambda i$ for some constant λ (independent of i). Obviously, the events $U_1, U_2, \ldots, U_n$ are mutually exclusive and exhaustive. So, their probabilities add up to 1. This gives

$$\sum_{i=1}^{n} P(U_i) = \sum_{i=1}^{n} \lambda i = \lambda \sum_{i=1}^{n} i = \frac{\lambda n(n+1)}{2} = 1 \qquad (1)$$

from which we get $\lambda = \dfrac{2}{n(n+1)}$ and hence

$$P(U_i) = \frac{2i}{n(n+1)} \quad \text{for } i = 1, 2, \ldots, n \qquad (2)$$

(Note that λ depends on n. Therefore it would perhaps be better to denote it by λ_n. But that does not matter because now that we have found $P(U_i)$ explicitly, we shall not not need λ anymore.)

The next task is to determine $P(W)$ as a function of n. The event W is the disjunction of the mutually exclusive events $U_1W, U_2W, \ldots, U_nW$. Therefore

$$P(W) = \sum_{i=1}^{n} P(U_iW) \tag{3}$$

Now, the event U_iW is the conjunction of the events U_i and W. Therefore by the law of conditional property, we have

$$P(U_iW) = P(U_i)P(W|U_i) \text{ for } i = 1, 2, \ldots, n \tag{4}$$

where $P(W|U_i)$ is the conditional probability of W given U_i. This is precisely the probability of drawing a white ball from the i-th urn. As the i-th urn contains $n+1$ balls of which i are white, we have

$$P(W|U_i) = \frac{i}{n+1} \text{ for } i = 1, 2, \ldots, n \tag{5}$$

Putting (2) and (5) into (4), we get

$$P(U_iW) = \frac{2i^2}{n(n+1)^2} \text{ for } i = 1, 2, \ldots, n \tag{6}$$

and hence from (3),

$$\begin{aligned} P(W) &= \sum_{i=1} \frac{2i^2}{n(n+1)^2} \\ &= \frac{2}{n(n+1)^2} \sum_{i=1}^{n} i^2 \\ &= \frac{2n(n+1)(2n+1)}{6n(n+1)^2} \\ &= \frac{2n+1}{3n+3} \end{aligned} \tag{7}$$

It is now immediate that $\lim_{n\to\infty} P(W) = \frac{2}{3}$. (A very special case of Exercise (6.41), part (ii).)

The problem is a hotch-potch of several parts jumbled together. The essential idea is to express $P(W)$ as a function of n. (It would be more logical to denote the event W by W_n, because it is a different event for each n. But this might cause even more confusion.) The event W can occur in any one of n ways, depending upon from which urn the ball is drawn. Finding the probability of each of these n sub-events requires conditional probability of W given U_i. Finally, the probability of U_i is given in a rather clumsy manner. Apparently, the idea was to test the knowledge of the formula for the sum of the first n positive integers.

Q. 22 If $P(U_i) = c$ where c is a constant, then $P(U_n/W)$ equals

(A) $\dfrac{2}{n+1}$ (B) $\dfrac{1}{n+1}$

(C) $\dfrac{n}{n+1}$ (D) $\dfrac{1}{2}$

Answer and Comments: (A). This problem is a little more straightforward at the start than the last one because $P(U_i)$ is given in a less clumsy way. (Of course, instead of saying that $P(U_i)$ is a constant, a better wording would have been to say that all urns are equally likely to be selected.) Since $P(U_i) = c$ for every $i = 1, 2, \ldots, n$, instead of (1) we have a much simpler equation, viz. $nc = 1$ from which we get

$$P(U_i) = c = \frac{1}{n} \text{ for } i = 1, 2, \ldots, n \tag{8}$$

instead of (2). Equation (3) still remains valid. Our interest now is in determining the conditional probability $P(U_n/W)$, i.e. the probability that we had picked the n-th urn, it being given that the ball we picked happens to be white. Since all urns are equally likely, it is tempting to think that the answer is $\frac{1}{n}$. This temptation betrays a common confusion about the concept of conditional probability. (For elaboration, see Comment No. 5 of Chapter 22.) Had $\frac{1}{n}$ been one of the given options, chances are that some candidates would have fallen for it. Apparently, the examiners have tried to help such candidates by alerting them that their way of thinking is not correct.

The correct way to find $P(U_n/W)$ is to use the law of conditional probability. It gives

$$P(U_n/W) = \frac{P(U_nW)}{P(W)} \tag{9}$$

To find the numerator, we note that the Equation (4) above still holds because it is independent of the probabilities with which the urns are chosen. So, applying (4) with $i = n$ we get

$$P(U_nW) = P(U_n)P(W/U_n) \tag{10}$$

By (8) we already know $P(U_n)$ as $\dfrac{1}{n}$. As for the second factor, we note again that Equation (5) is also still valid. So

$$P(U_nW) = \frac{1}{n} \times \frac{n}{n+1} = \frac{1}{n+1} \tag{11}$$

Let us now worry about the denominator of (9), viz. $P(W)$. As Equation (3) also remains valid, we have to compute $P(U_iW)$ for every $i = 1, 2, \ldots, n$. This is a generalisation of (10) with n replaced by i. Doing the same in (11), we get

$$P(U_iW) = \frac{1}{n} \times \frac{i}{n+1} = \frac{i}{n(n+1)} \tag{12}$$

Combining (3) and (12), we now have

$$P(W) = \sum_{i=1}^{n} P(U_iW) = \sum_{i=1}^{n} \frac{i}{n(n+1)} = \frac{n(n+1)}{2n(n+1)} = \frac{1}{2} \tag{13}$$

So, finally, putting (11) and (13) into (9) we get $P(U_n/W) = \dfrac{2}{n+1}$.

There is a more elegant way to arrive at the answer. Since all the urns are equally likely to be picked, we might as well empty them all into a single big urn and draw one ball from it at random. For each ball in this new urn, we keep track of which urn it came from. Now this new urn has $n(n+1)$ balls, of which half are white. Among these $\frac{1}{2}n(n+1)$ white balls, exactly n come from the n-th urn. So the probability of U_n given W is the ratio $\dfrac{n}{\frac{1}{2}n(n+1)}$ which is simply $\dfrac{2}{n+1}$.

Q. 23 If n is even, $P(U_i) = \dfrac{1}{n}$ for every i and E denotes the event of choosing an even numbered urn, then $P(W/E)$ equals

(A) $\dfrac{n+2}{2n+1}$ (B) $\dfrac{n+2}{2(n+1)}$

(C) $\dfrac{n}{n+1}$ (D) $\dfrac{1}{n+1}$

Answer and Comments: (B). Yet another computation of conditional probability. But this time the problem can be taken as a continuation of the last one because the values of $P(U_i)$ are the same as in the last problem. In fact, now that we know an elegant way to solve the last problem, let us do the present one by the elegant method only. (Those who cannot think of it are welcome to imitate the pedestrian solution to the last problem. In fact, this is recommended as a drill.) So let us suppose that all the urns are emptied in a big urn and a ball is drawn at random from it. We are given that it originally came from an even numbered urn. Since n is even, this means that the ball came from one of the urns numbered $2, 4, \ldots, n-2, n$. And we have to find the probability of its being white. The total number of balls in these even numbered urns is $\dfrac{n}{2}(n+1)$. The number of white balls among them is $2 + 4 + \ldots + n$. Calling $n/2$ as m this number is $2(1 + 2 + \ldots + m) = m(m+1) = \dfrac{n(n+2)}{4}$. So the desired probability is simply the ratio $\dfrac{n(n+2)/4}{n(n+1)/2}$ which equals $\dfrac{n+2}{2(n+1)}$.

It is not clear what the word 'comprehension' in the title means. In the context of examinations (especially in languages) it means that an excerpt from some article (or a poem) is given and questions are asked to test if the candidates have understood the passage correctly. But here, instead of a passage, all we see is a bunch of three problems with a common setting. Naturally, there is considerable duplication of ideas in their solutions. In fact, Q. 22 is completely subsumed by Q. 23 and it is not clear what is gained by this duplication. As for Q. 21, it is qualitatively different from the other two in the bunch. But the solution does require conditional probability. Moreover, the probabilities of the events U_i are given in an unnecessarily clumsy manner. Perhaps, by 'comprehension' the paper-setters mean the ability to make sense out of a clumsily worded problem!

Comprehension II

Now we define the definite integral using the formula $\int\limits_a^b f(x)dx = \dfrac{b-a}{2}(f(a)+f(b))$. For more accurate result, for $c \in (a,b)$, we let $F(c) = \dfrac{c-a}{2}(f(a)+f(c)) + \dfrac{b-c}{2}(f(b)+f(c))$. When $c = \dfrac{a+b}{2}$, $\int\limits_a^b f(x)dx = \dfrac{b-a}{4}(f(a)+f(b)+2f(c))$.

Q. 24 $\int\limits_0^{\pi/2} \sin x dx$ equals

(A) $\dfrac{\pi}{8\sqrt{2}}(1+\sqrt{2})$ (B) $\dfrac{\pi}{8}(1+\sqrt{2})$

(C) $\dfrac{\pi}{4}(1+\sqrt{2})$ (D) $\dfrac{\pi}{4}(1+2\sqrt{2})$

Answer and Comments: (B). The first task is to correctly understand the problem. The integral that is asked here is not the usual definite integral, which can be evaluated by first finding an antiderivative, viz. $-\cos x$ of the integrand $\sin x$. We now have to follow the definition of the integral as given in the statement of the problem. The trouble, however, is that the problem itself gives two different definitions for $\int\limits_a^b f(x)dx$, one at the beginning and then one at the end (for more accurate result!) and it is not clear which of these two is to be used.

If we follow the first definition of $\int\limits_a^b f(x)dx$, viz. $\dfrac{b-a}{2}(f(a)+f(b))$, then $\int\limits_0^{\pi/2} \sin x dx$ equals $\dfrac{\pi}{4}(\sin 0 + \sin\frac{\pi}{2}) = \dfrac{\pi}{4}$. But this is not given as one of the alternatives. So, we interpret the question to mean the second definition, viz. $\int\limits_a^b f(x)dx = \dfrac{b-a}{4}(f(a)+f(b)+2f(c))$. With this definition, $\int\limits_0^{\pi/2} \sin x dx = \dfrac{\pi/2}{4}(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}) = \dfrac{\pi}{8}(1+\dfrac{2}{\sqrt{2}}) = \dfrac{\pi}{8}(1+\sqrt{2})$ which matches with (B).

Q. 25 If $f''(x) < 0$ for every $x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

(A) $\dfrac{f(b) - f(a)}{b - a}$ (B) $\dfrac{2(f(b) - f(a))}{b - a}$

(C) $\dfrac{2f(b) - f(a)}{2b - c}$ (D) 0

Answer and Comments: (A). Like the last question, the present one is straightforward once you know what it is asking. We do not know the function $f(x)$. But we have a new function $F(c)$ defined in terms of $f(c)$ by

$$F(c) = \frac{c-a}{2}(f(a) + f(c)) + \frac{b-c}{2}(f(b) + f(c)) \tag{1}$$

In this expression $a, b, f(a), f(b)$ are constants. But c varies over the interval $[a, b]$. So, this expression is a function of c and we have to maximise it for $c \in (a, b)$. As $f(c)$ is given to be differentiable as a function of c, we can differentiate (2) w.r.t. c and get

$$\begin{aligned} F'(c) &= \frac{f(a) + f(c)}{2} + \frac{(c-a)f'(c)}{2} - \frac{f(b) + f(c)}{2} + \frac{(b-c)f'(c)}{2} \\ &= \frac{f(a) - f(b)}{2} + \frac{(b-a)f'(c)}{2} \end{aligned} \tag{2}$$

It follows that $F'(c) = 0$ only when $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. So, if at all one of the given alternatives is correct, it must be (A). Note, incidentally, that such a point $c \in (a, b)$ does exist by the Lagrange's Mean Value Theorem. Moreover, it is unique because the hypothesis $f''(x) < 0$ implies that $f'(x)$ is strictly monotonically decreasing and hence a one-to-one function. Of course, we have not proved that at F will have a maximum at this point c. In a time constrained examination of the multiple choice type, it would be foolish to spend time on it. But an honest answer must verify it. To do so, we note that F has no other interior critical point and so the second derivative test can be applied. Therefore we shall be through if we can show that $F''(c) < 0$ if c is such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. This is easy, because differentiating (2) we have $F''(c) = \dfrac{b-a}{2} f''(c)$ which is always negative since we are given that $f''(x) < 0$ for all $x \in (a, b)$.

Q. 26 If $f''(x)$ is continuous everywhere and $\lim\limits_{t \to a} \dfrac{\int_a^t f(x)dx - \frac{t-a}{2}(f(t) + f(a))}{(t-a)^3} = 0$ for all a, then $f(x)$ is a polynomial in x whose maximum possible degree is

(A) 1 (B) 2
(C) 3 (D) 4

Answer and Comments: (A). Once again the correct interpretation of the problem is half the solution. If we interpret the integral in the numerator according to the first definition given in the statement of the problem, then the numerator is identically zero and so the hypothesis will always hold no matter what the function $f(x)$ is. In this case, no conclusion can be drawn about the degree of $f(x)$. So we have to discard this interpretation.

Let us, therefore, take the second definition of the integral. Then the numerator in the problem equals

$$\frac{(t-a)}{4}(f(a) + f(t) + 2f(\frac{a+t}{2})) - \frac{t-a}{2}(f(a) + f(t))$$

which reduces to $\dfrac{(t-a)}{4}(2f(\dfrac{a+t}{2}) - f(a) - f(t))$ after simplification. So our hypothesis now becomes

$$\lim_{t \to a} \frac{2f(\frac{a+t}{2}) - f(a) - f(t)}{(t-a)^2} = 0 \tag{3}$$

for all a. As $f(x)$ is given to be twice differentiable, this limit, say L, can be evaluated applying L'Hôpital's rule twice, giving

$$\begin{aligned} L &= \lim_{t \to a} \frac{f'(\frac{a+t}{2}) - f'(t)}{2(t-a)} \\ &= \lim_{t \to a} \frac{\frac{1}{2}f''(\frac{a+t}{2}) - f''(t)}{2} \\ &= -\frac{1}{4}f''(a) \end{aligned} \tag{4}$$

where in the last step we have used the continuity of f'' at a.

So, the hypothesis now means that $f''(a) = 0$ for all a or, in other words, that f'' is identically zero. Integrating, we get

$$f'(x) = A \tag{5}$$

for some constant A. (Here, by integration we mean the ordinary integration and not the one given in the problem!) One more integration gives

$$f(x) = Ax + B \tag{6}$$

for some constants A and B. Hence $f(x)$ is a polynomial of degree 1 or 0 (depending upon whether $A \neq 0$ or $A = 0$).

Because of the unavailability of the original question paper, it is not clear how the preamble to this section was worded. Instead of *defining* $\int\limits_a^b f(x)dx$ as $\dfrac{b-a}{2}(f(a) + f(b))$, a better wording would have been to say that the latter expression is an *estimate* of the ordinary definite integral $\int\limits_a^b f(x)dx$. Also, some motivation of how $F(c)$ is arrived at could have been given. It is obtained by splitting the original integral $\int\limits_a^b f(x)dx$ into two parts, viz. as $\int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$ and then applying the earlier estimate to each of the two integrals. In general, $F(c)$ would be a more accurate estimate of the integral $\int\limits_a^b f(x)dx$ if the intermediate point c is suitably chosen. But the right choice will depend on the function f and the interval $[a, b]$. In absence of any other information about f, one standard choice for c is to take it as the mid-point of the interval $[a, b]$.

Had the preamble explained these things, the three questions would have been less obscure. The first question (Q. 24) simply asks the (revised) estimate for a particular definite integral. The hypothesis in Q. 25 makes the function strictly concave downwards on the interval $[a, b]$ (see Comment No. 18 of Chapter 13). In this case, no matter how you choose $c \in (a, b)$, the estimate $F(c)$ will always fall short of the exact value of the integral

$\int\limits_a^b f(x)dx$. The significance of the problem is that it tells you that the best possible estimate is obtained by taking c to be the 'mean value' of $f(x)$ over $[a, b]$ as given by the Lagrange Mean Value Theorem. The third question (Q. 26) deals with the difference between the revised, more accurate estimate $\dfrac{b-a}{4}(f(a) + f(b) + 2f(\frac{a+b}{2}))$ and the original crude estimate $\dfrac{b-a}{2}(f(a) + f(b))$. As the length of the interval tends to 0, this difference obviously tend to 0. The hypothesis deals with how rapidly it tends to 0. (See Comment No. 6 of Chapter 15 for an elaboration of this concept.) The problem amounts to showing that unless the function $f(x)$ is a linear function, the difference between these two estimates cannot tend to 0 more rapidly than the cube of the length of the interval of integration.

The topic in this section is very interesting. The standard method to evaluate a definite integral $\int_a^b f(x)dx$ is to first find an antiderivative, say $F(x)$, of the integrand and then evaluate the integral as $F(b) - F(a)$. However, as explained in Comment No. 25 of Chapter 17, many times we run into situations where we cannot express the antiderivative in a closed form, as happens, for example, when $f(x) = e^{x^2}$ or $\sin(x^2)$. In such cases we have to resort to methods such as the trapezoidal rule and the Simpson's rule to evaluate the integral approximately. The two estimates given in the present section are among the simplest such approximations.

In fact, instead of giving some numerical problems, the paper-setters could have asked some questions to test the ability of the candidate to digest concepts. Here are some examples, along with the expected answers:

1. **What is the geometric significance of the first estimate of the integral, viz. $\dfrac{b-a}{2}(f(a) + f(b))$ in relation to the graph of f?**

   Ans: It is the area of the trapezium with vertices $(a, 0), (b, 0), (a, f(a))$ and $(b, f(b))$. The last two vertices lie on the graph of the function f while the first two are their projections on the x-axis. So, the given estimate is the area below the chord of the graph of the function joining the points $(a, f(a))$ and $(b, f(b))$ on the graph. It is approximately the area below the portion of the graph of $f(x)$ between $x = a$ and $x = b$.

2. **How is the revised estimate $F(c)$ related to the first estimate?**

Ans: It is obtained by splitting the original integral $\int_a^b f(x)dx$ into two parts, viz. as $\int_a^c f(x)dx + \int_c^b f(x)dx$ and then applying the earlier estimate to each of the two integrals.

3. **How is the first estimate related to the ordinary definite integral $\int_a^b f(x)dx$ defined in terms of the limit of Riemann sums?**

   Ans: Both $f(a)(b-a)$ and $f(b)(b-a)$ are certain Riemann sums of the function $f(x)$ for the partition of $[a, b]$ into a single interval, viz. $[a, b]$ itself. The first estimate is the arithmetic mean of these two Riemann sums.

4. **If $f(x)$ is concave upwards on $[a, b]$, how does the revised estimate $F(c)$ compare with the exact value of the integral $\int_a^b f(x)dx$? Also, how does $F(c)$ compare with the first estimate?**

   Ans: Since $f(x)$ is concave upwards, the chords always lie above the graphs. Therefore, no matter which $c \in (a, b)$ is chosen, the estimate $F(c)$ always exceeds the integral, i.e. $\int_a^b f(x)dx < F(c)$. But we also have $F(c) < \frac{b-a}{2}(f(a) + f(b))$. So, it is a better estimate of the integral than the first estimate.

5. **For which functions $f(x)$ will the estimate $\frac{b-a}{2}(f(a)+f(b))$ be exact (i.e. coincide with the integral $\int_a^b f(x)dx$) for every choice of the interval $[a, b]$?**

   Ans: When $f(x)$ is a linear function, i.e. a function of the form $f(x) = Ax + B$ where A, B are some constants.

Had some such questions been designed, they would have tested ‘comprehension’ in the true sense of the term. It is a pity that the ‘comprehension’ in this bunch as it now stands amounts to trying to make some sense out of some obscure, clumsily stated problems.

Comprehension III

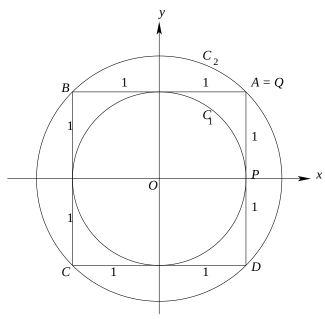
Let $ABCD$ be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square $ABCD$. L is a line through A.

Q. 27 If P is a point on C_1 and Q is a point on C_2, then $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ equals

(A) 0.75 (B) 1.25
(C) 1 (D) 0.5

Answer and Comments: (A). An excellent example of how a multiple choice question *should not* be. The real crux of the problem is to show that the ratio in the statement of the problem is constant, i.e. independent of the points P and Q chosen, as long as they lie on the circles C_1, C_2 respectively. Once this is shown, finding the numerical value of this ratio is a relatively minor matter. In fact, it will come built-in in the proof of constancy of the ratio.

But since it is a multiple choice question, a smart student may simply *assume* that the ratio is independent of the particular points P and Q. To find its numerical value, he can choose P and Q to be any points for which both the numerator and the denominator can be calculated easily. One such choice is to take P as a point of contact of C_1 with a side of the square as shown in the figure and take Q as the vertex A itself.



With this choice we have $PA = PD = 1$ and $PB = PC = \sqrt{5}$. So,

$$PA^2 + PB^2 + PC^2 + PD^2 = 1 + 5 + 5 + 1 = 12 \text{ sq. units} \qquad (1)$$

Similarly, $QA = 0, QB = QD = 2$ and $QC = \sqrt{4+4} = \sqrt{8}$. Hence

$$QA^2 + QB^2 + QC^2 + QD^2 = 0 + 4 + 4 + 8 = 16 \text{ sq. units} \qquad (2)$$

Therefore, the ratio asked in the problem is $\frac{12}{16} = 0.75$. Hands down!

Fortunately, the work involved in an honest solution to the problem is not prohibitively lengthy either. With a suitable choice of coordinates as shown in the figure, take the vertices A, B, C, D as $(1, 1), (-1, 1), (-1, -1)$ and $(1, -1)$ respectively. Let $P = (x_1, y_1)$ be any point on the circle C_1 and $Q = (x_2, y_2)$ be any point on the circle C_2. Then we have

$$\begin{aligned} x_1^2 + y_1^2 &= 1 & (3) \\ \text{and} \quad x_2^2 + y_2^2 &= 2 & (4) \end{aligned}$$

Now a straightforward computation gives

$$\begin{aligned} PA^2 &= (x_1 - 1)^2 + (y_1 - 1)^2 = 3 - 2x_1 - 2y_1 & (5) \\ PB^2 &= (x_1 + 1)^2 + (y_1 - 1)^2 = 3 + 2x_1 - 2y_1 & (6) \\ PC^2 &= (x_1 + 1)^2 + (y_1 + 1)^2 = 3 + 2x_1 + 2y_1 & (7) \\ \text{and} \quad PD^2 &= (x_1 - 1)^2 + (y_1 + 1)^2 = 3 - 2x_1 + 2y_1 & (8) \end{aligned}$$

where we have used (3)

Adding these four equations, we have

$$PA^2 + PB^2 + PC^2 + PD^2 = 12 \tag{9}$$

By an entirely analogous computation using (4), we get

$$QA^2 + QB^2 + QC^2 + QD^2 = 16 \tag{10}$$

(9) and (10) together give the answer. Actually, we have shown a little more than the problem asks. We have not only shown that the ratio in the problem is a constant, but we have shown that the numerator as well as the denominator are constants (i.e. independent of P and Q).

It is obvious that this is the way the paper-setters intended the problem to be solved. Otherwise there was no need to give the side of the square as 2 units length. Indeed the answer is independent of the side of the square. Had the side been some other length, then by a similarity transformation all lengths would get multiplied by the same constant of proportionality. Therefore their squares would get multiplied by the square of this constant. As a result the ratio in the problem would remain intact.

By giving the side of the square as 2 units, the paper-setters have apparently tried to help the honest students. This is commendable. But what they probably failed to realise is that a crook does not need their help and can get the answer in the sneaky manner given at the beginning. Because the honest solution is also easy, the resulting saving may be only a minute or so. But in a severely time-constrained, highly competitive examination, one minute is significant.

But then again, perhaps the paper-setters did think of the possibility of sneaking but decided to let it pass. A candidate who can think of it is probably intelligent enough to do the problem in the conventional way too. In fact, from a practical point of view he is more intelligent. And it is possible that the paper-setters wanted to reward this quality.

Although nobody is likely to do this problem by pure geometry, we give here one such solution for the lovers of pure geometry (a fast dying creed). We shall prove separately that both the numerator and the denominator of the ratio $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ are constants (i.e. independent of P and Q, as long as P lies on C_1 and Q on C_2). Interestingly, it is easier to do this for the denominator because then Q lies on the circumcircle of the square $ABCD$. Since AC and BD are diameters of this circle, the triangles AQC and BQD are both right-angled. (We do not draw a separate diagram here. We refer the reader to the diagram in the earlier solution and ask him to do the necessary fillings.) So, by Pythagoras theorem,

$$\begin{aligned} QA^2 + QC^2 = AC^2 &= 4r_2^2 \qquad (11)\\ \text{and} \quad QB^2 + QD^2 = BD^2 &= 4r_2^2 \qquad (12) \end{aligned}$$

where r_2 is the radius of C_2. Adding these two equations we get that the denominator is a constant equal to $8r_2^2$ (which equals 16 in the present problem).

But this argument cannot be applied for the numerator because the $\angle APC$ and $\angle BPD$ need not be right angles. In this case we need an extension of the Pythagoras theorem called Apollonius theorem (see the end of Comment No. 4 of Chapter 11). The point O is the midpoint of both AC and BD. So PO is a median of both ΔAPC and ΔBPD. So, Apollonuis theorem gives the following analogues of (7) and (8) respectively.

$$PA^2 + PC^2 = \frac{1}{2}AC^2 + 2OP^2 \quad = \quad 2r_2^2 + 2OP^2 \qquad (13)$$

$$\text{and} \quad PB^2 + PD^2 = \frac{1}{2}BD^2 + 2OP^2 \quad = \quad 2r_2^2 + 2OP^2 \qquad (14)$$

Adding these two equations we see that the numerator equals $4r_2^2 + 4OP^2$ (which equals $8 + 4$ i.e. 12 in the present problem.)

Actually, the pure geometry solution is not substantially different from the earlier one based on coordinate geometry. In the earlier solution, we added the four Equations (5) to (8) together. Instead, suppose we first add only (5) and (7). Then we get

$$PA^2 + PC^2 = 3 \qquad (15)$$

which is a special case of (13) with $r_2 = \sqrt{2}$ and $OP = 1$. Similarly, (14) is the equivalent of adding (6) and (8). So, in principle, the two solutions are not different. In the coordinate geometry proof, when we add PA^2 and PC^2, certain terms cancel because of algebra. In the pure geometry solution, we need the Apollonius theorem to add them. But if you look at the proof of the Apollonius theorem, it is essentially algebraic, because it consists of expressing each of PA^2 and PC^2 using the Pythagoras theorem and then doing some cancellation.

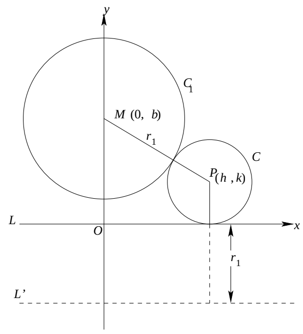
Q. 28 A variable circle touches the line L and the circle C_1 externally and both the circles are on the same side of the line L. Then the locus of the centre of the circle is

(A) an ellipse (B) a hyperbola
(C) a parabola (D) parts of a straight line

Answer and Comments: (C). In the preamble of this comprehension, L is given to be a line through the vertex A of the square $ABCD$. This line figured nowhere in the last problem. In this problem, there is a reference to it with a further restriction that the circle C_1 lies entirely on one side of L. This can be confusing to a candidate. If after reading the preamble, he has made a rough sketch in which the line L cuts the circle C_1 (which was perfectly legitimate to do as nothing was specified about the line L other than that it passed through A), he will now have to draw another sketch in which L is completely outside C_1, thereby resulting in an unnecessary waste of precious time. It would have been far better to say nothing about L in the preamble and introduce it only now along with the restriction on it.

As for the circle C_1, in the preamble it was given to be the incircle of a certain square of side 2 units length. But that introduction is totally irrelevant in the present problem. Basically, the present problem is about any fixed circle C_1 and any line L not cutting it. The problem deals with the locus of the centre of a variable circle which lies on the same side of L as C_1 does and touches the line L and also the circle C_1 externally. Note that since the equations of C_1 and L are not given, we cannot determine the locus completely. But we can determine what type of a curve it represents and that is all that the problem asks.

Let M and r_1 be the centre and the radius of the fixed circle and P and r be the centre and the radius of the variable circle C. We want to find the locus of P. Because of the data, $MP = r_1 + r$. Also, the perpendicular distance of P from L is r. If the two distances were equal then the locus of P would have been a parabola with focus M and directrix L. But as it stands the distance of P from the line L falls short of its distance from the point M by a fixed constant, viz. r_1.



This difficulty can be resolved neatly by finding another fixed line L' whose distance from P is $r + r_1$. The correct choice is to let L' be the line parallel to L at a perpendicular distance r_1 from it and lying on the opposite side of L as the circle C_1. We now see without any computations that the locus of P is a parabola with focus M and directrix L'.

Note that there is hardly any computation in the solution. The success depends on the construction of the line L'. That puts this particular problem in a class above the rest. Conceiving such elegant constructions is akin to an artistic experience and the ability to come up with them is a valuable asset. In pure geometry problems, there is ample scope for this ability.

What if you cannot think of this trick on the spur of the moment? Then everything is not lost. The ever so reliable coordinates are always available for help. Take the given line L as the x-axis and choose the y-axis to pass through the point M, the centre of the given circle C_1. Then we can take M as $(0, b)$ where we may suppose $b > 0$ without loss of generality. As is a usual practice in locus problems, let (h, k) be the current coordinates of the point P. As P and M lie on same side of L, we have $k > 0$. Therefore, the distance of P from L is k and this is also the radius of the circle C. As C touches C_1 externally, we have

$$\sqrt{h^2 + (k-b)^2} = r_1 + k \tag{16}$$

On squaring and simplifying, this becomes $h^2-(2b+2r_1)k+b^2-r_1^2 = 0$. Hence the locus of P is $x^2-(2b+2r_1)y+b^2-r_1^2 = 0$ which is a parabola as the quadratic terms form a perfect square. Actually, one need not even do this much. The question does not ask you to identify the locus but the type of curve it represents. It is clear just by inspecting (16) that when you square it the k^2 terms on both the sides will cancel out. As the only other quadratic term is h^2, the curve is going to be a parabola.

Of course, this solution is not as elegant as the earlier one. But that's the very price you pay for the reliability of coordinates. In the case of the last problem, we remarked that there was no substantial difference in the pure geometry proof and the one based on coordinates. But in the present problem, the coordinate geometry solution is no match to the pure geometry solution in terms of elegance.

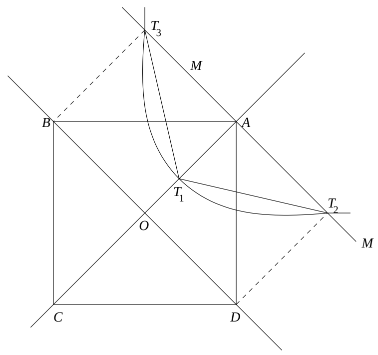
The problem is of the same spirit as a 2001 JEE question (see Exercise (9.15 (c)) where you had to identify the locus of the centre of a variable circle, which touches two given circles, one internally and the other externally. But its solution was not so tricky. That question carried 5 points in a two hour paper with total 100 points. On a proportionate basis that gives 6 minutes. The present problem is for 5 points in a two hour paper with 184 total marks. So, now you get considerably less time. Of course, you do not have to write your reasoning now.

Q. 29 A line M through A is drawn parallel to BD. A point S moves such that its distances from the line BD and the vertex A are equal. If the locus of S cuts M at T_2 and T_3 and AC at T_1, then the area of $\Delta T_1T_2T_3$ is

(A) $\dfrac{1}{2}$ sq. units (B) $\dfrac{2}{3}$ sq. units

(C) 1 sq. units (D) 2 sq. units

Answer and Comments: (C). Another problem on parabola. The locus of S is a parabola with focus A and directrix the line BD. As AC is perpendicular to BD and hence to the directrix M, it follows that the axis of the parabola is along AC. Let O be the centre of this square. Then T_1 is the midpoint of OA. (This fact was also used in the solution to Q. 10.) The points T_2 and T_3 lie on the parabola and also on the line M. As the perpendicular distance between M and BD is AC, the perpendiculars from T_2 and T_3 to BD must meet at D and B (or vice versa). So $\Delta T_1T_2T_3$ is an isosceles triangle with base T_2T_3. So its area is simply $AT_1.AT_2$.



It only remains to find these lengths. From the preamble, the side of the square is given as 2 units length. So its diagonal is $2\sqrt{2}$. Therefore $OA = \sqrt{2}$. Since $AT_1 = \frac{1}{2}OA$ while $AT_1 = OA$, we get the area of the triangle $T_1T_2T_3$ as $\frac{1}{2}OA^2 = 1$ square units. The segment T_2T_3 is popularly called the latus rectum of the parabola and we could have used the fact that its length is four times the distance between the vertex T_1 and the focus A. But that does not simplify the calculations.

Unlike in the last comprehension, the three problems in the present one are totally unrelated to one another. The last problem has considerable duplication of ideas with Q. 10. As there is no common theme to the three problems in the present bunch, it is difficult to comprehend what 'comprehension' means. In fact, as pointed out at the beginning of the solution to Q. 28, introducing the line L in the preamble and then imposing a restriction on it later can only result in a waste of time.

Comprehension IV

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and U_1, U_2, U_3 be column matrices satisfying the equations $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Let U be the 3×3 matrix whose columns are U_1, U_2, U_3.

Q. 30 The value of $|U|$ is

(A) 3 (B) -3
(C) $3/2$ (D) 2

Answer and Comments: (A). The straightforward way is to begin by identifying each of the column vectors U_1, U_2, U_3. Each requires us to solve a system of three equations in three unknowns. For example, suppose $U_1 = \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix}$. Then the equation $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is equivalent to the system

$$\begin{aligned} u_{11} &= 1 & (1) \\ 2u_{11} + u_{21} &= 0 & (2) \\ 3u_{11} + 2u_{21} + u_{31} &= 0 & (3) \end{aligned}$$

which can be solved by inspection to get

$$U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad (4)$$

Similar calculations yield

$$U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \text{ and } U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \tag{5}$$

Put together this gives the matrix U as

$$U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \tag{6}$$

Now that we have identified the matrix U explicitly, we can answer any questions about it. The present problem asks for its determinant $|U|$. A straightforward calculation gives

$$|U| = 1 \times (-1) + 2 \times (-7) + 2 \times 9 = 3 \tag{7}$$

A slightly better approach is to put together the three equations given in the statement of the problem to give the single matrix equation

$$AU = B \tag{8}$$

where A and B are the matrices given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \tag{9}$$

and,

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The matrix A is non-singular since its determinant is 1 which is non-zero. Therefore A^{-1} exists and so from (8) we get

$$U = A^{-1}B \tag{11}$$

This gives us an alternate method for calculating the matrix U. But the computations involved in finding A^{-1} are not substantially different from those in solving the three equations (1), (2) (3) simultaneously. Moreover, we still have to multiply the two matrices A^{-1} and B. So there is not much saving in this approach.

But there is a slicker way to find $|U|$ even without finding U. Taking determinants of both the sides in (8),

$$|A||U| = |B| \tag{12}$$

From (9) we know $|A| = 1$. Also, from (10) we compute $|B| = 3$. Putting this into (12) gives $|U| = 3$. This solution is certainly more elegant in terms of its approach, which is based on the multiplicativity property of determinants (i.e. the fact that the determinant of a product of two matrices is the product of the determinants.) Surely, it is much easier to multiply determinants (which are some real numbers) than to multiply two 3×3 matrices which is a nasty job.

Of course, evaluating a 3×3 determinant simply by expanding all terms is not a very exciting job either. But in this particular problem, this part too is drastically simplified because of certain particular features of the matrices A and B. Note that for the matrix B all the entries below the main diagonal are zero. A matrix with this property is called **upper triangular**. Similarly, the matrix A given by (9) is **lower triangular** because all entries above the main diagonal vanish. In either case, the determinant of the matrix has only one non-zero term, viz. the product of the diagonal elements. So the determinants of B and of A can be written down simply by inspection.

This problem is excellent because it rewards those who look for elegant solutions before proceeding with brute force computations.

Q. 31 The sum of the elements of U^{-1} is

(A) -1 (B) 0
(C) 1 (D) 3

Answer and Comments: (B). Once again, the most straightforward approach would be to begin by calculating the matrix U^{-1} explicitly. If we have already identified U by (6) in the solution to the previous problem, then using the standard formula for the inverse in terms of the cofactors of the elements of a matrix, one can show, by sheer computation, that

$$U^{-1} = \frac{\text{adj }(U)}{|U|} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix} \tag{13}$$

from which it is trivial to verify that the sum of the elements of U^{-1} is 0.

But the computations involved in finding U^{-1}, while inherently simple, are highly repetitious and prone to errors. So once again we look for some better method. Note that we are not asked to find the matrix U^{-1} *per se* but only the sum of its elements. Just as in the last problem, we were able to find the determinant of the matrix U even without finding U explicitly, let us see if we can do something similar here.

We begin by an observation which is applicable to any square matrix $C = (c_{ij})$ of order 3 (or of any order n for that matter). It is well-known that the columns of C are precisely the product matrices $C\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $C\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively. Adding these three, we get

$$C\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_{11} + c_{12} + c_{13} \\ c_{21} + c_{22} + c_{23} \\ c_{31} + c_{32} + c_{33} \end{bmatrix} \tag{14}$$

Verbally, multiplying a matrix on the right by the column vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives a column vector whose entries are the row sums of the matrix C. An entirely analogous argument shows that $[1\ 1\ 1]C$ is a row vector whose entries are the column sums of the matrix C.

What if we perform both the operations, i.e. premultiplication by the row vector $[1\ 1\ 1]$ and also post-multiplication by the column vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$? Then, as expected, we get a 1×1 matrix whose lone entry is the sum of all the entries of C. It is customary to denote a 1×1 matrix by its lone entry. We then have

$$[1\ 1\ 1]\ C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} \tag{15}$$

This simple formula allows us to recast our goal. We want the sum, say S, of the elements of the matrix U^{-1}. Because of (15) we can rewrite this as

$$S = [1\ 1\ 1]\ U^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{16}$$

Of course, merely recasting a problem does not solve it! Indeed, if we have already calculated the matrix U^{-1}, then it is foolish to use (16). Instead, it is far better just to look at the entries of U^{-1} and add. The true worth of a formula like (16) comes when we have not explicitly calculated the matrix U^{-1} but can nevertheless identify it in some other way, e.g. as the product of some simpler matrices. This is indeed the case in the present problem. For, since both A and B are invertible, from Equation (8) we have $U = A^{-1}B$ and hence $U^{-1} = B^{-1}A$. Putting this into (16), we now have

$$S = [1\ 1\ 1]\ B^{-1}A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{17}$$

Here S, which is simply a real number, is expressed as a product of four matrices! Using associativity of matrix multiplication we are free to group any of these together as long as we do not change their order. In particular, we can rewrite S further as

$$S = PQ \tag{18}$$

where

$$P = [1\ 1\ 1]\ B^{-1} \tag{19}$$

$$\text{and} \quad Q = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{20}$$

We already know that Q is a column vector whose entries are the row sums of A. From (9), we have

$$Q = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \tag{21}$$

Unfortunately, P cannot be computed so readily. We know it is a row vector whose entries are the column sums of the matrix B^{-1}. The trouble is that even though we know B from (10), we have not yet found its inverse. It may thus appear that we have not really gained anything by this approach over the earlier approach. The only difference is that instead of the matrix U we now have to find the inverse of the matrix B. But this is no small gain. First, we are given B whereas we have to *find* U (which so far we have escaped). Secondly, even if we compute U as in (6), computing its inverse is considerably more laborious than finding the inverse of B. This happens again because B is an upper triangular matrix. It is easy to show that the inverse of every such matrix (if it exists) is also upper triangular and further that its diagonal entries are precisely the reciprocals of the corresponding diagonal entries of the original matrix. Even if we do not know this result, we can find B^{-1} simply by solving some equations.

So, let $B^{-1} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$. Then $BB^{-1} = I$ gives

$$\begin{bmatrix} x_1 + 2x_2 + 2x_3 & y_1 + 2y_2 + 2y_3 & z_1 + 2z_2 + 2z_3 \\ 3x_2 + 3x_3 & 3y_2 + 3y_3 & 3z_2 + 3z_3 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

which is a system of 9 equations in 9 unknowns. A similar system for finding U^{-1} would be horrendous. But not the present one. For we immediately get $x_3 = 0, y_3 = 0, z_3 = 1$ and thereafter $x_2 = 0, y_2 = 1/3, z_2 = -1$ and finally the first row entries, viz. $x_1 = 1, y_1 = -2/3$ and $z_1 = 0$. Hence

$$B^{-1} = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

(Note, incidentally, that B^{-1} is upper triangular as we had remarked it would have to be and, further also, that its diagonal entries are the reciprocals of the corresponding diagonal entries of B. In fact, those who know this already could have incorporated this information to reduce the number of unknowns in (22) from 9 to 3 and thereby simplify the calculations still further.)

Putting (23) into (19), we get

$$P = [1 \quad -1/3 \quad 0] \tag{24}$$

Finally, putting (18), (21) and (24) together, we have

$$S = PQ = [1 \quad -1/3 \quad 0] \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = 0 \tag{25}$$

which completes the solution.

As in the last problem, in this problem too, the gap of labour needed between the elegant and the brute force solutions is remarkable. But the difference is that in the last problem the key idea in the elegant solution was a familiar one, viz. the multiplicativity property of determinants. And once you hit it, hardly any numerical work had to be done. Comparatively, the key idea behind the elegant solution to the present problem, viz. Equation (15), is not so frequently used and hence not easy to come up with. Moreover, you have to do some numerical work to get B^{-1} anyway. As a result, even though the present problem is also very good, it is a shade below the last one. True elegance lies in using something usual in an unusual way.

Q. 32 The value of $[3 \;\; 2 \;\; 0]\, U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(A) 5 (B) 5/2
(C) 4 (D) 3/2

Answer and Comments: (A). In the last problem, even if we had calculated the matrix U explicitly, there was considerable more work to do in the brute force method to find the sum of all the entries of

U^{-1}. In the present problem, on the other hand, if we have calculated U already, then the best way to get the answer is to carry out the multiplications. Thus,

$$
\begin{aligned}
[3\ \ 2\ \ 0]\, U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} &= [3\ \ 2\ \ 0] \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \\
&= [-1\ \ 4\ \ 4] \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \\
&= -3 + 8 = 5 \qquad (26)
\end{aligned}
$$

In the last two problems, we could avoid the computation of the matrix U by resorting to some elegant tricks. There seems to be no way to do so in the present problem. The only thing we can possibly try is to write U as $A^{-1}B$ by (11). Then reasoning as in the elegant solution to the last problem, the desired number, say T, equals

$$
\begin{aligned}
T &= [3\ \ 2\ \ 0]\, U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = [3\ \ 2\ \ 0]\, A^{-1} B \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \\
&= LM \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
L &= [3\ \ 2\ \ 0]\, A^{-1} \qquad (28) \\
\text{and}\quad M &= B \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \qquad (29)
\end{aligned}
$$

As we already know B by (10), computation of M is immediate.

$$
M = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} \qquad (30)
$$

The computation of L is also equally easy. But first we must find A^{-1}. As A is a lower triangular matrix, its inverse can be found by a method which is analogous to the one we used in the last problem for finding the inverse of B which was upper triangular. In fact, we now use the comments made there after finding B^{-1} to start with a considerably simplified format for A^{-1}. Since we already know that A^{-1} is going to be lower triangular and also that its diagonal entries will be the reciprocals of the corresponding diagonal entries of A, right at the start, we take A^{-1} in a highly simplified form, viz. $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ where we have only three unknowns, viz. a, b and c. Then the requirement $AA^{-1} = I$ spells out as

$$\begin{bmatrix} 1 & 0 & 0 \\ 2+a & 1 & 0 \\ 3+2a+b & 2+c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

from which we get $a = -2, c = -2$ and $b = 1$. Therefore

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \tag{32}$$

We are now in a position to compute L using (28), giving

$$\begin{aligned} L &= [3 \;\; 2 \;\; 0]\, A^{-1} \\ &= [3 \;\; 2 \;\; 0] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\ &= [-1 \;\; 2 \;\; 0] \end{aligned} \tag{33}$$

Putting (33) and (30) into (27), we finally get

$$\begin{aligned} T = LM &= [-1 \;\; 2 \;\; 0] \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} \\ &= -7 + 12 = 5 \end{aligned} \tag{34}$$

which is the same answer as before. Unlike in the last two problems, in the present problem, the difference between the two approaches (one with finding U and the other without finding it explicitly) is not very dramatic. The fact that the row vector and the column vector appearing in the statement of the problem are transposes of each other would have been helpful had the matrix U been symmetric. In particular, this would have been the case if A^{-1} equaled the transpose of B (or, equivalently, B^{-1} were the transpose of A). For, in that case, the expression asked would have been of the form $[3\ \ 2\ \ 0]\ B^tB \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ which means the row vector L above would have been the transpose of the column vector M. But in that case LM would have been simply the sum of the squares of the entries of M. Summing up, if B^{-1} (which we calculated in (23)) had come out to be the transpose of A (which is given to us in the preamble), then there would indeed have been a significant short cut to the answer. (It is possible that the data in the original problem was so designed as to make U symmetric, but some of the figures got changed in its retrieval.)

The simplifications in the second problem in this comprehension were due to the fact that that the matrix U could be expressed as $A^{-1}B$ where the matrices A^{-1} and B are, respectively, lower and upper triangular. Such a factorisation of a square matrix is called a ***LU*-decomposition** of it. Here L stands for 'lower triangular' and U for 'upper triangular'. If the matrix is symmetric then the two factors can further be chosen to be the transposes of each other. It takes some work to find an LU-decomposition of a matrix. But once it is obtained, it considerably simplifies the solution of systems of linear equations.

Of all the four comprehensions, the bunch of the problems in the last one is the best. The three problems are intimately related to each other unlike in the comprehension about geometry, where there is hardly any common theme. There is a common theme in the problems about approximations of definite integrals. But in absence of any elaboration of it in the preamble, the problems look obscure and clumsy.

Sadly, the paper-setters have not done justice to the purpose of 'comprehension' as it is understood in the context of examinations. Traditionally, examinations like the JEE are highly problems oriented. As a result, while preparing for them, the students concentrate heavily on solving problems, often by analogy with some familiar problems from the huge banks of solved problems at their disposal. They often totally ignore the ‘theory’ or the thought that goes into the solutions. So the adept ones among them can pull every conceivable trick needed to solve a problem but often fail to comprehend even half a page of a mathematical text expounding some concepts. Comprehensions were expected to correct this anomaly. None of the four comprehensions given here does this job. Each is as problem-oriented as the rest of the paper.

SECTION IV

Answers are to be given as four-digit integers.

Q. 33 If the roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d (where a, b, c, d are distinct numbers), then the value of $a + b + c + d$ is ....

Answer and Comments: 1210. The given conditions readily translate into a system of four equations in the four unknowns a, b, c, d, viz.

$$\begin{aligned} a + b &= 10c && (1) \\ ab &= -11d && (2) \\ c + d &= 10a && (3) \\ cd &= -11b && (4) \end{aligned}$$

So the straightforward approach to the solution would be to solve this system for a, b, c, d and add their values. But one has to be wary that in doing so one is not aiming too high. The focus should be on what is asked. (See the first tip given in Comment No. 3 of Chapter 24.) Here we are not asked the individual values of a, b, c, d but only their sum. So there may be a simpler way to get $a+b+c+d$ without first finding each of a, b, c, d.

Indeed the symmetry of the equations suggests one such way. If we add (1) and (3) we get

$$b + d = 9(a + c) \qquad (5)$$

So, we would be through if we can find $a + c$. Similarly, if we multiply (2) and (4), we get $abcd = 121bd$. If b and d are both non-zero, this gives

$$ac = 121 \qquad (6)$$

What happens if either b or d vanishes? Suppose, for example that $b = 0$. Then by (2), d would also vanish, contradicting that a, b, c, d

are given to be distinct. Similarly, vanishing of d would force that of b too by (4), and hence a contradiction. So, (6) is true. (This is a sad feature of the objective type tests. A scrupulous student will spend time eliminating the degenerate possibilities. An easy going candidate who cannot even think of them gets rewarded in terms of the time saved, because no reasoning is to be shown. Only the final answer matters.)

To get more relationship between a and c, we use the fact that a is a root of the first quadratic and c is a root of the second. Coupled with (6), these give, respectively,

$$\begin{aligned} a^2 &= 1210 + 11d & (7) \\ \text{and} \quad c^2 &= 1210 + 11b & (8) \end{aligned}$$

(We could also have gotten (7) by multiplying (1) by a and then using (2) and (6). Similarly, there is an alternate derivation for (8). Mathematically, this is not surprising because the equations (1) to (4) together are equivalent to the data in the problem. But depending on the inclination of a particular student, it may happen that one method strikes him as more natural.)

We add these (7) and (8) and further add $2ac$ (=242) to get

$$(a+c)^2 = 2652 + 11(b+d) = 99(a+c) + 2652 \qquad (9)$$

where we have used (5) too. The problem is now essentially solved. We treat (9) as a quadratic in $(a+c)$, solve it and then by (5) get the value of $b+d$ too.

But the details need to be attended to. In writing the discriminant of the quadratic, it is foolish to expand $(99)^2$. Instead, let us not forget that the figure 2652 is divisible by 121 from its very construction and 121 is the square of 11 which is a factor of 99. Because of these simplifications, we get

$$\begin{aligned} a+c &= \frac{99 \pm 11\sqrt{(9)^2 + 4 \times 22}}{2} = \frac{99 \pm 11\sqrt{169}}{2} & (10) \\ &= \frac{99 \pm 11 \times 13}{2} = \frac{11(9 \pm 13)}{2} \\ &= 121 \ \text{ or } \ -22 & (11) \end{aligned}$$

Once again, a scrupulous student will worry what happens if $a + c = -22$. In that case, because of (6), we shall get $a = c = -11$. But this contradicts that a, b, c, d are all distinct. So he will conclude that

$$a + c = 21 \tag{12}$$

The unscrupulous one will, naturally, take the express train to (12). If at all the thought of considering the possibility $a + c = -22$ occurs to him, he will discard it on the practical ground that the answer is to be filled in must be a positive integer!

After reaching (12), both the scrupulous and the unscrupulous candidates will use (5) to get $b + d = 121 \times 9$. The stupid ones will compute this to 1089. The smart ones will retain it as it is and add it to 121 to get $121 \times (9 + 1) = 1210$ as the value of $a + b + c + d$. Ultimately all get the same answer!

The problem is a reasonable problem on quadratic equations in a conventional type examination, where some partial credit could have been reserved for dismissing the degenerate cases $bd = 0$ and $a + c = -22$. In an examination where it is only the final, numerical answer that matters, a sincere student who ponders on such fine points effectively wastes his time. To discourage sloppiness, occasionally there should be some problems where the answer lies in the degenerate cases.

Q. 34 The value of $\dfrac{5050 \int_0^1 (1 - x^{50})^{100} dx}{\int_0^1 (1 - x^{50})^{101} dx}$ is ..... .

Answer and Comments: 5051. Evidently, it is unthinkable to evaluate the integrals individually by expanding the integrands by the binomial theorem. We must resort to some trick to evaluate them. But as in the last problem, it pays to carefully focus on what is asked. The problem merely asks for the ratio of the two integrals rather than their individual values. So what really matters is the mutual relationship of the two integrals. They are strikingly similar except for the exponents occurring in their integrands. Moreover these exponents are consecutive integers.

All this suggests that each integral is a member of a sequence of similar integrals and we are to find a formula for the relationship between two consecutive terms of this sequence. Such a relationship is popularly called a reduction formula. They are discussed in Chapter 18.

This gives one possible line of attack. For every positive integer n we let

$$I_n = \int_0^1 (1 - x^{50})^n dx \tag{1}$$

The problem asks us to evaluate the ratio $\dfrac{I_{100}}{I_{101}}$. This gives an implicit hint that the reduction formula for I_n must be of the form where the ratio I_n/I_{n+1} is expressed as a function of n, say $f(n)$. Our interest is in $f(100)$. The factor 5050 in the numerator is evidently inserted to make the answer come out to be a whole number. It gives a clue that $f(100)$ should be a rational number with 5050 (or some factor of it) in its denominator. As there is nothing special about 100, we expect, more generally, that $f(n)$ should come out to be a ratio of two polynomial expressions in n. Moreover, since $5050 = 50 \times 101$, in general we expect that the ratio I_n/I_{n+1} should have a factor $n + 1$ in the denominator.

With this preamble, we now proceed to obtain a reduction formula for $I_n = \int\limits_0^1 (1-x^{50})^n$. To relate I_{n+1} to I_n, we must decrease the exponent $n+1$ in the integrand of I_{n+1} by 1. For this we have to take derivatives. So, integration by parts is the right tool. If we apply it, we get

$$\begin{aligned}
I_{n+1} &= \int_0^1 (1 - x^{50})^{n+1} dx \\
&= x(1 - x^{50})^{n+1}\Big|_0^1 + 50(n + 1) \int_0^1 x^{50}(1 - x^{50})^n dx \\
&= 50(n + 1) \int_0^1 x^{50}(1 - x^{50})^n dx \qquad (2)
\end{aligned}$$

The trouble is that the new integral we got has a factor of x^{50} besides the power $(1 - x^{50})^n$ in its integrand. So it is not quite I_n. But we can come out of this difficulty if we rewrite x^{50} as $(x^{50} - 1) + 1$ and hence rewrite the product $x^{50}(1 - x^{50})^n$ as $(1 - x^{50})^n - (1 - x^{50})^{n+1}$. This is precisely the difference of the integrands in I_n and I_{n+1}. As a result, we can reduce (2) further to get

$$I_{n+1} = 50(n+1)(I_n - I_{n+1}) \tag{3}$$

Or, in a simplified form,

$$\frac{I_n}{I_{n+1}} = \frac{50(n+1)+1}{50(n+1)} \tag{4}$$

Putting $n = 100$, $\dfrac{5050 I_{100}}{I_{101}} = 5051$ which is what is asked.

The problem is reasonable once you get the key idea needed in it, viz. the reduction formulas, which is not difficult to arrive at. But the paper-setters have played a little dirty game by giving the integrand as a power of $(1 - x^{50})$. Obviously, the exponent 50 is related to the numbers 100 and 101 in a very obvious way. But the solution is equally applicable if we replace 50 by any positive integer m. The only change would be that in (3), the coefficient 50 would be replaced by m. A candidate who tries to infer anything from the relationship between 50 and 100 will be only wasting his time. Maybe this was intended. Sometimes, the ability not to get carried away by false clues is an asset.

Q. 35 If $a_n = \frac{3}{4} - (\frac{3}{4})^2 + (\frac{3}{4})^3 - \ldots + (-1)^{n-1}(\frac{3}{4})^n$ and $b_n = 1 - a_n$, then the least natural number n_0 such that $b_n > a_n$ for all $n > n_0$ is ....

Answer and Comments: 5. This is a straightforward problem about geometric progressions and inequalities, especially of powers. The purpose of introducing b_n is not clear. Since $b_n = 1 - a_n$, the inequality $b_n > a_n$ is equivalent to $a_n < \frac{1}{2}$. The problem could have as well be framed directly in terms of the latter inequality. Leaving this reduction to the candidate does not serve to test any great quality on his part, except possibly the ability not to get bogged down by needlessly clumsy formulation of a problem.

Now, coming to the solution, a_n is the sum of a G.P. with the first term $\frac{3}{4}$ and common ratio $-\frac{3}{4}$. Therefore by the formula for the sum of the terms of a G.P., we have

$$\begin{aligned} a_n &= \frac{\frac{3}{4}(1-(-\frac{3}{4})^n)}{1+\frac{3}{4}} \\ &= \frac{3}{7}\left(1-(-\frac{3}{4})^n\right) \end{aligned} \tag{1}$$

Since $|-\dfrac{3}{4}| < 1$, we know from (1) that $a_n \to \dfrac{3}{7}$ as $n \to \infty$. In particular, since $\dfrac{3}{7} < \dfrac{1}{2}$, there will be some positive integer n_0 such that $a_n < \dfrac{1}{2}$ for all $n > n_0$. The problem asks us to find the least integer n_0 having this property.

We begin by recasting the inequality. Note that $a_n < \dfrac{1}{2}$ if and only if $1-(-\frac{3}{4})^n < \dfrac{7}{6}$, which in turn, is equivalent to

$$-\frac{1}{6} < \left(-\frac{3}{4}\right)^n \tag{2}$$

The L.H.S. is negative while the R.H.S. is positive for even n. So (2) holds for all even values of n. For odd values of n, say $n = 2m+1$, (2) reduces to $-\dfrac{1}{6} < -\left(\dfrac{3}{4}\right)^{2m+1}$, or equivalently,

$$\left(\frac{3}{4}\right)^{2m+1} < \frac{1}{6} \tag{3}$$

The L.H.S. decreases as m increases because the base $\dfrac{3}{4}$ is less than 1. In fact the L.H.S. tends to 0 as $m \to \infty$. So, there is some value of m, say m_0 for which (3) holds. By monotonicity, it will then also hold for all $m \geq m_0$.

The least non-negative integer m for which (3) holds can be found by taking logarithms w.r.t. some base greater than 1. Suppose we take logarithms with base 10. Then keeping in mind that the logarithms of both the sides are negative, we first reverse this inequality by taking reciprocals before applying logarithms. Thus, (3) holds if and only if

$$\left(\frac{4}{3}\right)^{2m+1} > 6 \tag{4}$$

or equivalently,

$$2m+1 > \frac{\log 6}{\log(4/3)} = \frac{\log 6}{\log 4 - \log 3} \tag{5}$$

and finally, if and only if,

$$m > \frac{1}{2}(\frac{\log 6}{\log 4 - \log 3} - 1) = \frac{1}{2}(\frac{2\log 3 - \log 2}{2\log 2 - \log 3}) \tag{6}$$

Thus the smallest integer m for which (3) holds is the smallest integer which exceeds the R.H.S. of (6). But to evaluate this, we must be given the values of the common logarithms of 2 and 3 (or logarithms w.r.t. some other base). As these are not given, it is best to find the least m for which (3) holds simply by trial. For $m = 0, 1, 2$ the L.H.S. of (3) equals $\frac{3}{4}, \frac{27}{64}, \frac{243}{1024}$ respectively. They are all bigger than $\frac{1}{6}$. But the last one is already less than $\frac{1}{4}$. So, for the next value of m, viz. $m = 3$ we have $(\frac{3}{4})^7 < \frac{1}{4} \times \frac{9}{16} = \frac{9}{64} < \frac{1}{6}$. Hence the smallest odd value of n for which (2) holds is $n = 7$. But we already saw that it holds for all even n anyway. Therefore if (2) is to hold for all $n > n_0$, then the smallest such n_0 is 5. If instead of $n > n_0$ we were given $n \geq n_0$, the answer would be $n_0 = 6$. (With a calculator, the numerical value of the R.H.S. of (6) comes out to be 2.614131259. Hence the least integer exceeding it is indeed 3 as we found by trial.)

The crux of the problem involves standard concepts and techniques. The last part where we have to find the answer by trial is somewhat unusual. When the answer involves logarithms and devices such as log tables or calculators are not allowed, it is a standard practice to leave the answer in a form like (6). (See for example, the JEE 2000 question, given in Comment No. 6 of Chapter 19.) Alternatively, the statement of the question itself may give the relevant logarithms. In the present problem, giving the values of $\log 2$ and $\log 3$ would have enabled a candidate to get the answer from (6) without having to resort to trial and error. But probably, most students would have found the answer by trial anyway and since it is an objective type question, the difference would never be known. If the requirement $a_n < \frac{1}{2}$ appearing in the question (in a twisted from) had been replaced by something like $a_n < \frac{3}{7} + 10^{-3}$, then finding the least n by trial would have been too time-consuming and the use of logarithms would have been mandatory. Or, the data in the problem could have been retained as it is, but instead of finding the integer n_0, the problem could have been asked in a multiple choice format with the alternatives containing expressions involving $\log 2$ and $\log 3$.

Q. 36 If $f(x)$ is a twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2$ and $f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeros of $g(x) = (f'(x))^2 + f''(x)f(x)$ in the interval $[a, e]$ is ....

Answer and Comments: 6. We are not given the function $f(x)$ explicitly. Nor are we given anything (such as a differential equation) from which we can identify $f(x)$ or $g(x)$. If we could identify $g(x)$ explicitly, then we can answer the question using the particular properties of that function, just as we answer questions about zeros of trigonometric functions by using various trigonometric identities.

Besides the existence of $f''(x)$, the only information we have about $f(x)$ is its values at the points a, b, c, d, e, which too are unknown except for their relative order. So, this is an excellent example of a problem where the lack of information is itself a clue. (See Comment No. 2 of Chapter 24 for an elaboration of this apparently paradoxical situation.)

So, our only recourse is some theorems from calculus which assert the existence of zeros of abstract functions under certain conditions such as continuity and differentiability. At the JEE level, there are only two such theorems. One is the Intermediate Value Property (IVP) which says that if $h(x)$ is continuous on an interval $[p, q]$ and $h(p), h(q)$ are of opposite signs, then $h(x)$ has at least one zero in (p, q). (See Comment No. 3 for a proof and Comment No.s 4,5 and 6 for applications.) Another theorem, called Rolle's theorem (Comment No. 8 of Chapter 16) asserts that if $h(x)$ is continuous on $[p, q]$ and differentiable on (p, q) and $h(p) = h(q) = 0$, then the derivative $h'(x)$ has at least one zero in (p, q). A slightly improved version of this says that all we need (besides the continuity and differentiability requirements) is that h has equal values at the end-points, i.e. that $h(p) = h(q)$ and not that these values be 0. The improved version follows by applying the earlier version to the function $h(x) - h(p)$, or, as a very special case of Lagrange's Mean Value Theorem (MVT).

Let us now see which of these two results can help us in finding the zeros of the function $g(x) = (f'(x))^2 + f''(x)f(x)$. The IVP is not of much help directly because although we are given the values of $f(x)$ at some points, from that we are not in a position to find the values of $g(x)$ (or even to determine their signs) at any point. Let us, therefore, look at the only remaining hope, viz. Rolle's theorem. For this we must first recognise $g(x)$ as a derivative of some function. And this is indeed the case. In fact,

$$g(x) = (f'(x))^2 + f''(x)f(x) = \frac{d}{dx}(f(x)f'(x)) \tag{1}$$

This recasting is the key to the solution. By Rolle's theorem, between any two zeros of $f(x)f'(x)$ there is at least one zero of $g(x)$. So the problem is now reduced to finding the minimum number of zeros of the function $f(x)f'(x)$. Note that $f(p)f'(p) = 0$ if and only if either $f(p) = 0$ or $f'(p) = 0$ (or both). So the problem further reduces to counting the minimum numbers of zeros of each of the two functions $f(x)$ and $f'(x)$. If we can show that they have, say, at least m and n zeros respectively in $[a, e]$, and further that none of these zeros is a common zero of both, then it will follow that $f(x)f'(x)$ has at least $m + n$ zeros and consequently that its derivative $g(x)$ has at least $m + n - 1$ zeros.

Let us tackle these two tasks separately. First the zeros of $f(x)$ in $[a, e]$. We are already given that $f(a) = 0$ and $f(e) = 0$. So, these are two zeros of $f(x)$. The remaining zeros have to come from the IVP. Since $f(b) = 2 > 0$ and $f(c) = -1 < 0$, there exists $z_1 \in (b, c)$ such that $f(z_1) = 0$. Similarly, from $f(c) = -1 < 0$ and $f(d) = 2 > 0$, there exists $z_2 \in (c, d)$ such that $f(z_2) = 0$. Further,

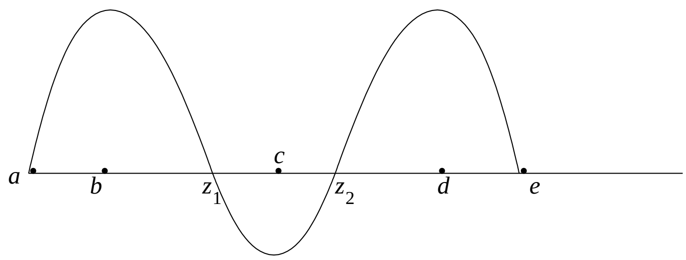
$$a < b < z_1 < c < z_2 < d < e \tag{2}$$

So these four zeros of $f(z)$, viz. a, z_1, z_2, e are distinct. By Rolle's theorem between every consecutive two of them, there is at least one zero of $f'(x)$. Hence there exist $z_3 \in (a, z_1), z_4 \in (z_1, z_2)$ and $z_5 \in (z_2, e)$ such that $f'(z_3) = f'(z_4) = f'(z_5) = 0$. Further,

$$a < z_3 < z_1 < z_4 < z_2 < z_5 < e \tag{3}$$

Thus we have found at least seven distinct zeros of $f(x)f'(x)$. By Rolle's theorem again, its derivative, which is $g(x)$ has at least 6 zeros in (a,e).

Strictly speaking, the solution is not yet complete. We have shown by some argument that if $f(x)$ satisfies the conditions in the problem, then the corresponding function $g(x) = (f'(x))^2 + f(x)f''(x)$ has at least 6 zeros in (a,e). But conceivably, using some other argument we (or somebody else) can show that under the same conditions $g(x)$ must have at least 7 distinct zeros. The problem demands that we show that this is impossible, or in other words, that in general 6 is the best lower bound on the number of zeros of $g(x)$. To do this it suffices to give an example of a function $f(x)$ which satisfies all the conditions of the problem and has exactly 6 and no more zeros. The graph of one such function is shown below. Note that here z_3, z_4, z_5 coincide with b, c, d respectively.



Constructing such a function by an explicit formula is a little messy. If instead of $f(c) = -1$ we had $f(c) = -2$, then the function $f(x) = 2\sin x$ with $a = 0, b = \pi/2, c = 3\pi/2, d = 5\pi/2$ and $e = 3\pi$ would work. Here $g(x)$ comes out to be $\frac{d}{dx}(4\sin x\cos x) = 4\cos 2x$ and it has exactly 6 zeros in the interval $[0, 3\pi]$, viz. all odd multiples of $\frac{\pi}{4}$ from $\frac{\pi}{4}$ to $\frac{11\pi}{4}$.

Unlike the last three questions, here the examiners have attempted to pose a question of theoretical calculus as a fill in the blank question. There are some inherent limitations on the success of such attempts. The real crux of such problems is the arguments that are needed in justifying the steps rather than some number associated incidentally with the problem. When the correct answer is a four digit number like that of Q. 33 or Q. 34, a correct answer is unlikely to be arrived at by wrong reasoning. But that is not the case for a problem like the present one where the answer is a small one, viz. 6. The figure 6 is easy to come up with by a totally wrong argument. For example, a candidate may think that since $f(x)$ has two given zeros, so must be the case with $f'(x)$ and also with $f''(x)$. Therefore the function $(f'(x))^2 + f(x)f''(x)$ has 6 zeros since it involves all three, viz. $f(x), f'(x)$ and $f''(x)$.

But the real loss is that a candidate who has the fineness of realising that the solution is not over until 6 is shown to be the best answer stands to get no reward. Similarly, as commented earlier, even though Q. 33 was purely computational, a candidate who scrupulously dismisses the degenerate cases there is only wasting his time.

Fill-in-the-blank type questions were asked in the JEE long time ago. That time they were evaluated manually. As a result, the examiner could adjudicate whether to allow $\sin \frac{\pi}{2}$ as a correct answer even though it was not simplified to 1. But then controversies would arise as to what degree of simplification is expected from the candidates. For example, in a combinatorial or probability problem, it is reasonable to expect that something like $\binom{9}{3}$ be simplified to 84 and equally reasonable not to expect any simplification for something like $\binom{365}{20}$. But what about something like $\binom{13}{7}$?

Asking that the answer be a whole number eliminates such controversies. This, in fact, is close to the style adopted for the answers of the Main Problem in each chapter of *Eductive JEE Mathematics*. The merits and demerits of this practice are discussed in Comment No. 1 of Chapter 2. The major demerit is that a candidate who has correctly done all the work except for a minor numerical slip loses heavily as there is little to warn him of his mistake.

For the past several years, the objective part (the Screening Paper) consisted wholly of multiple choice questions. But some questions are simply not suited to be asked in this format. So they had to be asked only in the Main Paper, where the candidates would have to show the reasoning. Now the second round is eliminated completely. Apparently, the intention is to use the fill in the blank type questions to fill the gap thus created. But out of the four questions in this Section only one (viz. Q. 34) can be said to fully meet the demand.

SECTION V

Match the entries given in the two columns in each question. One entry in the first column may have more than one matching in the second and vice versa.

Q. 37 Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at $(3, 0)$. Then

(i)	Area of ΔPQR	(A) 2
(ii)	Radius of circumcircle of ΔPQR	(B) 5/2
(iii)	Centroid of ΔPQR	(C) $(5/2, 0)$
(iv)	Circumcentre of ΔPQR	(D) $(2/3, 0)$

Answer: (i) $\leftrightarrow$ (A), (ii) $\leftrightarrow$ (B), (iii) $\leftrightarrow$ (D), (iv) $\leftrightarrow$ (C).

Comments: It is customary (and logical) to specify areas in terms of square units rather than a mere number. This convention has been followed in the wording of Q. 29. Had it been followed in this question too while giving the options in the second column, then the matching option for (i) in the left column could have been found without doing any work, because it is the only option which is expressed in terms of sq. units. That would be a superb tribute to the ability to get at the right answer by merely looking at the format of the answer, much the same way that a Sherlock Holmes can identify a murderer correctly simply by looking at the clothes of the suspects.

It is perhaps to avoid this unwarranted short cut that the paper-setters have given the area as a pure number. But even as the question stands, there is ample room for sneaking in. The first column lists four geometric attributes of the triangle PQR. Two of these ((iii) and (iv)), are certain points, viz. the centroid and the circumcentre of the triangle. When they are specified by coordinates, the corresponding entries in the right column have to be ordered pairs of real numbers. There are only two such entries, viz. (C) and (D). So, by sheer common sense, one of these two has to match (iii). Moreover, the centroid and the circumcentre coincide only for an equilateral triangle. So, if we can eliminate this possibility, which is often easy to do, we automatically get the match of (iv) as soon as we have found that of (iii). Also, if we can tell by inspection that one of (C) and (D) is outside the triangle, then it cannot possibly be the centroid.

In fact, in the traditional form of questions asking for matching the pairs, it was inherent that the matching be a bijective one. And the charm of such questions was that after making all but one matches correctly, the last one came in as a bonus. Apparently, the paper-setters have decided to disallow this bonus. That is why it is given that the same entry in the first column may have several matches in the second. This effectively means that in the first column, we are having four separate problems, each with one or more correct answers in the second column. And, we have to tackle them one-by-one.

A typical point on the parabola $y^2 = 4x$ is of the form $(t^2, 2t)$ for some value of the real parameter t. The slope of the tangent at this point is $\frac{1}{t}$ and so that of the normal is $-t$. Therefore the equation of the normal at this point is

$$y - 2t = -t(x - t^2) = t^3 - xt \tag{1}$$

Let the three points P, Q, R on the parabola correspond to t_1, t_2 and t_3 respectively. Then for these values of t, the normal is given to pass through the point $(3, 0)$. Combined with (1), this means that t_1, t_2, t_3 are the roots of the cubic equation

$$t^3 - t = 0 \tag{2}$$

(Points on a parabola the normals at which are concurrent are called **co-normal points.** So the present problem deals with a triangle whose vertices are co-normal points on a given parabola. We could also have started with the equation of the normal to a parabola $y^2 = 4ax$ in terms of its slope m, viz. $y = mx - 2am + am^3$. In that case, instead of (2) we would have gotten the cubic equation

$$m^3 - m = 0 \tag{3}$$

But thereafter we would have to express the points P, Q, R in terms of m, for which we would have to express m in terms of the parameter

t anyway. And that would take us to (2). The point is that such specialised formulas as the equation of a chord in terms of its midpoint, or the equation of a normal in terms of its slope, should be applied with discretion. Sometimes they do save precious time. But they are not golden tools all the time.)

Although a general cubic equation is not very easy to solve, (2) is an exception. By inspection, its roots are $0, 1$ and -1. As a result, we can take $P = (0, 0)$, $Q = (1, 2)$ and $R = (1, -2)$. The parabolical part of the problem is now over. From now onwards it is a simple problem in coordinate geometry. Taking the A.M. of the x-coordinates and also that of the y-coordinates of the vertices, we immediately get that the centroid of ΔPQR is at $(2/3, 0)$. As noted before, this leaves the circumcentre no other choice but $(5/2, 0)$. Still, to get this honestly, we note that the x-axis is the perpendicular bisector of the segment QR. So, the circumcentre must be of the form $(h, 0)$ for some h. Equating its distances with the three vertices we get $h^2 = (h - 1)^2 + 4$ which gives $h = 5/2$. Honesty has paid, because the circumradius is now h which is $5/2$. As for the area, ΔPQR is isosceles with base $QR = 4$ and altitude 1. So the area is 2 sq. units.

Basically, this is a trivial problem. It would have been a good problem if instead of (2), we ran into a cubic equation which was not easy to solve. In that case, the centroid would have been $(\dfrac{t_1^2 + t_2^2 + t_3^2}{3}, \dfrac{2(t_1 + t_2 + t_3)}{3})$. As the expressions in the numerators are symmetric functions of the roots t_1, t_2, t_3, they can be evaluated even without solving the cubic. (See Exercise (9.61) for a far more challenging problem of this spirit. Gone are the days when gems like this were asked in the JEE.)

Another criticism of this problem is that in the given triangle, side QR is the latus rectum of the parabola and the third vertex P is also the vertex of the parabola. This was also exactly the case in Q. 29 above, where too, the area of the triangle was asked. This duplication in the same paper is shocking to say the least. And to make it worse, Q. 29 itself has some overlap with Q. 10 as remarked there. So what we have is not a duplication but a triplication of ideas!

Q. 38 Match the following:

(i)	$\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \ln[(\sin x)^{\sin x}]dx$	(A)	1
(ii)	Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$	(B)	0
(iii)	Cosine of the angle of intersection of the curves $y = 3^{x-1}\ln x$ and $y = x^x - 1$	(C)	$6\ln 2$
(iv)	Some problem about the differential equation $\dfrac{dy}{dx} = \dfrac{2}{x+y}$ with $y(1) = 0$	(D)	$4/3$

Answer: (i) $\leftrightarrow$ (A), (ii) $\leftrightarrow$ (D), (iii) $\leftrightarrow$ (A), (iv) $\leftrightarrow$ (uncertain)

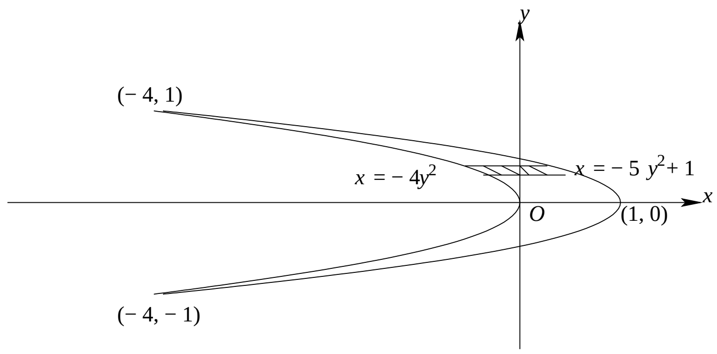
Comments: Unlike in the last question where all the four entries in the first column had a common setting, here it is a bunch of four totally unrelated problems. The details of the problem in (iv) were irretrievable. Had this happened in a question of matching the pairs in the traditional form, it would not have mattered. Once the other three entries in the first column are found matches, the match of the last one is fixed in heaven no matter what the problem is. That is not the case here. So, let us tackle the first three problems fully and the fourth one to the extent we can.

The integrand in (i) looks horrendous because of the exponentials in it with bases other than e. Let us convert it to an exponential with base e so that it looks less formidable. Since $\sin x = e^{\ln \sin x}$, we get $(\sin x)^{\cos x} = e^{\cos x \ln \sin x}$. If we differentiate this, we get $e^{\cos x \ln \sin x}\dfrac{d}{dx}(\cos x \ln \sin x) = e^{\cos x \ln \sin x}(\cos x \cot x - \sin x \ln \sin x)$. But if we convert the two powers to the base $\sin x$, we get

$$\frac{d}{dx}((\sin x)^{\cos x}) = (\sin x)^{\cos x}(\cos x \cot x - \ln[(\sin x)^{\sin x}]) \qquad (1)$$

But the R.H.S. is precisely the integrand in (i). So we are lucky. The integral then equals $(\sin x)^{\cos x}\Big|_0^{\pi/2} = 1^0 - 0^1 = 1$.

(ii) is a typical problem of evaluating an area bounded by two curves. Both the curves are parabolas. Going by the number of problems they have asked, it looks like the paper-setters this year are especially fond of parabolas! The two given parabolas meet when $-4y^2 = 1 - 5y^2$, which gives $y = \pm 1$ and $x = 4$. So the region bounded by them is as shown in the figure below. Clearly it is symmetric about the x-axis since both the curves are so. Therefore it suffices to find the area of the upper half. It is more convenient to do this by horizontal, rather than vertical slicing.



By horizontal slicing as shown the desired area, say A, equals

$$\begin{aligned} A &= 2\int_0^1 (-5y^2+1) - (-4y^2)dy \\ &= 2\int_0^1 1 - y^2 dy \\ &= 2(y - y^3/3)\Big|_0^1 = 4/3 \end{aligned} \tag{2}$$

As compared to some area problems in the past, the one this year is quite straightforward. But in the past, such questions would get about 5 to 8 minutes. For the present problem, the time allowed is less than one minute! Once again, it is no consolation that only the final answer needs to be shown. There is no smart way of simply guessing the answer. To arrive at it, you have to do a lot of work anyway, whether you show it or not.

In the case of Item (iii), by inspection we can identify $(1, 0)$ as a point of intersection of the two curves $y = 3^{x-1}\ln x$ and $y = x^x - 1$. The angle of intersection of the two curves is the angle between their tangents at the point of intersection. Now, by direct calculation,

$$\frac{d}{dx}(3^{x-1}\ln x) = \frac{3^{x-1}}{x} + 3^{x-1}(x-1)\ln 3 \ln x \tag{3}$$

whose value at $x = 1$ is 1. Hence the slope of the tangent to the first curve at the point $(1, 0)$ is 1. Similarly,

$$\begin{aligned} \frac{d}{dx}(x^x - 1) &= \frac{d}{dx}(e^{x \ln x} - 1) \\ &= x^x(\frac{d}{dx}(x \ln x)) \\ &= x^x(1 + \ln x) \qquad (4) \end{aligned}$$

whose value at $x = 1$ is also 1. Therefore the two curves have the same tangents at their point of intersection. Hence the angle between them is 0 and and its cosine is 1. Note that both the entries (i) and (iii) have the same match, viz. (A) in the second column, even though they are totally unrelated to each other.

Strictly speaking, the solution is not complete. We have found one point of intersection of the two curves by inspection. But there could be others which are not easy to find. (Occasionally, the paper-setters do miss some possible solutions, thereby making the problem almost impossible to solve. See the JEE 1996 and JEE 1987 problems discussed in Comment No. 14 of Chapter 10.) Moreover, in the present problem, the angles of intersection of the two curves at these other points may be different than the one at $(1, 0)$. As the question is framed to have a single answer, we simply have to *assume* that there are no other points of intersection. A candidate who tries to find them or to prove their non-existence is wasting his time. Yet another instance where scruples do not pay.

Finally, let us tackle Item (iv) to the extent we can by solving the differential equation given, viz.

$$\frac{dy}{dx} = \frac{2}{x + y} \qquad (5)$$

with the initial condition that $y = 0$ when $x = 1$.

As it stands (5) cannot be cast in the separate variables form. Nor does a substitution like $y = vx$ work as one sees by trial. But there is a trick. If the R.H.S. were $\frac{x+y}{2}$ instead of $\frac{2}{x+y}$, then the equation would have been a linear first order d.e. which can be solved by the method given in Comment No. 12 of Chapter 20. That suggests the trick. Why don't we interchange the roles of x and y? Normally, we take x as the independent variable and y as the dependent variable, i.e. a function of x. But there is nothing sacrosanct about this if we are only looking for solution curves which are of the form $f(x, y) = c$ where the roles of the two variables are on par. So, we take reciprocals of both the sides of (5) and get

$$\frac{dx}{dy} = \frac{x+y}{2} \tag{6}$$

(In JEE 2005 too, the Screening Paper had one d.e. which was amenable to the very same trick. See Q. 25 in the Educative Commentary on JEE 2005 by the author. But that time an alternate approach was also possible. That does not seem possible with the present problem.)

Rewriting (6) as

$$\frac{dx}{dy} - \frac{x}{2} = \frac{y}{2} \tag{7}$$

we get $e^{-y/2}$ as an integrating factor (see Comment No. 12 of Chapter 19) and

$$x = -y - 2 + ce^{y/2} \tag{8}$$

as the general solution, where c is an arbitrary constant. The initial condition $y = 0$ when $x = 1$ determines c as 3. Therefore the particular solution of the d.e. in Item (iv) is

$$x + y + 2 = 3e^{y/2} \tag{9}$$

We cannot proceed further because it is not known what is asked.

Q. 39 Match the following.

(i)	Two rays $x + y = \|a\|$ and $ax - y = 1$ intersect each other at a point in the first quadrant for all a in the interval (a_0, ∞). The largest value of such a_0 is	(A)	2
(ii)	Point (α, β, γ) lies on the plane $x + y + z = 2$. If $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ and $\hat{k} \times (\hat{k} \times \vec{a}) = \vec{0}$, then γ equals	(B)	1
(iii)	$\left\|\int\limits_0^1 (1 - y^2)dy\right\| + \left\|\int\limits_1^0 (y^2 - 1)dy\right\|$	(C)	$\left\|\int\limits_0^1 \sqrt{1-x}\, dx\right\| + \left\|\int\limits_{-1}^0 \sqrt{1+x}\, dx\right\|$
(iv)	If $\sin A \sin B \sin C + \cos A \cos B = 1$, then $\sin C$ equals	(D)	4/3

Answer: (i) $\leftrightarrow$ (B), (ii) $\leftrightarrow$ (A), (iii) $\leftrightarrow$ (C) and (D), (iv) $\leftrightarrow$ (B).

Comments: Another bunch of four totally unrelated problems. But there is even more. The normal practice in a match the pairs question is that the first column contains some problems and the second one contains their (usually numerical) answers. But in this question item (C) in the second column is an integral. When evaluated, it could as well come out to be the value of $\sin C$ in item (iv) in the first column. We would not know until we find both. So, in effect, in this single question there are *five* different problems, to be solved for 6 points. In terms of proportional time this means less than four minutes! In effect, then, each one of these five questions is to be understood and worked out in less than 48 seconds. Moreover, even though the problems are totally unrelated to each other, to claim the 6 marks for the question, *all* have to be answered correctly. This is almost like saying that in order to enter the I.I.T.'s you must not only do well in the JEE, but simultaneously you (or perhaps your sister) must also qualify for a medical admission!

Anyway, getting down to business, let us tackle these five problems, one-by-one. The word 'ray' in the statement of (i) is a little misleading. No harm would arise if it is replaced by 'line', or better still, by 'straight line'. A 'ray' is a half-line, i.e. a portion of a straight line lying entirely on one side of some point P on it. (The point P is then called the initial point of the ray.) In the present instance, the portion of the line $x + y = |a|$ in the first quadrant is not a ray but a line segment with end points $(|a|, 0)$ and $(0, |a|)$.

Assuming, therefore, that the word 'ray' simply means a 'line' we solve the two equations simultaneously to get the coordinates of their point of intersection as

$$x = \frac{|a| + 1}{a + 1} \tag{1}$$

$$\text{and} \quad y = \frac{a|a| - 1}{a + 1} \tag{2}$$

Since the point of intersection lies in the first quadrant, we have $x > 0$ and $y > 0$. As the numerator of x is always positive, we must have $a + 1 > 0$, which gives $a > -1$ or equivalently a lies in the interval $(-1, \infty)$. But we must not hastily conclude $a_0 = -1$, for, the positivity of y may restrict a still further. This indeed happens, because as the denominator of y is now known to be positive, we must also have $a|a| > 1$. This automatically rules out negative values of a and thereby implies $a|a| = a^2$. So the positivity of the numerator of y now implies $a > 1$, i.e. $a \in (1, \infty)$. Hence we must take a_0 as 1. (As in Q. 36, in a problem like this a complete solution does not end here. The interval of values which a could assume under the conditions of the problem was already narrowed down from $(-1, \infty)$ to $(1, \infty)$. Strictly speaking, we must now show that no further narrowing is possible. That is, for every $a \in (1, \infty)$, we must show that the two given lines intersect at a point in the positive quadrant. In the present case, no further work is needed for this, because for every $a \in (1, \infty)$, (1) and (2) show that x, y are both positive.)

Unlike the entry (i) in the first column, entry (ii) is quite clear and straightforward. The first condition simply means

$$\alpha + \beta + \gamma = 2 \tag{3}$$

(It is really difficult to see what is gained by giving this piece of data in such a twisted form. It does not test intelligence. It only slows down an intelligent student and thereby becomes more a test of his speed.) As for the vector triple product, we have, by a standard identity,

$$\hat{k} \times (\hat{k} \times \vec{a}) \quad = \quad (\hat{k} \cdot \vec{a})\hat{k} - (\hat{k} \cdot \hat{k})\vec{a} \tag{4}$$

Since $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ we have $\hat{k} \cdot \vec{a} = \gamma$. Also $\hat{k} \cdot \hat{k} = 1$. Putting these values in (4),

$$\hat{k} \times (\hat{k} \times \vec{a}) = \gamma\hat{k} - \vec{a} = \gamma\hat{k} - (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) = -\alpha\hat{i} - \beta\hat{j} \tag{5}$$

As we are given that $\hat{k} \times (\hat{k} \times \vec{a}) = \vec{0}$, we now get $\alpha = 0 = \beta$. Combining this with (2) we get $\gamma = 2$.

Entry (iii) in the first column contains two very easy integrals. Evaluating them, we get

$$\begin{aligned} \left| \int_0^1 (1 - y^2)dy \right| + \left| \int_1^0 (y^2 - 1)dy \right| &= \left| (y - \frac{y^3}{3})\Big|_0^1 \right| + \left| (\frac{y^3}{3} - y)\Big|_1^0 \right| \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \end{aligned} \tag{6}$$

Thus we get that (iii) matches with (D). Instead of going to Item (iv), let us first finish off with the two integrals in (C). Calculation of these two integrals is almost as easy as the ones in (iii). As both the integrands are positive (and the lower limits are smaller than the upper ones) the absolute value signs are redundant. Therefore,

$$\begin{aligned} \left| \int_0^1 \sqrt{1 - x}dx \right| + \left| \int_{-1}^0 \sqrt{1 + x}dx \right| &= \int_0^1 \sqrt{1 - x}dx + \int_{-1}^0 \sqrt{1 + x}dx \\ &= (-\frac{2}{3}(1 - x)^{3/2})\Big|_0^1 + (\frac{2}{3}(1 + x)^{3/2}\Big|_{-1}^0 \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \end{aligned} \tag{7}$$

So, we see that (iii) and (C) also match each other. We did this by actually computing both of them. And since all the four integrals are easy to evaluate, any improvement seems pointless. The problem would have been more interesting if the integrals on each side did not permit easy evaluations individually but their sum was easy to evaluate. And the problem would be still more interesting if the integrals were such that their evaluations were not easy but by suitable substitutions the integrals in (C) could be transformed to those in (iii). But all this can hardly be expected in a problem which is to be solved in 48 seconds.

We now come to the last part, viz. (iv) in the first column. We are given an equation

$$\sin A \sin B \sin C + \cos A \cos B = 1 \tag{8}$$

and are asked to determine the value of $\sin C$. Most people would instinctively take A, B, C to be the angles of a triangle. But this ought to have been specified in the statement of the problem. Without this hypothesis, $\sin C$ cannot be determined uniquely. Suppose, for example that $A = B = 0$. Then (8) holds for every value of C and hence nothing can be said about the value of $\sin C$ (except, of course, that it lies in $[-1, 1]$). So, we assume that that A, B, C are the angles of a triangle. In that case, the problem is a virtual replication of a 1986 JEE problem, solved in Comment No. 10 of Chapter 14 on trigonometric optimisation. Even the notations have not been changed! (In terms of time allowed, the difference is staggering. In 1986, you were given the answer. Specifically, you were asked to show (with reasoning) that the $a : b : c = 1 : 1 : \sqrt{2}$ and you got 9 minutes for it. Now you get 48 seconds to come up with $\sin C = 1$.)

Normally, when we have three unknowns A, B and C we need three equations to determine them. One of these is (8) while the other is given by our assumption, viz. $A + B + C = \pi$. In general two equations cannot determine the unknowns uniquely. The only way this can happen is that one side of the equation represents the optimum value of the other. As a simple example, if α, β lie in, say $[0, \pi]$, then from the value of $\cos\alpha + \cos\beta$ we cannot determine them uniquely. But if this value happens to be 2, then it is the maximum value of $\cos\alpha + \cos\beta$ and it can occur only when both $\cos\alpha$ and $\cos\beta$ both equal 1 each. That determines α, β.

We apply a similar reasoning here. Since $\sin A, \sin B$ are non-negative and $\sin C \leq 1$, the L.H.S. of (8) is at most $\sin A \sin B + \cos A \cos B$ which is simply $\cos(A - B)$, which can never exceed 1. So (8) forces the equality of $\cos(A - B)$ and 1. The only way this can happen is if $A = B$. Therefore, (8) now becomes

$$\sin^2 A \sin C + \cos^2 A = 1 \tag{9}$$

From this we see that $\sin C = \dfrac{1 - \cos^2 A}{\sin^2 A} = 1$.

Q. 40 Match the following:

(i)	If $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then $\tan t$ equals	(A)	$2\sqrt{2}$
(ii)	Sides a, b, c of a triangle ABC are in A.P. and $\cos\theta_1 = \frac{a}{b+c}$, $\cos\theta_2 = \frac{b}{c+a}$ and $\cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2}$ equals	(B)	1
(iii)	a line is perpendicular to $x + 2y + 2z = 0$ and passes through $(0, 1, 0)$. Then the perpendicular distance of this line from the origin equals	(C)	$\frac{\sqrt{5}}{3}$
(iv)	A plane passes through $(1, -2, 1)$ and is perpendicular to the two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$. Then the distance of the plane from the point $(1, 2, 2)$ is	(D)	$2/3$

Answer: (i) $\leftrightarrow$ (B), (ii) $\leftrightarrow$ (D), (iii) $\leftrightarrow$ (C), (iv) $\leftrightarrow$ (A).

Comments: Entry (i) is a direct continuation of Exercise (10.20), which asks for the partial sum, say S_n of the given series. In other words,

$$S_n = \sum_{i=1}^{n} \tan^{-1}(\frac{1}{2i^2}) = \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{8}) + \ldots + \tan^{-1}(\frac{1}{2n^2}) \tag{1}$$

There is no standard or obvious formula for this sum. But we can learn a little by experimenting with a few small values of n. The well-known identity $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ translates into

$$\tan^{-1}(x) + \tan^{-1}(y) = \frac{x + y}{1 - xy} \tag{2}$$

which is valid at least when both x, y lie in $[0, 1)$. Repeated applications of this give,

$$\begin{aligned}
S_1 &= \tan^{-1}(\frac{1}{2}) && (3)\\
S_2 &= \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{8}) = \tan^{-1}(\frac{1/2 + 1/8}{1 - 1/16}) = \tan^{-1}(\frac{2}{3}) && (4)\\
S_3 &= \tan^{-1}(\frac{2}{3}) + \tan^{-1}(\frac{1}{18}) = \tan^{-1}(\frac{2/3 + 1/18}{1 - 1/27}) = \tan^{-1}(\frac{39}{52})\\
&= \tan^{-1}(\frac{3}{4}) && (5)
\end{aligned}$$

The pattern is now clear. For every $n \in I\!N$, we have

$$S_n = \tan^{-1}(\frac{n}{n+1}) \qquad (6)$$

This can be proved by induction on n using (2). But in an objective type test that hardly matters. By taking the limit as $n \to \infty$, we get

$$\begin{aligned}
t = \sum_{i=1}^{\infty} \tan^{-1}(\frac{1}{2i^2}) &= \lim_{n\to\infty} S_n\\
&= \lim_{n\to\infty} \tan^{-1}(\frac{n}{n+1})\\
&= \tan^{-1}(\lim_{n\to\infty} \frac{n}{n+1}) && (7)\\
&= \tan^{-1}(1)\\
&= \frac{\pi}{4} && (8)
\end{aligned}$$

where in (7) we have used the continuity of the arctan function at 1 and Theorem 1 of Comment No. 3 of Chapter 16. So, $t = \pi/4$ and hence $\tan t = 1$.

In a conventional examination, a proof of (6) would have to be given. With it this would be a good problem, especially because it tests the ability to make experiments, recognise some pattern, come up with a guess and finally to prove the guess.

Item (ii) is also a good problem in trigonometry. But its beauty is marred by the objective type format of the test, exactly the same way as that of Q. 27. The real crux of the problem is to show that if the sides a, b, c are in an A.P. then $\tan^2(\frac{\theta_1}{2}) + \tan^2(\frac{\theta_3}{2})$ is a constant. Finding the value of this constant is a relatively minor matter. But in an objective test, this minor matter is all that matters. So, to find the value of this expression, all that we have to do is to take a *particular* triangle ABC of our choice and compute $\tan^2(\frac{\theta_1}{2}) + \tan^2(\frac{\theta_3}{2})$ directly for it. All we have to ensure is that a, b, c are in an A.P. We could, for example, take ABC to be a right angled triangle with $a = 3, b = 4$ and $c = 5$. But nothing can beat the case of an equilateral triangle. This is perfectly legitimate because it is nowhere required that three numbers in an A.P. have to be distinct. So, taking $a = b = c$, we have $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \frac{1}{2}$. Therefore, $\theta_1 = \theta_2 = \theta_3 = 60°$. Hence $\tan^2(\frac{\theta_1}{2}) + \tan^2(\frac{\theta_3}{2}) = 2\tan^2(30°) = \frac{2}{3}$. The answer is complete!

As in the case of Q. 27, maybe the paper-setters intended this short cut because that is all you can reasonably expect in a minute. Still, for the sake of completeness, we give a proof of the constancy of $\tan^2(\frac{\theta_1}{2}) + \tan^2(\frac{\theta_3}{2})$ under the conditions of the problem. We begin by expressing $\tan^2(\frac{\theta}{2})$ in terms of $\cos\theta$ by

$$\tan^2(\frac{\theta}{2}) = \frac{2\sin^2(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})} \quad = \quad \frac{1-\cos\theta}{1+\cos\theta} \tag{9}$$

Applying this for the given θ_1 and θ_3 and adding

$$\begin{aligned}
\tan^2(\frac{\theta_1}{2}) + \tan^2(\frac{\theta_3}{2}) &= \frac{1-\cos\theta_1}{1+\cos\theta_1} + \frac{1-\cos\theta_3}{1+\cos\theta_3} \\
&= \frac{b+c-a}{b+c+a} + \frac{a+b-c}{a+b+c} \\
&= \frac{2b}{a+b+c} = \frac{2b}{3b} = \frac{2}{3} \qquad (10)
\end{aligned}$$

since $a + c = 2b$. So, an honest proof was not, after all, difficult. But when you have to get it in less than a minute, dishonesty is tempting.

Item (iii) deals with the distance of a point from a line. In Q. 10 too, we encountered the distance of a point from a line. But that time both were in the xy-plane. Now they are in the three dimensional space. So this problem is qualitatively different. Let L be the given line. Since it is perpendicular to the plane $x+2y+2z=0$, it is parallel to the normal to this plane, and hence to the vector $\hat{i}+2\hat{j}+2\hat{k}$. Since it passes through $(0,1,0)$ its parametric equation in the vector form can be written down as

$$\vec{r} = \vec{r}(t) \quad = \quad t\hat{i} + (1+2t)\hat{j} + 2t\hat{k} \tag{11}$$

Let the perpendicular from the origin O to L fall at the point $P = \vec{r}(t_0)$. Then $OP \perp L$ which gives

$$\vec{r}(t_0) \cdot (\hat{i} + 2\hat{j} + 2\hat{k}) = 0 \tag{12}$$

(11) and (12) together give $t_0 + 2(1+2t_0) + 4t_0 = 0$ i.e. $t_0 = -\frac{2}{9}$, which determines P as $(-\frac{2}{9}, \frac{5}{9}, -\frac{4}{9})$. Therefore the distance of O from L, i.e. the length of $\overrightarrow{OP}$, equals $\frac{\sqrt{4+25+16}}{9} = \frac{\sqrt{5}}{3}$.

Item (iv) asks for the distance of a point from a plane. Since the plane passes through $(1,-2,1)$, its equation can be written in the form

$$a(x-1) + b(y+2) + c(z-1) = 0 \tag{13}$$

where a, b, c are some constants to be determined. The perpendicularity conditions give the following system of two homogeneous, linear equations in the three unknowns a, b, c.

$$\begin{aligned} 2a - 2b + c &= 0 && (14) \\ \text{and} \quad a - b + 2c &= 0 && (15) \end{aligned}$$

This system has no unique solution. But by Theorem 7 in Comment No. 17 in Chapter 3, it determines the relative proportions of a, b, c. Specifically, we get that a, b, c are proportional to the numbers $\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ and $\begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix}$ respectively. So, we may take $a = -3, b = -3$ and $c = 0$, or still better, $a = 1, b = 1$ and $c = 0$. Putting these in (13), we get the equation of the plane as

$$x + y + 1 = 0 \tag{16}$$

The distance of the point $(1, 2, 2)$ from this plane is $\dfrac{1+2+1}{\sqrt{1^2+1^2+0^2}} = 2\sqrt{2}$.

Essentially the same solution can be presented a little differently using the concept of cross product. Since the plane is perpendicular to the plane $2x-2y+z=0$, it is parallel to the normal to the latter plane. Thus we get that the given plane is parallel to the vector $2\hat{\imath} - 2\hat{\jmath} + \hat{k}$. Call this vector $\vec{u}$. Similarly the perpendicularity of the plane with the plane $x - y + 2z = 0$ makes it parallel to the vector $\hat{\imath} - \hat{\jmath} + 2\hat{k}$ ($= \vec{v}$) (say). Therefore the plane is perpendicular to the cross product of these two vectors, viz. to

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} \\ &= -3\hat{\imath} - 3\hat{\jmath} \end{aligned} \tag{17}$$

So, this vector is perpendicular to the plane. Further the plane passes through the point $(1, -2, 1)$. So its equation is

$$-3(x-1) - 3(y+2) + 0(z-1) = 0 \tag{18}$$

which reduces to (16). The rest of the work is the same.

When approached in this manner, the problem overlaps considerably with Q. 20 above, where too, the key idea was that a vector perpendicular to each of two given vectors is parallel to their cross product. Yet another duplication, which seems to have no purpose.

Whichever method is used in the solution, the problem is extremely straightforward. The formula for the distance of a point from a plane is the direct analogue of that for the distance of a point from a line in a plane. The latter was already used in the solution to the problem above about a parabola (Q. 10). This duplication could have been avoided. Apparently it is necessitated because in order to bring Item (iv) on par with the other three items in the first column, its answer has to be a single real number. Had it been left at merely asking for the equation of the plane, then the answer in the second column would have been $x + y + 1 = 0$ and it would have been obvious that this cannot be the answer to any of the other three items. So this is another example how the inherent constraints of formatting compel the paper-setters to append a problem with relatively unimportant ancillaries, thereby increasing the possibility that a candidate who has got the conceptual part of the problem correctly gets a rude jolt because of a silly slip in handling these appendages.

SUMMARY AND CONCLUDING REMARKS

The year 2006 is remarkable in the long history of the Joint Entrance Examination. Right from its start more than forty years ago, the JEE remained a conventional type examination till 1978. That is, you had to show not only the answers but the working for them.

The objective type questions made their first appearance in 1978. Their nature, purpose and importance varied considerably over the years. For example, initially the multiple choice questions had only one correct answer. Later on MCQs with one or more correct answers were introduced. Negative marking was introduced, then withdrawn for many years and re-introduced from 2002 onwards. Mechanical evaluation was started a few years earlier.

From 2000 to 2005, the JEE was conducted in two rounds. The first or the Screening round was fully objective. Only those who cleared it were allowed to appear for the second or the Main round. The papers in the Main round were the conventional types and the final selection was made solely on the basis of the performance in the second round. In effect, this meant that the screening was done efficiently with objective type questions but the ultimate selection was based solely on conventional types of questions.

It is for the first time in 2006 that the selection is based entirely on a single, fully objective type paper in each subject. This unprecedentedness warrants a more quantitative analysis of the question paper than we have been doing for the past three years (2003 to 2005) in the respective educative commentaries.

In all, the Mathematics question paper, to be completed in two hours, carries 184 marks. (On a proportionate time scale, this means you get less than 40 seconds per mark.) Only 24 marks out of these (Q. 33 to Q. 36) are for the fill in the blank questions, where the answers are to be filled in by circling the appropriate digits of a four digit number. The remaining 160 marks are for multiple choice questions. The last four questions (Q. 37 to Q. 40) ask for matching entries in one column with those in the second. But the same entry is allowed to have several matches. As a result, even if you know the matches of all but one, you do not automatically know the match of the last one. In effect this means that each entry is like an independent multiple choice question with one or more correct answers from the given ones. Also as in Q. 39, occasionally even the second column contained an item which first had to be evaluated before deciding if it matched with any entry in the first column.

Counting all these possibilities, the present JEE 2006 Mathematics question paper has 53 different problems to be worked out in two hours. The following tables give a classification of these problems. Listed against each problem are its location (L) i.e. its question number (as listed in this commentary), the number of marks (M) it carries and the broad area(s) of mathematics it comes from. This is followed by seven columns headed by the letters T, C, S, I, U, F and R. A tick mark ($\surd$) under these columns symbolises the following qualities:

T : (**Thought oriented**). This means that the success requires a certain key idea which will come only with a correct line of thinking.

C : (**Computational**). This means that the problem involves a fair amount of computation which can go wrong because of silly slips.

S : (**Sneaky approach**). This means that there is an efficient but wrong way to get to the right answer, or a part of it which gets rewarded because the candidate does not have to show the reasoning. (Not to be confused with an elegant solution, which is perfectly legitimate, and comes under T.)

I : (**Improper**). This means there is some impropriety such as some ambiguity or redundancy (not necessarily a mathematical mistake) or a poor notation or clumsiness in the statement of the problem or annexation of tidbits irrelevant to the main theme of the problem.

U : (**Unscrupulous**). This means that during or even after getting the correct answer, some additional justification will have to be given to complete the solution scrupulously had the same problem been asked in a conventional type examination.

F : (**Familiar problem**). This means a substantially similar problem has been asked in the past or is included in *Educative JEE Mathematics* by the author. (The exact reference is given in the comments on that problem.)

R : (**Repetitious problem**). This means that the work needed overlaps considerably (except for numerical differences) with another problem in this paper (whose serial number appears in the column).

Needless to say that such categorisation is open to controversies. Many problems require a combination of thought and computation and sometimes it is a matter of opinion which of the two dominates. If the thought needed is a commonplace (e.g. that of finding a vector perpendicular to a given plane), then the problem is not classified under T. In some problems both a thought and a computation are needed strongly and so the tick mark occurs in both the columns. When an elegant approach is possible, the problem is classified under T. Similarly whether a problem falls under U or not depends on the degree of expected completeness. The standards adopted here are those in the author's *Educative JEE Mathematics*. Also, some problems not listed under F here may be familiar because of inclusion in some other commonly used source. As for the area covered by a problem, sometimes it is a combination of two areas. But when one of them is superficial, we do not list it.

S. No.	Q. No.	M	Area(s)	T	C	S	I	U	F	R
1	1	3	Trigonometry, inequalities	$\surd$						
2	2	3	Evaluation of Limits		$\surd$					
3	3	3	Solution of triangles	$\surd$	$\surd$					
4	4	3	Trigonometric optimisation	$\surd$			$\surd$		$\surd$	

S. No.	Q. No.	M	Area(s)	T	C	S	I	U	F	R
5	5	3	Quadratic inequalities, trigonometric equations	$\surd$	$\surd$					
6	6	3	Complex numbers	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		
7	7	3	Number theory, combinatorics	$\surd$	$\surd$					
8	8	3	Finding antiderivatives	$\surd$	$\surd$	$\surd$				
9	9	3	Differential equations	$\surd$		$\surd$		$\surd$	$\surd$	
10	10	3	Equation of parabola	$\surd$	$\surd$		$\surd$			53
11	11	3	Integrals, maxima/minima	$\surd$	$\surd$			$\surd$		
12	12	3	Vector projection		$\surd$					20
13	13	5	Tangents to parabola		$\surd$	$\surd$		$\surd$		
14	14	5	Ellipse, hyperbola		$\surd$		$\surd$			
15	15	5	Solution of triangles	$\surd$					$\surd$	
16	16	5	Continuity, differentiability	$\surd$				$\surd$		
17	17	5	Finding a cubic		$\surd$		$\surd$		$\surd$	
18	18	5	Skew-symmetric matrix	$\surd$						
19	19	5	Differential equations	$\surd$	$\surd$		$\surd$		$\surd$	
20	20	5	Angle between two vectors		$\surd$					12, 53
21	21	5	Probability, limits	$\surd$	$\surd$		$\surd$			
22	22	5	Conditional probability	$\surd$						23
23	23	5	Conditional probability	$\surd$						22
24	24	5	Approximate integration	$\surd$			$\surd$			
25	25	5	Maxima/minima	$\surd$			$\surd$	$\surd$		
26	26	5	Theoretical calculus	$\surd$	$\surd$		$\surd$			
27	27	5	Incircle and circumcircle of a square	$\surd$		$\surd$	$\surd$			
28	28	5	Identifying locus type	$\surd$			$\surd$			
29	29	5	Triangle inscribed in a parabola	$\surd$	$\surd$					10, 37
30	30	5	Determinant of a matrix	$\surd$						
31	31	5	Sum of matrix entries	$\surd$	$\surd$					
32	32	5	Product of matrices		$\surd$					
33	33	6	Quadratic equations	$\surd$	$\surd$			$\surd$		
34	34	6	Reduction formula for definite integrals	$\surd$	$\surd$		$\surd$			
35	35	6	G.P., inequalities		$\surd$					
36	36	6	Theoretical calculus	$\surd$				$\surd$		
37	37(i)	1.5	Parabola, area of triangle		$\surd$					29
38	37(ii)	1.5	Circumradius of triangle		$\surd$					
39	37(iii)	1.5	Centroid of triangle		$\surd$	$\surd$				
40	37(iv)	1.5	Circumcentre of triangle	$\surd$	$\surd$	$\surd$				

S. No.	Q. No.	M	Area(s)	T	C	S	I	U	F	R
41	38(i)	1.5	definite integral		√					
42	38(ii)	1.5	area between parabolas		√					
43	38(iii)	1.5	angle between curves		√			√		
44	38(iv)	1.5	differential equation		√				√	
45	39(i)	1.2	coordinates, inequalities		√		√			
46	39(ii)	1.2	vectors		√		√			
47	39(iii)	1.2	definite integrals		√					
48	39(C)	1.2	definite integrals		√					
49	39(iv)	1.2	trigonometric optimisation	√			√		√	
50	40(i)	1.5	trigonometric identity, infinite sum	√	√				√	
51	40(ii)	1.5	solution of triangle	√		√				
52	40(iii)	1.5	distance from a line	√	√					
53	40(iv)	1.5	distance from a plane		√		√			12, 20
			Total counts	33	37	8	17	9	8	11

Setting a good JEE question paper is a tough examination where the paper-setters are the candidates and the answer-book they write is the question paper which they set for over three lakhs JEE aspirants, vying for every single mark. Because of the keen competition, even a difference of a few marks can mean the difference between being an electrical engineer and being a chemical engineer. And, if this happens because of flaws in the question paper, that is most regrettable.

Fortunately, the 2006 JEE Mathematics question paper does not have any serious mathematical mistakes. There are some obscure problems, such as those in Comprehension II (about estimation of integrals). Sometimes, as in Q. 39(i), the wording is confusing. Also, in Q. 39(iv), it is not given explicitly that A, B, C are the angles of a triangle. But generally, anybody would take this to be the case and so no harm is done.

But, even though there are no mathematical mistakes in the question paper, there are many other undesirable features as the tables above show. If the metaphor of the paper-setters being candidates at the paper-setting examination is to be stretched further, then the tables above are evaluations of their performance. Every tick mark in any of the columns marked under S (sneaky answer), I (impropriety), U (unscrupulous) and F (familiar problem) is like a red mark on the evaluated answer-book. For a good problem, all the last five columns should contain blanks. In fact, just as a cartoon without a verbal caption is considered as the best type of a cartoon, ideally a JEE question should have only a blank in the column under C (computation) too. That is, the computation involved should be so marginal that there is virtually no possibility of going wrong because of computational mistakes. Q. No.s 1, 16, 18, 22, 23, 30 and 36 are indeed of this type. Q. 28 would also qualify had the line L in it been introduced properly. The paper-setters deserve to be commended for setting these beautiful questions. Together they carry 39 marks out of 184.

It is obviously too much to expect that all problems be of this type. Many times computations are inevitable to give effect to the key idea. And as long as the computations needed can be reasonably completed in the time allowed (which is 40 seconds per mark as calculated earlier) there is no reason to shun a problem simply because it involves some computation. Moreover, the ability to carry out certain computations is also a prerequisite and has to be tested somewhere.

In the present paper there are a few problems where the computations can be reasonably completed in the time allowed, with some time left for checking their accuracy. They include Q. No.s 5, 7, 10, 11, 14, 15, 27, 28, 29, 31, 33, 34 and 35. Together they account for 60 marks.

The rest of the questions, which add up to nearly half the credit, are marred either because there is something confusing in the framing of the question (thereby forcing the candidates to spend extra time just to see what the question means) or because the computations involved are either tricky or straightforward but far too lengthy. Glaring examples of the latter type are Q. No.s 6, 8, 17, 19, 20, 21, 32, and all parts of Q. 37 to 40. It is ridiculous to expect that the area of the region between two parabolas (Q. 38(ii)) can be evaluated in less than a minute.

The trouble arises because the persons who participate in setting a question are usually poor judges of how long it takes to solve it. They have already done the thinking part and expect the others to get the key idea in a jiffy. They also often make the mistake of thinking that since no work needs to be shown, very little time is needed for the computations. Also, since they already know the correct answers to the computations, they probably think that the very first attempt is free of any numerical mistakes and hence no time is needed to check its accuracy. For most mortals this is far from the case.

To get a realistic idea of the time needed to answer a question completely, the paper-setters would do well to try candidate simulation. This means that one or two members of the paper-setting team should stay away completely while the remaining ones draft the question paper. Thereafter, these two members should attempt the problems as if they are candidates, with absolutely no help from those who have set the questions. Besides giving a realistic idea of the time needed, this will also serve to expose ambiguities and/or mistakes in the questions.

In terms of the topics covered, there are some avoidable duplications of work, listed in the last column of the tables above. The objective type format rules out the proofs of many identities and inequalities. Not surprisingly, binomial identities (or even the binomial theorem for that matter) find no representation in the entire paper. Trigonometric identities about a triangle feature in Q. 3 and in Q. 15. But at both the places, those who can recollect a relatively less known identity get an unfair advantage over those who have to come up with it. In a conventional examination, the questions could have given these identities and asked their proofs. In that case, the gap between these two types of candidates would have narrowed down considerably.

Also totally absent are identities about vectors. As remarked in the comments on Q. 20, a few of them could have been catered to, instead of asking the candidates to do qualitatively same computations in all questions about vectors. Among the conics, the ellipse and the hyperbola figure only once, viz. in Q. 14. The parabola, on the other hand, comes up in Q. No.s 10, 13, 28, 29, 37 and again in 38(ii). This unevenness could have been corrected. In the case of probability too, all the three problems asked (viz. Q. 21, 22 and 23) had the same underlying idea, viz. conditional probablility. Instead, a question involving either a complementary or a binomial probability would have been welcome.

All the four questions about matrices (viz. Q. 18, 30, 31, 32) are good. In fact, two of them are among the best questions in the whole paper. Together these four questions carry 20 marks, which is a disproportionately heavy credit for a single topic. Two of these four questions (e.g. Q. 31 and 32) could have been dropped to make some more room for number theory (which is barely touched upon in Q. 7) or binomial identities (which are not catered at all).

The overall picture is that speed will dominate all other qualities in the selection. That also increases the importance of memory, previous preparation and, most importantly, the strategy. Although Q. 13 to 36 carry more marks than Q. 1 to 12, there is not much difference in the level of difficulty (except for those that are thought oriented). So those who attempted the questions in the order they are asked stand to lose. The best strategy in a paper like this would be to attempt Q. 13 to 36 first and then to give the remaining time to Q. 1 to 12. If any time is left, try Q. 37 where the four parts have a common theme. In each of the remaining three questions, (viz. Q. 38 to 40), you have to answer four (or five) totally unrelated parts and even if you make a single mistake, you get nothing. It is a costly gamble to go for them.

The biggest disappointment comes from the comprehensions. This is elaborated at the end of the comments on Q. 32. Also, the author's concept of a comprehension is illustrated with a list of five hypothetical questions about the second comprehension (Q. 24 to 26).

OFFICIAL TEXT AND CONSEQUENT CHANGES

For JEE 2006, candidates were not allowed to take with them the question papers. This is in sharp contrast with a comparable examination like the AIEEE (meant for selection to the National Institutes of Technology) where the candidates are allowed to take away the test booklet (which contains the question paper). Nor were the JEE question papers displayed on the websites of the IITs, as is done by some state boards that conduct the CET (Common Entrance Test) for admission to the engineering colleges in their states. It is shocking that examinations of similar types and comparable purposes should differ so drastically in terms of their degree of transparency.

As a result, memory based versions of the questions had to be relied upon while preparing the present commentary. Given the large number of questions and the multiple answers to each, the text of the question paper became so voluminous that it was humanly impossible, even with a concerted effort, for any individual or any team to reproduce the full text verbatim. As a result, many obscurities remained (e.g. Q. 10, 31, 32 and 38(iv)) as discussed in the comments above.

It took more than two months and an intervention of the Right to Information Act to get a photocopy of the JEE 2006 Mathematics question paper. Many of the obscurities can now be resolved and the comments pertaining to the respective questions have to be revised accordingly. But we have preferred to keep them as they are, as a grim testimony of the ill effects of a needlessly secretive policy. Instead, we now give the official versions of the questions and the consequent revisions in the comments. We omit the texts where they substantially coincide with the memorised versions given above.

It may be noted that within each section, the questions are often permuted among themselves in different versions of the same question paper. We stick to the numbering in which the questions have appeared above. We shall only change the text where it is substantially different and then comment on the revised question.

SECTION I

Q. 1 No significant change.

Q. 2 No significant change.

Q. 3 No change except that instead of 'incircle' the words 'circle inscribed' are used. Also there is no mention of units, either for the length of the radius or for the area.

Q. 4 No significant change. Hence the improprieties discussed earlier remain.

Q. 5 For $0 < \theta < 2\pi$, if $2\sin^2\theta - 5\sin\theta + 2 > 0$, then θ lies in

(A) $(\frac{5\pi}{6}, \frac{9\pi}{8})$ (B) $(\frac{\pi}{8}, \frac{41\pi}{48})$
(C) $(0, \frac{\pi}{8}) \cup (\frac{\pi}{6}, \frac{41\pi}{48})$ (D) $(0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, 2\pi)$

Comment: This wording is better because the word 'interval(s)' could cause confusion when what is really meant is the union of intervals. The incorrect answers given here are different than the earlier ones. But that makes little difference because in this problem you cannot get the correct answer by eliminating the wrong answers. The correct answer remains the same except for the letter allotted to it.

Q. 6 Let $w = \alpha + i\beta$, $\beta \neq 0$ be a complex number. Then the set of complex numbers z, $(z \neq 1)$, such that

$$\frac{w - \overline{w}z}{1 - z} \text{ is a real number}$$

is

(A) $\{z : z \neq 1\}$ (B) $\{z : z \neq 1 \text{ and } |z| = 1\}$
(C) $\{z : |z| \neq 1\}$ (D) $\{z : z = \overline{z}\}$

Comment: This formulation is more direct. As in the last question, some of the wrong alternatives are different. In a way that makes it easier to do the problem in the sneaky way, i.e. by eliminating the wrong answers. For example, the point $z = 0$ satisfies (A), (C) and (D) all but not the given condition. So the correct answer must be (B). In the earlier version, the correct answer is the same, except that its letter was (D).

Q. 7 Let r, s, t be three distinct prime numbers. If p and q are two positive integers whose least common multiple is $r^2s^4t^2$, then the number of pairs (p, q) is equal to

(A) 252	(B) 242
(C) 225	(D) 224

Comment: There are two changes in the text of the question: (i) the three primes r, s, t are given to be distinct and (ii) the pairs (p, q) are not given to be ordered pairs. In the earlier solution, we had assumed (i) tacitly anyway. But the second change calls for some comment. (p, q) is a standard notation for the *ordered* pair whose first entry is p and the second entry is q. So, even though the word 'ordered' is not used explicitly, one may take the question implicitly to mean that it asks only for ordered pairs, In that case, the earlier solution will apply. The standard notation for the *unordered* pair consisting of p and q is $\{p, q\}$. So, if by (p, q) we mean the unordered pair, then the solution will have to be modified accordingly. For example, in finding the pairs of a and u in the solution, $(2, 0)$ and $(0, 2)$ represent the same pair. Similarly, $(2, 1)$ will equal $(1, 2)$. Hence, instead of 5, we have only three pairs, viz. $(2, 0), (2, 1)$ and $(2, 2)$. Similarly the number of pairs (b, v) is not 9 but 5 while the number of pairs (c, w) would come down from 5 to 3. Hence the answer to the problem would be $3 \times 5 \times 3$, i.e. 45. As this is not one of the given alternatives, we have to *assume* that (p, q) stands for an ordered pair. But this way of arriving at the intended interpretation is time-consuming to say the least. It would have been far better to clearly specify in the text of the question that (p, q) stands for an ordered pair.

Q. 8 No significant change. But the correct answer is listed as (B) instead of (A). So, those who attempt the solution simply by differentiating the given alternatives one by one, have to wait a little longer!

Q. 9 No substantial change.

Q. 10 A parabola has its axis along the line $y = x$ with vertex and focus in the first quadrant. If the distances of vertex and focus from the origin are $\sqrt{2}$ and $2\sqrt{2}$ respectively, then the equation of the parabola is

(A) $(x - y)^2 = 8(x + y - 2)$	(B) $(x + y)^2 = 2(x - y + 2)$
(C) $(x - y)^2 = 4(x + y - 2)$	(D) $(x + y)^2 = 2(x + y - 2)$

Comment: This formulation is unambiguous and simpler too. It coincides with the second possible interpretation given earlier. So, the equation of the parabola is $(x-y)^2 = 8(x+y-2)$, i.e. (A). It is strange that such a simply worded question was memorised with an unnecessarily clumsy wording.

Q. 11 No significant change except that the function $f(x)$ is defined by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

As before, $g(x)$ is defined by $g(x) = \displaystyle\int_0^x f(t)dt$, $0 \leq x \leq 3$. The four alternatives given for the answer are same as before. (And, luckily, their order is also the same.)

Comment: Because of the change in the definition of $f(x)$, instead of (1) in the earlier solution, we now have

$$g'(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

Similarly, instead of (2) we now have

$$g''(x) = \begin{cases} 1, & 0 \leq x < 1 \\ -e^{x-1}, & 1 < x < 2 \\ 1, & 2 < x \leq 3 \end{cases}$$

But these change do not affect the answer because the increasing/decreasing behaviour of $g(x)$ remains unaffected throughout the interval $[0,3]$. So, the correct answer is still (A). The last remark made in the earlier solution also remains intact. That is, if in the question as it is given now, the function $f(x)$ were defined by $f(x) = e^{x-1} - 2$ instead of $2 - e^{x-1}$ on the interval $(1,2]$, then the problem would have been far more interesting, because in that case $f(x)$ would have changed its sign at all the four points $1, 1+\ln 2, 2$ and e and both (A) and (B) would have been correct.

Q. 12 Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$ be three vectors. Then a vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is of magnitude $1/\sqrt{3}$ is

(A) $2\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$ (B) $4\hat{\imath} - \hat{\jmath} + 4\hat{k}$

(C) $4\hat{\imath} - 7\hat{\jmath} - 4\hat{k}$ (D) $3\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$

Comment: The alternatives given for the answer are different (numerically) from those given earlier. But the data is the same, except for the difference that we are now only given the *magnitude* of the projection on the vector $\vec{c}$ and not the projection itself. As a result, the actual projection can also equal $-1/\sqrt{3}$ instead of only $1/\sqrt{3}$. This does not affect most of the working. In particular, if we take a desired vector as $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ then we get $v_1 = v_3$ exactly as before. It is ironic that among the four given alternatives only (B) satisfies this condition. So, without doing anything further, a candidate can mark it as the correct answer! Note that here we have only used that $\vec{v}$ lies in the plane of $\vec{a}$ and $\vec{b}$. The part involving its projection on $\vec{c}$ does not come into the picture at all! To avoid this sneaky path, the paper-setters ought to have included at least one more alternative where the coefficients of $\hat{\imath}$ and $\hat{k}$ are equal.

For an honest answer, we must not, of course, capitalise on such lapses on the part of the paper-setters. Instead, let us see what the condition about the projection of $\vec{v}$ on $\vec{c}$ means. As in the earlier solution, the projection of $\vec{v}$ on $\vec{c}$ is $\dfrac{\vec{v}\cdot\vec{c}}{\sqrt{3}}$. But we can no longer set this equal to $1/\sqrt{3}$. Instead, we first take its magnitude, i.e. its absolute value and set it equal to $1/\sqrt{3}$. In other words, we get $\dfrac{\vec{v}\cdot\vec{c}}{\sqrt{3}} = \pm\dfrac{1}{\sqrt{3}}$, and hence $v_1 + v_2 - v_3 = \pm 1$ instead of the equation $v_1 + v_2 - v_3 = 1$ obtained earlier. As before, since we already know that $v_1 = v_3$, this equation simply means $v_2 = \pm 1$. Among the given alternatives, (B) is the only one where this condition is satisfied.

SECTION II

Q. 13 Tangents common to both the parabolas $y = x^2$ and $y = -x^2 + 4x - 4$ are

(A) $y = 4(x-1)$	(B) $y = 0$
(C) $y = -4(x-1)$	(D) $y = -30x - 20$

Comment: There is no substantial change, except that the in the equation of the second parabola, we now first have to complete the square. The purpose of adding such complications is far from clear. Surely, they are not relevant to the main theme of the problem. They only add to its drudgery. But then, possibly as a compensation, the paper-setters have also been kind enough to tell us that there are more than one tangents, instead of the earlier version which was fussily non-committal in this respect.

Q. 14 No significant change, except that in the last alternative, the focus of the parabola is $(3\sqrt{5}, 0)$ instead of $(5\sqrt{3}, 0)$. But that makes little difference because anyway it is a wrong alternative and in a problem like this you can't get the answer merely by eliminating the wrong alternatives sneakily.

Q. 15 No significant change, except that in the statement of the problem, the line through D perpendicular to AD is given to meet AC in E and AB produced in F. In other words, the word 'produced' is added. This is an old practice in geometry where a fussy distinction was made between a point on the side AB and a point on the side AB produced, the latter indicating that the point lies on the *line* AB but not on the *segment* AB. Sometimes the phrase 'produced if necessary' was used to allow both the possibilities. Nowadays, people are not so fussy about these trifles. In fact, in the present problem, the given possibility occurs when $b > c$, i.e. when the side AC is bigger than the side AB. If it were the other way, then the perpendicular to AD through D will meet the side AB and the side AC produced. But all the four alternative answers are independent of which of these two sides is bigger. So there was really no need to take sides here. A perfectly safe wording which encompasses both the possibilities would have been to simply say that the line through D perpendicular to AD meets the *lines* AC and AB in points E, F respectively.

Q. 16 No significant change.

Q. 17 Let $f(x)$ be a polynomial of degree 3 having a local maximum at $x =$

-1. If $f(-1) = 2, f(3) = 18$, and $f'(x)$ has local minimum at $x = 0$, then

(A) the distance between $(-1, 2)$ and $(a, f(a))$, which are the points of local maximum and local minimum on the curve $y = f(x)$ is $2\sqrt{5}$

(B) $f(x)$ is a decreasing function for $1 \leq x \leq 2\sqrt{5}$

(C) $f'(x)$ has a local maximum at $x = 2\sqrt{5}$

(D) $f(x)$ has a local minimum at $x = 1$

Comment: Qualitatively, the nature of the problem remains the same, viz. to determine a cubic polynomial from four pieces of data (because there are four unknowns associated with a general polynomial of degree 3) and then to answer questions about the known cubic.

But the numerical data is somewhat different than the earlier version. As a result, some of the calculations will change, although not in spirit. Exactly as before, because of the first and the last piece of the data, we take the cubic as $f(x) = ax^3 - 3ax + d$. But after this the calculations change numerically because the other two given conditions are slightly different. So, instead of the equations in (1) of the earlier solution, we now have

$$2a + d = 2 \quad \text{and} \quad 18a + d = 18$$

solving which we get $a = 1$ and $d = 0$. These values are, in fact, simpler than the ones we obtained earlier. So, the polynomial $f(x)$ is simply $x^3 - 3x$. This gives $f'(x) = 3x^2 - 3x$ and $f''(x) = 6x - 3$, from which we further get that f has a local maximum at $x = -1$ (which is given to us anyway) and a local minimum at $x = 1$ (which is new to us). So, we get $a = 1$ and $(a, f(a)) = (1, f(1)) = (1, -2)$. The distance of this point from $(-1, 2)$ is $\sqrt{4 + 16} = 2\sqrt{5}$. So the statement (A) is true. From $f'(x) = 3x(x - 1)$ we see that $f(x)$ is increasing on $[1, \infty)$ and hence, in particular on $[1, 2\sqrt{5}]$. So (B) is false. As for (C), $f''(x) = 6x - 3$ does not vanish at $x = 2\sqrt{5}$. So $f'(x)$ has neither a local minimum nor a local maximum at $x = 2\sqrt{5}$. Hence (C) is also false. The truth of (D) is already known.

Q. 18 No significant change.

Q. 19 Let C be a curve such that the tangent at any point P on it meets the x-axis and the y-axis at A and B respectively. If $BP : PA = 3 : 1$ and the curve passes through the point $(1, 1)$, then

(A) The curve passes through $(2, 1/8)$

(B) Equation of normal to the curve at $(1, 1)$ is $3y - x = 2$

(C) The differential equation for the curve is $3y' + xy = 0$

(D) The differential equation for the curve is $xy' + 3y = 0$

Comment: The question is basically the same. But the alternatives are slightly different. The wording of the data is a little faulty. The first sentence does not specify any property of the curve. It would do so only if the points A and B were introduced earlier. Instead, what you are saying is that you are considering a typical point, say P, on the curve C and introducing A and B as the points where the tangent at P meets the x- and the y-axes respectively. You can do this to *any* curve. So, it is wrong to use phrases like 'such that' or 'with the property that' in a sentence like this. This is as silly as saying, "Let A be a man such that his father is F." A better wording is to say, "Let A be a man and let F be his father."

In the given question, it is only the second sentence which specifies the vital properties which characterise the curve. A better wording would have been

"Let C be a curve with the property that whenever the tangent at any point P on it meets the x- and the y- axes at points A and B respectively, $BP : PA = 3 : 1$. Suppose also that C passes through the point $(1, 1)$."

In this formulation, the first sentence implies that the curve C satisfies a certain geometric property. This property makes the curve belong to a certain class of curves (specifically, to a one parameter family of curves). The second sentence then tells us which member of this class is the given curve C. If the first sentence in this formulation appears too complicated, here is a simpler one where the contents of the first sentence are split into two.

"Let P be a variable point on a curve C and let the tangent to C at P meet the x-axis at A and the y-axis at B. Suppose, for every position of the point P on the curve, $BP : PA = 3 : 1$. If further, C passes through the point $(1, 1)$, then"

While the given formulation is not likely to cause any confusion, one hopes that a certain linguistic discipline is followed in framing the text of a question.

Mathematically, the data is exactly the same as before and so the working also remains the same. We already derived $xy' + 3y = 0$ as the differential equation satisfied by the curve C. So (D) is a correct statement and (C) is false. We also solved this d.e. earlier and showed that (A) is true. But the alternative (B) here is different than the earlier. The slope of the normal at $(1, 1)$ is $\frac{1}{3}$ as calculated earlier. Hence the equation of the normal is $y = \frac{1}{3}x + \frac{2}{3}$. Thus we see that (B) is true now.

Q. 20 No significant change. (Hats off to persons who can correctly remember so much numerical data when the particular numbers occurring in it are essentially arbitrary and follow no logical pattern.)

SECTION III

Questions from Q. 21 to Q. 32 have been grouped into four groups. Each bunch of three consecutive questions is preceded by a 'paragraph' to be read first. (Earlier we called it the 'preamble'.) The paragraph is supposed to give the common setting for the three questions appearing under it.

The paragraph for Q. 21 to Q. 23 reads :

" There are n urns numbered $1, 2, \ldots, n$ each containing $(n+1)$ balls. Urn i contains i white balls and $(n + 1 - i)$ red balls, $i = 1, 2, \ldots, n$. An urn is selected and a ball is drawn at random from it. Let U_i denote the event that urn numbered i is selected and let W denote the event that a white ball is drawn from the selected urn. Further suppose that E denotes the event that an even numbered urn is selected."

Q. 21 No significant change.

Q. 22 No significant change.

Q. 23 No significant change.

Comment: The 'paragraph' above is worded in a language which is easy to understand. But there is little 'comprehension' in it because it introduces no new concept. The three problems are like any other problems and the 'paragraph' is merely a statement of the hypothesis.

The paragraph for Questions from 24 to 26 is a little more detailed. Also, there is indeed something conceptual in it, viz. approximation of a definite integral. It reads as follows:

"Let $y = f(x)$ be a twice differentiable, non-negative function defined on $[a, b]$. The area $\int_a^b f(x)dx,\ b > a$ bounded by $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ can be approximated as

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}\{f(b) - f(a)\}.$$

Since $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ c \in (a, b)$, a better approximation to $\int_a^b f(x)dx$ can be written as

$$\int_a^b f(x)dx \approx \frac{(c-a)}{2}\{f(a) + f(c)\} + \frac{(c-b)}{2}\{f(c) + f(b)\} \equiv F(c).$$

If $c = \dfrac{a+b}{2}$, then this gives :

$$\int_a^b f(x)dx \approx \frac{b-a}{4}\{f(a) + 2f(c) + f(b)\}. \tag{1}$$

Q. 24 The approximate value of $\int_0^{\pi/2} \sin x\ dx$ using rule (1) given above is

(A) $\dfrac{\pi}{8\sqrt{2}}(1 + \sqrt{2})$ (B) $\dfrac{\pi}{4\sqrt{2}}(1 + \sqrt{2})$

(C) $\dfrac{\pi}{8}(1 + \sqrt{2})$ (D) $\dfrac{\pi}{4}(1 + \sqrt{2})$

Comment: Now the question is unambiguous because it clearly tells you which formula to apply.

Q. 25 No significant change.

Q. 26 No significant change. But now the ambiguity is gone because the symbol $\int_a^b f(x)dx$ means only one thing, viz. the integral of $f(x)$ over $[a, b]$.

The paragraph for Q. 27 to 29 reads :

" The length of a side of a square $ABCD$ is 2. A circle C_1 inside the square touches all the four sides of the square and another circle C_2 passes through all the four vertices of the square. Let P and Q be any two points on the circles C_1 and C_2 respectively. A point S moves in the coordinate plane such that it is always at an equal distance from a fixed line L and a fixed point R. A circle C in the coordinate plane always touches the circle C_1 externally and the line L.

Q. 27 No significant change.

Q. 28 Let L be the line passing through any two adjacent vertices of the square. If circles C_1 and C are on the same side of the line L, then the locus of the centre of C is

(A)	an ellipse	(B)	a hyperbola
(C)	a parabola	(D)	a pair of straight lines

Comment: The same objections given earlier apply here. It is unfair to introduce the line L in the paragraph without any restriction and then put a restriction on it later. However, the particular restriction on L given here simplifies the problem slightly and makes the co-ordinate geometry solution more lucrative. (The pure geometry solution undergoes little change.)

Without loss of generality, we take L to pass through the vertices A and B. So, if we set up the coordinate system as in the solution to Q. 27, the equation of L comes out to be $y = 1$. Let (h, k) be the centre of the variable circle C and r its radius. As C and C_1 lie on the same side of L, we have $k \leq 1$ and so the perpendicular distance of (h, k) from L is $1 - k$. Since C is given to touch L, we have

$$r = 1 - k \tag{1}$$

But C also externally touches C_1 whose centre is $(0,0)$ and radius 1. Therefore we also have

$$1 + r = \sqrt{h^2 + k^2} \tag{2}$$

The desired locus is obtained by eliminating r from (1) and (2). It comes out as

$$2 - k = \sqrt{h^2 + k^2} \tag{3}$$

which simplifies to

$$h^2 = 4 - 4k \tag{4}$$

which is a parabola.

Q. 29 Let the point R be at A and L be the line passing through the vertices B and D, adjacent to A. If the locus of the point S intersects AC at T_1 and the line passing through A parallel to L at T_2 and T_3, then the area of the triangle $T_1T_2T_3$ is

(A) 4	(B) 1
(C) 3	(D) 2

Comment: There is no mention of units for the area. The definition of the line L given here is different from that in the last question. Also the manner in which it is given is unnecessarily clumsy. It would have been easier to say that let L be the line along the diagonal BD. Also it is not clear what is gained by introducing R as a 'fixed point' in the paragraph, and by identifying it with A now. Neither of the other two questions make any reference to the point R. So it would have been far better to simply drop any mention to it in the paragraph and word the present question suitably.

The contents of the question remain the same and so does the answer.

The paragraph for Questions from 30 to 32 reads:

“ Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. Suppose U_1, U_2, U_3 are three column vectors such that

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ and } AU_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

U is a 3×3 matrix whose first, second and third columns are U_1, U_2 and U_3 respectively. ”

Q. 30 The value of the determinant of U is

(A) 2 (B) 3
(C) 6 (D) 12

Comment: The phrase ‘column matrices’ used earlier has been replaced by the more customary ‘column vectors’. Conceptually the problem is the same. But the entries of the matrix A as well as those of AU_3 are different from the earlier ones. As a result, most of the computations have to be redone. For the present question, we follow the short cut of finding the determinant of U without first finding U itself. We are given that

$$AU = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

from which we readily compute $|AU| = 3$ quickly because this is an upper triangular matrix. With A we are not so lucky. Still, by straight expansion we see that its determinant is 1. Substituting these values in the equation

$$|AU| = |A||U| \tag{2}$$

we get $|U| = 3$. This is the same answer as before. But that is a coincidence because the data is now numerically different.

Q. 31 The sum of the elements of U^{-1} is

(A) $\dfrac{1}{12}$ (B) $\dfrac{1}{6}$

(C) $\dfrac{1}{3}$ (D) $\dfrac{1}{4}$

Comment: Once again, we try to avoid having to compute U. As in the earlier solution, if we let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

then (1) reads as $AU = B$. Hence $U = A^{-1}B$. Taking the inverses of both the sides we get

$$U^{-1} = B^{-1}A \tag{4}$$

and hence the desired sum, say S, of all entries of U^{-1} is given by

$$S = [1\ 1\ 1]\ B^{-1}A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{5}$$

which is exactly same as (17) in the earlier solution except that the matrices A and B are different now. So we have do the computations over again. But luck is still with us, because the matrix B remains upper triangular and therefore its inverse can be computed a lot more efficiently than that of a general matrix. We skip the computations and only write down the result, viz.

$$B^{-1} = \begin{bmatrix} 1 & -2/3 & -5/3 \\ 0 & 1/3 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

As before, instead of computing $B^{-1}A$ (which would give us U^{-1}), we use associativity of matrix multiplication in (5) and calculate P and Q where where

$$P = [1\ 1\ 1]\ B^{-1} \tag{7}$$

$$\text{and} \quad Q = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{8}$$

both of which can be done by inspection from the entries of B^{-1} and A. Thus $P = [1 \quad -1/3 \quad -4/3]$ and $Q = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$. Multiplying the two we finally get $S = 3 + 0 - (8/3) = 1/3$. Hence the correct answer is (C).

Q. 32 The value of $[3\ 2\ 0]U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(A) 13 (B) 26
(C) 12 (D) 24

Comment: As observed in the comments on the earlier version of the question, there would be an elegant way to do this problem if B^{-1} were the transpose of the matrix A. For, in that case A^{-1} would equal B^t, the transpose of B. Since $U = A^{-1}B$, the desired matrix product would equal $[3\ 2\ 0]B^tB \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, which can be evaluated simply by computing the row vector $[3\ 2\ 0]B^t$ and then taking the sum of the squares of its entries.

But unfortunately, we do not have $B^{-1} = A^t$. So there is apparently no elegant way to do the problem. The only simplification we can do is to rewrite the expression asked as $[3\ 2\ 0]A^{-1}B \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and hence as the product LM where L is a row vector and M is a column vector, defined respectively by

$$L = [3\ 2\ 0]A^{-1} \tag{9}$$

$$\text{and} \quad M = B \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \tag{10}$$

There is no elegant way to compute A^{-1} either because the matrix A is not of a particularly simple type such as a triangular matrix. So, we compute A^{-1} by taking the transpose of the matrix of the cofactors of its entries and dividing by the determinant $|A|$, which, fortunately equals 1. The answer comes out to be

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ -1 & -2 & 2 \end{bmatrix} \tag{11}$$

A direct computation gives

$$L = [3\ 2\ 0] \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ -1 & -2 & 2 \end{bmatrix} = [1\ 1\ 1] \tag{12}$$

$$\text{and} \quad M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} \tag{13}$$

and hence, finally,

$$[3\ 2\ 0]U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = LM = [1\ 1\ 1] \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} = 7 + 6 + 0 = 13 \tag{14}$$

So the correct answer is (A).

As there is no elegant way to answer this question, the work becomes laborious and very prone to numerical errors. Also, it is highly repetitious. When you multiply two matrices of order 3 or find the inverse of a general 3×3 matrix, you have to do qualitatively same computation 9 times. The purpose of testing this ability is questionable. Comparatively, the computations in the last question were not such a drudgery if you followed the elegant approach. Still, in terms of proportionate time (viz. just about 3 minutes for a 5 points question), they were too many. It is only Q. 30 which can truly be called a good question.

SECTION IV

Q. 33 No significant change, except that a, b, c, d are given to be real numbers, which is nowhere needed in the solution.

Q. 34 No significant change.

Q. 35 No significant change.

Q. 36 No significant change.

SECTION V

Q. 37 No significant change.

Q. 38 There is no change in the first three entries of the first column and the entries (A), (B) and (D) of the second. But entry (C) is given as $2 \log 6$ instead of $6 \ln 2$. Item (iv) of the first column reads :

" A continuous function $f : [1, 6] \longrightarrow [0, \infty)$ is such that $f'(x) = \dfrac{2}{x + f(x)}$ and $f(1) = 0$. Then the maximum value of f cannot exceed "

Comment: The differential equation satisfied by the function $f(x)$ is the same as before (except for the change of notation). The initial condition is also the same. So, the solution also remains the same, viz.

$$x + f(x) + 2 = 3e^{f(x)/2} \tag{1}$$

The problem involves the maximum of the function $f(x)$ over the interval $[0, 6]$. It is impossible to solve (1) explicitly for $f(x)$. Nor is it needed. From the statement of the question we have to assume that the denominator $x + f(x)$ never vanishes in the interval $[0, 6]$. So, by continuity, it has to be either positive throughout or negative throughout. The latter possibility is excluded because we are given that $1 + f(1) = 1 > 0$. So we conclude that $x + f(x)$ and hence also $f'(x)$ are positive throughout the interval $[0, 6]$. Therefore $f(x)$ is strictly increasing on $[0, 6]$. Hence the maximum of $f(x)$ on $[0, 6]$ occurs at $x = 6$. This maximum, say M $(= f(6))$ is given by

$$M + 8 = 3e^{M/2} \tag{2}$$

Again, from this equation we cannot determine M explicitly. But that is not necessary either. All we are asked is to find which of the four given alternatives in the second column of the question are upper bounds on M. These alternatives (in ascending order) are $0, 1, 4/3$ and $2 \ln 6$. (We are not given an approximate value of $\ln 6$. However, for the purpose of the present comparison, it is enough that $\ln 6 > 1$ which is true since $6 > e$.)

The problem now boils down to deciding between which adjacent pair M lies. We have no way of computing the number M even approximately from (2). So, it will not be easy to compare M directly with other numbers. But we know that $M = f(6)$ and since $f(x)$ is known to be strictly increasing it is easier to compare M with the values of f at other points. Let us see if there is any value of x in the interval $[0, 6]$ at which $f(x) = 2$. If we can show that such a value exists, then it will follow that $M \geq 2$. To look for such x we put $f(x) = 2$ in (1) and solve the resulting equation for x to get $x = 3e - 4$. Since $2 < e < 3$, we see that this number lies between 2 and 5. So, it is in the interval $[0, 6)$ and therefore, as we just said, we know $M > 2$, which automatically implies that M is bigger than all the three numbers $0, 1$ and $4/3$ in the second column. Hence none of these three is a correct match for the entry (iv) in the first column of the question.

It only remains to compare M with the number $2 \log 6$. So, let us see if there is any value of x for which $f(x) = 2 \log 6$. Once again we put $f(x) = 2 \ln 6$ in (1) and solve to get $x = 16 - 2 \log 6$. Even though we do not know an approximate value of $\log 6$, from the knowledge that $2 < e$ we know that $6 < 8 < e^3$ and hence taking natural logarithms, $\log 6 < 3$. Therefore, $16 - 2 \ln 6 > 16 - 2 \times 3 = 10$. It follows that there is no value of x in $[0, 6]$ for which $f(x) = 2 \log 6$. From this we claim that $M < 2 \log 6$. For, otherwise we shall have $2 \log 6 \leq M = f(6)$. Since we also have $2 \log 6 > 0 = f(1)$, by the Intermediate Value Property, there would be some point in the interval $[1, 6]$ at which f would equal exactly $2 \ln 6$. But we just showed that there is no such point even in the larger interval $[0, 6]$.

So, finally we have proved that the maximum value of f on $[0, 6]$ cannot exceed $2 \log 6$. Hence the correct match for (iv) in the fist column is (C) in the second column. The reasoning needed to arrive at this is non-trivial and that makes the problem an excellent one in an examination where the ability to reason is tested. But in the present case, chances are that most students will simply not bother about this reasoning and will tick the answer $2 \log 6$ on the superficial ground that since the solution of the differential equation involves the exponential function, there is a good chance that a problem about it would have natural logarithms in it. Such gamblers will actually be rewarded. Yet another instance where a good problem is marred by the multiple choice format of the examination.

Q. 39 There is no change in the entries in the second column (except for their relative order). In the first column, (ii), (iii) and (iv) are the same. But there is a slight change in the first two entry. It reads :

" The set of values of a for which the lines

$$x + y = |a|, ax - y = 4,$$

intersect in the region $x > 0, y > 0$, is the interval (a_0, ∞). Then the value of a_0 is "

Comment: This formulation is much better than the earlier one because it avoids the word 'ray'. There is a slight change in the numerical data, viz. the R.H.S. of the equation of the second line is 4 instead of 1. So the calculations have to be redone. But the method remains the same. Solving the equations together, we get

$$x = \frac{|a| + 4}{a + 1} \quad (1)$$

$$\text{and} \quad y = \frac{a|a| - 4}{a + 1} \quad (2)$$

as the coordinates of their point of intersection. As before the positivity of x implies $a + 1 > 0$, i.e. $a > -1$. In view of this, the positivity of y means that $a|a| > 4$. This rules out negative values of a and therefore

$a|a|$ is simply a^2. So we have $a^2 > 4$ or equivalently, $a > 2$ as a is already known to be positive. So $a_0 = 2$. Thus the correct match for (i) is (A) and not (B) as in the earlier version.

Q. 40 Item (A) in the second column is 0 instead of $2\sqrt{2}$. In the first column, there is no substantial change in the first three entries. But the last entry is a little different numerically. It reads :

(iv) Let P be the plane passing through the point $(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x+2y+z = 2$. Then the distance from the point $(\sqrt{3}, 2, 2)$ to the plane P is "

Comment: As the changes are only numerical, we spare the details. To get the equation of the plane P we follow the second approach, based on the cross product of the normals to the two given planes. These normals are $2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ respectively. Their cross product is

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

So, a normal vector to P can be taken as $\hat{\imath} - \hat{\jmath} + \hat{k}$. As the plane passes through $(1, 1, -1)$, its equation is

$$(x - 2) - (y - 1) + (z + 1) = 0$$

i.e. $x - y + z = 0$. Hence its perpendicular distance from the point $(\sqrt{3}, 2, 2)$ is $\dfrac{\sqrt{3} - 2 + 2}{\sqrt{1 + 1 + 1}} = 1$. So the correct match for (iv) in the first column is (B). (This is also the correct match for (i). But the preamble to the Section V makes it clear that this is permissible.)